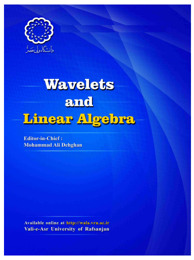
Wavelets and Linear Algebra 13(1) (2026) 57-65

Vali-e-Asr University
of Rafsanjan

Wavelets and Linear Algebra

http://wala.vru.ac.ir



Automatic continuity of linear mappings and homomorphisms on topological algebras

Ali Naziri-Kordkandi[a,*]

[a]*Department of Mathematics, Payame Noor University, Tehran, Iran*

ARTICLE INFO

Article history:

Accepted 16 May 2026
Available online 20 July 2026

Keywords:
Frames
Fundamental
topological algebras
Automatic continuity
Spectral radius
Radius of boundedness
Homomorphism

2000 MSC:
46H05
46H40

ABSTRACT

In this paper, we investigate the automatic continuity of linear mappings and homomorphisms on certain class of fundamental F-algebras and more generally on topological algebras. Some examples illustrating these results are presented as well.

1. Introduction

Throughout this paper, all topological algebras are assumed to be Hausdorff and defined over the field $\mathbb{C}$. The automatic continuity of linear mappings and homomorphisms plays a significant

*Corresponding author
Email address: ali_naziri@pnu.ac.ir (Ali Naziri-Kordkandi)

http://doi.org/10.22072/wala.2026.2081221.1484

role in the study of topological algebras and modern mathematical analysis. A classical starting point in this area is the well-known result that every character on a Banach algebra is automatically continuous [4]. Various extensions of automatic continuity in Banach and topological algebras have since been obtained by many authors; see, for example,[5, 6, 7]. A noteworthy class of topological algebras, namely fundamental topological algebras, exhibits several interesting structural properties that relate closely to automatic continuity. These properties allow the extension of several automatic continuity results to more general topological algebraic setting. In this work, we study the automatic continuity of linear mappings and homomorphisms on fundamental F-algebras and more generally on topological algebras. Some examples are discussed as well. Our approach focuses on identifying concise structural conditions expressed in terms of defining the spectral radius and the radius of boundedness under which algebraic homomorphisms between fundamental F-algebras are automatically continuous. By emphasizing how these quantities control the behavior of homomorphisms, we extend classical automatic continuity results and clarify the precise role of spectral and boundedness constraints in guaranteeing continuity.
This paper is organized as follows. Section 2 presents the necessary preliminaries, and Section 3 establishes automatic continuity results for homomorphisms on fundamental F-algebras. Section 4 extends these results to general topological algebras, and Section 5 contains concluding remarks.

2. Preliminaries

We begin this section by recalling a number of preliminary concepts and related results that will be used throughout the paper.

Definition 2.1. Let A be a unital algebra with a unit e. The spectrum of an element $x \in A$, denoted $sp_A(x)$, is the set

$$sp_A(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible}\}.$$

The spectral radius of an element x, written $r_A(x)$, is defined by

$$r_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\}.$$

If A does not have a unit element, the spectrum $sp_A(x)$ of $x \in A$ is defined by

$$sp_A(x) = \left\{\lambda \in \mathbb{C} - \{0\} : \frac{x}{\lambda} \text{ is not quasi-invertible}\right\} \cup \{0\}.$$

Lemma 2.2. *[10] If A is an algebra, then its Jacobson radical is*

$$\text{Rad}(A) = \{x \in A : r_A(xy) = 0; \textit{ for any } y \in A\}.$$

Definition 2.3. [1] Let x be an element of a topological algebra A. We say that x is bounded if there exists some $r > 0$ such that the sequence $\left(\frac{x^n}{r^n}\right)_n$ converges to zero. The radius of boundedness of x with respect to A is denoted by $\beta_A(x)$ and defined by

$$\beta_A(x) = \inf\left\{r > 0 : \left(\frac{x^n}{r^n}\right) \to 0\right\},$$

with the convention: $\inf \emptyset = +\infty$. Also, we denote

$$S(A) = \{x \in A : \beta_A(x) = 0\}.$$

Definition 2.4. [2] A topological algebra A is said to be a fundamental if there exists $b > 1$ such that for every sequence $(a_n)_n$ in A, whenever $b^n(a_n - a_{n-1})$ converges to zero, then $(a_n)_n$ must be Cauchy.

There are many topological algebras which are not fundamental (see [2]).

Example 2.5. [2] The space $M([0, 1))$ of all measurable functions on $[0, 1)$ with the convergence in measure topology is not fundamental.

It is well-known that every locally convex and locally bounded algebra is a fundamental topological algebra. [2].

Definition 2.6. A topological algebra that is complete and metrizable is referred to as an F-algebra.

Lemma 2.7. *[8] Let A be a fundamental F-algebra and $x \in A$. Then*

$$r_A(x) \leq \beta_A(x).$$

Definition 2.8. [6] Let A be a topological algebra.

(i) A is said to be strongly sequential if there exists a neighbourhood U of 0 such that for all $x \in U$ we have $x^k \to 0$ as $k \to \infty$.

(ii) A is said to be sequential if for each sequence $(x_n)_n, x_n \to 0$ there exists $x_m \in (x_n)_n$ such that $x_m^k \to 0$ as $k \to \infty$.

Proposition 2.9. *[6] With reference to the above definitions, (i) implies (ii) but the reverse implication need not hold.*

Definition 2.10. A Q-algebra is a topological algebra whose the set of invertible elements is open.

Definition 2.11. [4] Let A and B be F-spaces and $T : A \to B$ be a linear mapping. The separating space of T is defined by

$$S(T) = \{b \in B : \text{ there exists } (a_n)_n \text{ in } A \text{ s.t. } a_n \to 0 \text{ and } Ta_n \to b\}.$$

The separating space $S(T)$ is a closed linear subspace of B; moreover, by the Closed Graph Theorem, T is continuous if and only if $S(T) = \{0\}$.

3. Continuity on fundamental F-algebra

In this section, we study some results, in particular, the automatic continuity of homomorphisms on fundamental F-algebras.

Lemma 3.1. *Let A be a fundamental topological algebra, then the closure of A is also a fundamental topological algebra.*

Proof. straightforward. □

Theorem 3.2. *Let A be a fundamental topological algebra and B be a topological algebra. If $T : A \longrightarrow B$ is a injective dense range homomorphism such that T and T^{-1} are continuous, then B is a fundamental topological algebra.*

Proof. Our hypothesis implies that A and $T(A)$ are homeomorphic. So $T(A)$ is also fundamental topological algebra. By Lemma 3.1, B is a fundamental topological algebra. □

From the proof of [3, Theorem 4.5], we have the following lemma.

Lemma 3.3. *Let A be a fundamental F-algebra and φ be a character on A. If for some $b > 1$, $b^n x^n \to 0$ in A, $x \in A$, then $|\varphi(x)| < 1$.*

Theorem 3.4. *Let A be a unital fundamental F-algebra with a unit e and $x \in A$. Assume that $\varphi : A \longrightarrow \mathbb{C}$ is a continuous character on A such that for some $b > 1$, $b^n x^n \to 0$ as $n \to \infty$. If $\lambda \in \mathbb{C}$ is such that $|\lambda| > 1$, Then*

$$(\lambda e - x)^{-1} = \lambda^{-1} e + \sum_{n=1}^{\infty} \frac{x^n}{\lambda^{n+1}} \quad \textit{and} \quad |\varphi((\lambda e - x)^{-1})| \leq \frac{1}{|\lambda| - 1}.$$

Proof. If $|\lambda| > 1$, then

$$b^n \left(\frac{x}{\lambda}\right)^n \longrightarrow 0$$

By [3, Theorem 4.1], we have

$$(e - \frac{x}{\lambda})^{-1} = e + \sum_{n=1}^{\infty} x^n \lambda^{-n},$$

so

$$(\lambda e - x)^{-1} = \lambda^{-1} e + \sum_{n=1}^{\infty} x^n \lambda^{-n-1} = \lambda^{-1} e + \sum_{n=1}^{\infty} \frac{x^n}{\lambda^{n+1}}.$$

Now, the continuity of φ and Lemma 3.3 imply that

$$\begin{aligned}\left|\varphi((\lambda e - x)^{-1})\right| &= \left|\varphi\left(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}\right)\right| = \left| \sum_{n=0}^{\infty} \frac{\varphi(x^n)}{\lambda^{n+1}} \right| \leq \sum_{n=0}^{\infty} \frac{|\varphi(x^n)|}{|\lambda^{n+1}|} \\ &\leq \sum_{n=0}^{\infty} \frac{1}{|\lambda^{n+1}|} \\ &\leq \frac{1}{|\lambda| - 1}.\end{aligned}$$

□

The following theorem is fundamental for the automatic continuity of homomorphisms.

Theorem 3.5. *Let A be a strongly sequential fundamental F-algebra. Then every character $\varphi : A \longrightarrow \mathbb{C}$ is continuous.*

Proof. Let $(x_k)_k \subseteq A$ be a sequence such that $x_k \to 0$ and let U be a neighborhood of zero in A satisfying Definition 2.8. Let $\varepsilon > 0$. There exists $k_0 \in N$ such that $b\varepsilon^{-1}x_k \in U$ for all $k \geq k_0$ and some $b > 1$. So $(b\varepsilon^{-1}x_k)^n \to 0$ as $n \to \infty$, for every $k \geq k_0$. By Lemma 3.3, $|\varphi(x_k)| < \varepsilon$. Thus, $\varphi : A \longrightarrow \mathbb{C}$ is continuous. □

We now extend this result to homomorphisms.

Theorem 3.6. *Let A be a strongly sequential fundamental F-algebra and B be a commutative semisimple Banach algebra. Then every homomorphism $T : A \to B$ is continuous.*

Proof. Let $\varphi \in \Gamma_B$, where Γ_B denotes the space of continuous characters of B. Then $\varphi \circ T$ is a continuous character on A by Theorem 3.5. Suppose that $x_n \to 0$ in A and $Tx_n \to y$ in B. By the continuity of $\varphi \circ T$, we have $(\varphi \circ T)(x_n) \to 0$. On the other hand, $(\varphi \circ T)(x_n) = \varphi(Tx_n) \to \varphi(y)$. Consequently, $\varphi(y) = 0$. So we have

$$y \in \bigcap_{\varphi \in \Gamma_B} \ker \varphi = \mathrm{rad}(B) = \{0\}.$$

Hence, by the Closed Graph Theorem, T is continuous. □

Theorem 3.7. *Let $A$and B be F-algebras such that B is fundamental with a left (resp. right) uniformly bounded approximate identity. If $T : A \longrightarrow B$ is a surjective homomorphism, then the separating space $S(T)$ is a closed ideal in B.*

Proof. By [7, Proposition 5.1.2], the separating space $S(T)$ is a closed linear subspace of B. Let $y \in S(T)$ and $x \in A$. Then there exists a sequence $(x_n)_n$ in A such that $x_n \to 0$ and $T(x_n) \to y$. Since B is a fundamental F-algebra, by the Cohen's Factorization Theorem for fundamental F-algebras [2, Theorem 4.1], there exist $x_1, x_2 \in A$ such that $T(x) = T(x_1)T(x_2)$. Since $x_1x_2x_n \to 0$ and

$$T(x_1x_2x_n) = T(x_1)T(x_2)T(x_n) \to T(x_1)T(x_2)y = T(x)y,$$

it follows that $T(x)y \in S(T)$. Similarly, $yT(x) \in S(T)$ and so $S(T)$ is an ideal in B. □

Corollary 3.8. *Under the hypotheses of Theorem 3.7, if B is semisimple and $S(T) \subseteq Q(B)$, where $Q(B) = \{y \in B : r_B(y) = 0\}$, then T is continuous.*

Proof. Let $y \in S(T)$ and $b \in B$. Since $S(T)$ is an ideal in B, $by \in S(T)$ and so $by \in Q(B)$. Hence, $r_B(by) = 0$. It follows from [5, Lemma 2.1] that $y \in Rad(B)$ and hence $y = 0$. Therefore, T is continuous. □

We close this section by some examples.

Example 3.9. Let $\mathbb{R}$ be equipped with its Euclidean topology. If $(x_n)_n$ is a sequence in $\mathbb{R}$ such that $7^n(x_n - x_{n-1}) \longrightarrow 0$, then $(x_n)_n$ is a Cauchy sequence in $\mathbb{R}$.

Example 3.10. The algebra $C(\mathbb{R})$ of all continuous complex-valued functions on $\mathbb{R}$ with the sequence $(p_n)_n$ of seminorms defined by $p_n(f) = \sup_{|x|\leq n} |f(x)|$ is a fundamental algebra which is not strongly sequential.

Example 3.11. The algebra H_p, $0 < p < 1$ of holomorphic functions in the unit disc

$$x(\lambda) = \sum_{i=0}^{\infty} x_n \lambda^n$$

such that

$$\|x\|_p = \sum_{i=0}^{\infty} |x_n|^p < \infty,$$

with the pointwise multiplication is a fundamental topological algebra which is not a Banach algebra.

4. Continuity on topological algebras

In this section, we continue our study by further examining the conditions for automatic continuity on certain topological algebras.

Theorem 4.1. *Let A be a sequential topological algebra with $r_A \leq \beta_A$ and let $T : A \to \mathbb{C}$ be a linear functional on A. Assume that*

$$|T(x)| \leq r_A(x), \quad \text{for all } x \in A,$$

then T is bounded. Moreover, if A is metrizable, then T is continuous.

Proof. Assume, for the sake of contradiction, that T is unbounded. Then there exists a bounded sequence $(x_n)_n \subseteq A$ such that $|T(x_n)| > n$. Because $n^{-1}x_n \to 0$ and A is sequential, there exists $m^{-1}x_m \in (n^{-1}x_n)_n$ such that $(m^{-1}x_m)^k \to 0$ as $k \to \infty$. Hence, $\beta_A(x_m) \leq m < |T(x_m)|$. By the hypothesis, $r_A(x_m) \leq \beta_A(x_m)$. Thus, $r_A(x_m) < |T(x_m)|$, which is a contradiction. Therefore, T must be bounded. □

Using Lemma 2.7, we obtain the following corollary.

Corollary 4.2. *Let A be a fundamental F-algebra and $T : A \to \mathbb{C}$ be a linear functional on A. Assume that*

$$|T(x)| \leq r_A(x), \quad \text{for all } x \in A,$$

then T is continuous.

From [9, p. 281], we have the following proposition.

Proposition 4.3. *Let* (A, τ) *be a commutative topological algebra. Then*

$$\beta_A(xy) \leq \beta_A(x)\beta_A(y), \quad \text{for all } x, y \in A.$$

Lemma 4.4. *Let* A *be a commutative topological algebra whose elements are all bounded and* $r_A \leq \beta_A$*. Then*

$$S(A) \subseteq \mathrm{Rad}(A).$$

Proof. Let $y \in S(A)$ and let $x \in A$ be arbitrary. Since $r_A \leq \beta_A$, we have

$$r_A(yx) \leq \beta_A(yx) \leq \beta_A(y)\beta_A(x) = 0.$$

Thus, $r_A(yx) = 0$, and therefore $y \in \mathrm{Rad}(A)$. □

Using Lemma 4.4, we prove the following theorem.

Theorem 4.5. *Let* A *be an* F*-algebra and* B *be a semisimple commutative* F*-algebra with bounded elements such that* $r_B \leq \beta_B$*. Also, assume that* β_A *is continuous at zero and* β_B *is continuous on* B*. If* $T : A \to B$ *is a linear mapping satisfying*

$$\beta_B(Tx) \leq \beta_A(x), \quad \text{for all } x \in A.$$

Then T *is continuous.*

Proof. By the Closed Graph Theorem, to prove continuity of T, it is enough to show that whenever $(x_n)_n$ is a sequence in A with $x_n \to 0$ in A and $T(x_n) \to b$ in B, then $b = 0$. Since β_A is continuous at zero, $\beta_A(x_n) \to \beta_A(0) = 0$. Because β_B is continuous on B, $\beta_B(T(x_n)) \to \beta_A(b)$. But from the hypothesis,

$$\beta_B(T(x_n)) \leq \beta_A(x_n) \to 0.$$

Thus, $\beta_B(b) = 0$. Using Lemma 4.4, this implies

$$b \in \mathrm{Rad}(B) = \{0\}.$$

Therefore, T is continuous. □

Corollary 4.6. *Let* A *be an* F*-algebra and* B *be a semisimple commutative fundamental* F*-algebra such that* β_A *is continuous at zero and* β_B *is continuous on* B*. If* $T : A \to B$ *is a linear mapping satisfying*

$$\beta_B(Tx) \leq \beta_A(x), \quad \text{for all } x \in A.$$

Then T *is continuous.*

We now provide an example where all the hypotheses of Theorem 4.5 hold simultaneously.

Example 4.7. Let $A = C^\infty[0,1]$ be the Fréchet algebra of all infinitely differentiable complex-valued functions on $[0,1]$ endowed with the radius of boundedness

$$\beta_A(f) = \max_{0\le x\le 1} |f(x)| + \max_{0\le x\le 1} |f^{'}(x)|.$$

Then A is a non-Banach F-algebra and the radius of boundedness β_A is continuous at zero. Let $B = C_b(\mathbb{R})$ denote the algebra of bounded continuous functions on $\mathbb{R}$ equipped with the radius of boundedness

$$\beta_B(g) = \sup_{|x|\le 1} |g(x)|.$$

Since the spectrum of $g \in B$ is equal to the closure of $g(\mathbb{R})$, we have

$$r_B(g) \le \|g\|_\infty \le \beta_B(g),$$

and therefore B is a commutative semisimple F-algebra whose bounded elements satisfy $r_B \le \beta_B$. Define a linear mapping $T : A \to B$ by

$$T(f)(x) = \begin{cases} f(x) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

For every $f \in A$,

$$\beta_B(Tf) = \sup_{|x|\le 1} |T(f)(x)| = \sup_{0\le x\le 1} |f(x)|.$$

Since for every non-constant f, we have

$$\sup_{0\le x\le 1} |f(x)| \le \max_{0\le x\le 1} |f(x)| + \max_{0\le x\le 1} |f^{'}(x)|,$$

it follows that

$$\beta_B(Tf) \le \beta_A(f).$$

Thus, all assumptions of Theorem 4.5 are satisfied.

The proof of the following result on automatic continuity depends on the Open Mapping Theorem.

Theorem 4.8. *Let A be an F-algebra that is also a Q-algebra. If there exists a continuous surjective homomorphism from A onto another F-algebra B, then every character on B is automatically continuous.*

Proof. Let $T : A \to B$ be a continuous and surjective homomorphism. Then the Open Mapping Theorem implies that T is an open mapping. Let φ be a character on B. Then $\varphi \circ T$ is a character on A, which is continuous [6, Corollary 2.12]. Now, let U be an open subset of $\mathbb{C}$, then $(\varphi \circ T)^{-1}(U)$ is an open subset of A. Because T is an open mapping and

$$\varphi^{-1}(U) = ((T \circ T^{-1}) \circ \varphi^{-1})(U) = T((\varphi \circ T)^{-1}(U)),$$

it follows that $\varphi^{-1}(U)$ is open and so φ is continuous. □

Corollary 4.9. *Let A be a Banach algebra and B be an F-algebra. If there exists a continuous surjective homomorphism from A onto B, then every character on B is automatically continuous.*

5. Conclusion

In this paper, we obtained several results regarding the automatic continuity of linear mappings and homomorphisms on fundamental F-algebras and more generally on topological algebras.

Acknowledgments

The author would like to appriciate the referee for the valuable remarks.

References

[1] G. R. Allan. A spectral theory for locally convex algebras. *Proc. London Math. Sco.*, 15:399–421, 1965.

[2] E. Ansari-Piri. A class of factorable topological algebras. *Proc. Edin. Math. Soc.*, 33:53–59, 1990.

[3] E. Ansari-Piri. Topics on fundamental topological algebras. *Honam math. J.*, 23:59–66, 2001.

[4] H. G. Dales. *Banach Algebras and Automatic Continuity*, Clarendon Press, Oxford, 2000.

[5] T. G. Honary, H. Shayanpour. Automatic continuity of n-homomorphisms between topological algebras. *Bull. Aust. Math. Soc.*, 83:389-400, 2011.

[6] T. Husain. *Multiplicative Functionals on Topological Algebras*, Research notes in Math. 85, Pitmann Publishing, Boston, 1983.

[7] A. Naziri-Kordkandi. Automatic continuity on continuous inverse algebras. *Wavelets and Linear Algebra*, 8:53-61, 2022.

[8] A. Naziri-Kordkandi, A. Zohri, F. Ershad, B. Yousefi. Continuity in Fundamental Locally multiplicative topological Algebras. *Int. J. Nonlinear Anal. Appl*, 12:129–141, 2021.

[9] L. Oubbi. Further radii in topological algebras. *Bull. Belg. Math. Soc.*, 9:279-292, 2002.

[10] T. W. Palmer. *Banach Algebras and the General Theory of* **-Algebras.*, Volume I: Algebras and Banach algebras, Cambridge University Press, 1994.