Wavelets and Linear Algebra 13(1) (2026) 47-56

Vali-e-Asr University
of Rafsanjan

Wavelets and Linear Algebra

http://wala.vru.ac.ir

Results on operator relationships and their ranges equality in C^*-modules

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Article Info

Article history:
Received 12 January2025
Available online 20 July 2026

Keywords:
Range Equality
MoorePenrose inverse
Orthogonal Projection
Hilbert C^*-module.

2000 MSC:
47A05
46C05

Abstract

In this paper, by employing the concept of operator ranges and their equality, and utilizing the invertibility of certain operators, we establish the equality of ranges of related operators in Hilbert C^*-modules. The concept of operator ranges, combined with the invertibility of these operators, provides a tool for identifying and proving range equalities of related operators.

1. Introduction

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing inner products to take values in a C^*-algebra rather than in the field of real or complex numbers. Consequently, some

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http://doi.org/10.22072/wala.2026.2080618.1483

fundamental properties of ordinary inner product spaces may not hold in full generality for these modules. It is therefore essential, when studying Hilbert C^*-modules, to identify the conditions under which such properties remain valid, as well as the broader mathematical contexts in which they appear. A standard reference in this area is Lance's book [3]. Let us clarify our notation and terminology. Throughout this paper, $\mathscr{A}$ is a C^*-algebra. A Hilbert $\mathscr{A}$-module $\mathscr{X}$ is a right $\mathscr{A}$-module equipped with an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that $\mathscr{X}$ is complete with respect to the induced norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ for $x \in \mathscr{X}$. Throughout this paper, $\mathscr{A}$ denotes a C^*-algebra, and $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$, and $\mathscr{K}$ denote Hilbert $\mathscr{A}$-modules. Let $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ be the set of operators $A : \mathscr{X} \to \mathscr{Y}$ for which there exists an operator $A^* : \mathscr{Y} \to \mathscr{X}$ such that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for any $x \in \mathscr{X}$ and $y \in \mathscr{Y}$. It is known that any element $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ must be bounded and $\mathscr{A}$-linear. We refer to $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ as the set of adjointable operators from $\mathscr{X}$ to $\mathscr{Y}$. For any $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$, the range and null space of A are denoted by $\mathscr{R}(A)$ and $\mathscr{N}(A)$, respectively. In the case where $\mathscr{X} = \mathscr{Y}$, the space $\mathscr{L}(\mathscr{X}, \mathscr{X})$, abbreviated as $\mathscr{L}(\mathscr{X})$, forms a C^*-algebra.

A closed submodule $\mathscr{M}$ of $\mathscr{X}$ is said to be *orthogonally complemented* if $\mathscr{X} = \mathscr{M} \oplus \mathscr{M}^\perp$, where $\mathscr{M}^\perp = \{x \in \mathscr{X} : \langle x, y\rangle = 0 \text{ for any } y \in \mathscr{M}\}$. If $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ does not have a closed range, then neither $\mathscr{N}(A)$ nor $\overline{\mathscr{R}(A)}$ needs to be orthogonally complemented, as noted in [3, P. 24]. Furthermore, if $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $\overline{\mathscr{R}(A^*)}$ is not orthogonally complemented, it is possible that $\mathscr{N}(A)^\perp \neq \overline{\mathscr{R}(A^*)}$; see [2, 3]. These observations indicate that the theory of Hilbert C^*-modules is significantly different and more complex than that of Hilbert spaces.

It is well known that an operator $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ is inner invertible (i.e., there exists $B \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$ such that $ABA = A$) if and only if $\mathscr{R}(A)$ is closed in $\mathscr{Y}$. The Moore-Penrose inverse of $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ is the operator $X \in \mathscr{L}(\mathscr{Y}, \mathscr{X})$ that satisfies the Penrose equations:

$$(1) \quad AXA = A, \quad (2) \quad XAX = X, \quad (3) \quad (AX)^* = AX, \quad (4) \quad (XA)^* = XA. \tag{1.1}$$

The Moore–Penrose inverse of an adjointable operator $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ exists if and only if $\mathscr{R}(A)$ is closed in $\mathscr{Y}$. When it exists, it is unique and denoted by $A^\dagger$.

In this paper, by employing the concept of operator ranges, the relationships among operators in C^*-modules are analyzed in a structural manner. It is shown that the invertibility of certain operators plays a crucial role in determining the equality of the ranges of related operators, thereby elucidating the structure of their relationships within the framework of C^*-modules. We also present conditions under which the intersection of the ranges of the operators PQ and QP becomes trivial. Furthermore, equivalent conditions for the equality of two associated null spaces are formulated. The theorems presented are based on the fundamental properties of orthogonal projections in Hilbert C^*-modules. The concept of operator ranges, together with the invertibility of related operators, provides a tool for identifying and proving the equality of the ranges of related operators.

To achieve our main results, we need to use some supporting lemmas as follows.

Lemma 1.1. *(see [1, Lemma 2.1]) Let $A, B \in \mathscr{L}(\mathscr{X})$. The operator $I - AB$ is invertible if and only if $I - BA$ is invertible. In this case, we have:*

$$(I - AB)^{-1} = I + A(I - BA)^{-1}B.$$

Lemma 1.2. [3, Proposition 3.7] *Let $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$. Then $\overline{\mathscr{R}(AA^*)} = \overline{\mathscr{R}(A)}$*

Lemma 1.3. *(see [4]) Let $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $B, C \in \mathscr{L}(\mathscr{Z}, \mathscr{X})$ be such that $\overline{\mathscr{R}(B)} = \overline{\mathscr{R}(C)}$. Then $\overline{\mathscr{R}(AB)} = \overline{\mathscr{R}(AC)}$.*

2. Conditions for operators to have equal ranges

The following theorem establishes equivalent conditions for the equality of the null spaces of two operators

Theorem 2.1. *Let $A \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ and $U \in \mathscr{L}(\mathscr{Y})$. Then the following statements are equivalent:*

(1) $\mathscr{N}(U) \cap \mathscr{R}(A) = \{0\}$*;*

(2) $\mathscr{N}(UA) = \mathscr{N}(A)$.

Moreover, whenever these conditions hold and $\mathscr{R}(UA) = \mathscr{R}(A)$, the restriction

$$U|_{\mathscr{R}(A)} : \mathscr{R}(A) \to \mathscr{R}(A),$$

is invertible.

Proof. (1) ⇒(2) The inclusion $\mathscr{N}(A) \subseteq \mathscr{N}(UA)$ holds for all operators.

To prove the reverse inclusion, let $x \in \mathscr{N}(UA)$. Then

$$UAx = 0 \quad \Rightarrow \quad Ax \in \mathscr{N}(U).$$

Since $Ax \in \mathscr{R}(A)$, it follows that

$$Ax \in \mathscr{N}(U) \cap \mathscr{R}(A) = \{0\}.$$

Thus $Ax = 0$, hence $x \in \mathscr{N}(A)$. Therefore

$$\mathscr{N}(UA) = \mathscr{N}(A).$$

(2) ⇒(1) Let $y \in \mathscr{N}(U) \cap \mathscr{R}(A)$. Then $y = Ax$ for some $x \in \mathscr{X}$, and

$$0 = Uy = U(Ax) = UAx.$$

Thus $x \in \mathscr{N}(UA) = \mathscr{N}(A)$, so $Ax = 0$. Hence $y = Ax = 0$, proving

$$\mathscr{N}(U) \cap \mathscr{R}(A) = \{0\}.$$

Now, assume $\mathscr{R}(UA) = \mathscr{R}(A)$ and $\mathscr{N}(U) \cap \mathscr{R}(A) = \{0\}$.

Since $\mathscr{R}(UA) = \mathscr{R}(A)$ and $UA = U|_{\mathscr{R}(A)} \circ A$, every $y \in \mathscr{R}(A)$ can be written as $y = UAx = U(Ax)$ for some x, hence $y = U(z)$ with $z = Ax \in \mathscr{R}(A)$. Thus $U|_{\mathscr{R}(A)}$ is surjective onto $\mathscr{R}(A)$.

Let $y \in \mathscr{R}(A)$ satisfy $U(y) = 0$. Then $y \in \mathscr{N}(U) \cap \mathscr{R}(A)$, so by hypothesis $y = 0$. Hence $U|_{\mathscr{R}(A)}$ is injective.

Therefore $U|_{\mathscr{R}(A)} : \mathscr{R}(A) \to \mathscr{R}(A)$ is invertible.

□

Remark 2.2. It is worth noting that Theorem 2.1 also holds for general linear transformations between arbitrary vector spaces.

Theorem 2.3. *Let $\mathscr{X}$ and $\mathscr{Y}$ be Hilbert C^*-modules, and let*

$$A \in \mathscr{L}(\mathscr{X}, \mathscr{Y}), \qquad U, V \in \mathscr{L}(\mathscr{Y})$$

be adjointable operators such that

$$\mathscr{R}(UA) = \mathscr{R}(VA) \tag{2.1}$$

and

$$\mathscr{N}(U^*V) \cap \mathscr{R}(A) = \{0\}. \tag{2.2}$$

Then there exists a unique injective operator

$$W : \mathscr{R}(UA) \longrightarrow \mathscr{R}(VA)$$

such that

$$W(UAx) := VAx, \qquad x \in \mathscr{X}.$$

Proof. We define $W : \mathscr{R}(UA) \to \mathscr{R}(VA)$ by

$$W(UAx) := VAx.$$

Assume $UAx_1 = UAx_2$. Then

$$UA(x_1 - x_2) = 0.$$

Since $W(0) = 0$, we have

$$0 = WUA(x_1 - x_2),$$

which implies

$$VA(x_1 - x_2) = 0.$$

Thus

$$W(UAx_1) = VAx_1 = VAx_2 = W(UAx_2),$$

hence W is well-defined.

Let $UAx \in \mathscr{R}(UA)$ and suppose

$$W(UAx) = 0.$$

Then, by definition of W,

$$VAx = 0.$$

Applying U^* gives

$$U^*VAx = 0.$$

Since $Ax \in \mathscr{R}(A)$ and by assumption

$$\mathscr{N}(U^*V) \cap \mathscr{R}(A) = \{0\},$$

we obtain

$$Ax = 0.$$

Thus

$$UAx = U(Ax) = 0,$$

Therefore W is injective.

For the uniqueness, let $y \in \mathscr{R}(UA)$. Then there exists $x \in \mathscr{X}$ such that $y = UAx$. Using the defining relations for W_1 and W_2, we obtain

$$W_1(y) = W_1(UAx) = VAx = W_2(UAx) = W_2(y).$$

Thus $W_1 = W_2$ on $\mathscr{R}(UA)$. □

Some related results can be found in [5].

Theorem 2.4. *Let $\mathscr{X}$ be a Hilbert C^*-module, and consider an operator $A \in \mathscr{L}(\mathscr{X})$ that can be factorized as $A = BB^*C$, where B and C are operators with closed ranges satisfying $\mathscr{R}(B) = \mathscr{R}(C)$. Then*

(1) $\mathscr{R}(A) = \mathscr{R}(B)$

(2) $AC^\dagger(B^\dagger)^* = B$

(3) $\mathscr{N}(A) = \mathscr{N}(C)$

(4) $B^* = B^\dagger AC^\dagger$

(5) $A^\dagger B = C^\dagger(B^*)^\dagger$

(6) $CA^\dagger B = (B^*)^\dagger$

Proof. (1) Since $\mathcal{R}(B) = \mathcal{R}(C)$, it follows from Lemma 1.2 that

$$\mathcal{R}(B) = \mathcal{R}(BB^*) = B\mathcal{R}(B^*) = B\mathcal{R}(B^*B) = \mathcal{R}(BB^*B) = \mathcal{R}(BB^*C) = \mathcal{R}(A).$$

To prove (2), start with the factorization

$$A = BB^*C.$$

Rightmultiplying by the MoorePenrose inverse $C^\dagger$ gives

$$AC^\dagger = BB^*CC^\dagger = BB^*BB^\dagger = BB^*,$$

since $CC^\dagger$ and $BB^\dagger$ are the projections onto $\mathcal{R}(C) = \mathcal{R}(B)$.

Next, rightmultiplying by $(B^\dagger)^*$ yields

$$AC^\dagger(B^\dagger)^* = BB^*(B^\dagger)^* = B.$$

Thus the desired identity follows.

(3) If $x \in \mathscr{N}(C)$ then $Cx = 0$, and hence

$$Ax = BB^*Cx = 0,$$

so $x \in \mathscr{N}(A)$.

Now, let $x \in \mathscr{N}(A)$; then

$$0 = Ax = BB^*Cx.$$

Since $\mathscr{N}(BB^*) = \mathscr{N}(B^*)$, hence $Cx \in \mathscr{N}(BB^*) = \mathscr{N}(B^*)$.

But $\mathscr{N}(B^*) = \mathscr{R}(B)^\perp$, and by hypothesis $\mathscr{R}(B) = \mathscr{R}(C)$, so $\mathscr{N}(B^*) = \mathscr{R}(C)^\perp$. Since $Cx \in \mathscr{R}(C)$ and $Cx \in \mathscr{R}(C)^\perp$, we must have $Cx = 0$. Therefore $x \in \mathscr{N}(C)$. Therefore, we obtain the desired result.

(4) Since $A = BB^*C$ and $\mathcal{R}(C) = \mathcal{R}(B)$, we have

$$B^\dagger AC^\dagger = B^\dagger BB^*CC^\dagger = B^*CC^\dagger = B^*.$$

(5) By (2), we have $AC^\dagger(B^\dagger)^* = B$. Applying $A^\dagger$ on the left yields

$$A^\dagger B = A^\dagger A\, C^\dagger(B^\dagger)^*.$$

Since $A^\dagger A$ is the orthogonal projection onto $\mathscr{R}(A^*)$, it suffices to verify $\mathscr{R}(C^\dagger(B^\dagger)^*) \subseteq \mathscr{R}(A^*)$. From $A^* = C^*BB^*$ and equality $\mathscr{R}(B) = \mathscr{R}(C)$, we obtain $\mathscr{R}(A^*) = \mathscr{R}(C^*BB^*) = C^*\mathscr{R}(BB^*) = \mathscr{R}(C^*C) = \mathscr{R}(C^\dagger)$, the inclusion follows. Hence $A^\dagger A$ acts as the identity on this range, and we conclude

$$A^\dagger B = C^\dagger(B^\dagger)^*.$$

(6) By (5), we have $A^\dagger B = C^\dagger(B^\dagger)^*$. Left-multiplying by C yields

$$CA^\dagger B = CC^\dagger(B^\dagger)^*.$$

Since $CC^\dagger$ is the orthogonal projection onto $\mathscr{R}(C)$ and $\mathscr{R}(C) = \mathscr{R}((B^\dagger)^*)$, it acts as the identity on this range. Hence,

$$CA^\dagger B = (B^*)^\dagger,$$

which establishes the desired result. □

3. Results on range equality of the orthogonal projections

These theorems presented highlight key properties of orthogonal projections in Hilbert C^*-modules, particularly under conditions of range equality and invertibility. They establish relationships between the adjoints of projections, the closures of their ranges, and demonstrate how these properties are preserved through specific combinations of projections These theorems reveals a relationship between the closures of the ranges of two different combinations of the projections P and Q.

Theorem 3.1. *Let $P \in \mathscr{L}(\mathscr{X})$ be an orthogonal projection and let $A \in \mathscr{L}(\mathscr{X})$ be a positive operator. Then*

$$\overline{\mathscr{R}(PA)} = \overline{\mathscr{R}(PAP)}.$$

Proof. Since A is positive, by Lemma 1.2 we have $\overline{\mathscr{R}(A^{1/2})} = \overline{\mathscr{R}(A)}$. Applying Lemma 1.3, we obtain $\overline{\mathscr{R}(PA^{1/2})} = \overline{\mathscr{R}(PA)}$. Moreover, we have

$$\overline{\mathscr{R}(PA^{1/2})} = \overline{\mathscr{R}((PA^{1/2})(PA^{1/2})^*)} = \overline{\mathscr{R}(PA^{1/2}A^{1/2}P)} = \overline{\mathscr{R}(PAP)}.$$

Combining these equalities yields $\overline{\mathscr{R}(PA)} = \overline{\mathscr{R}(PAP)}$, which completes the proof. □

Corollary 3.2. *Let P, Q be orthogonal projections on a Hilbert C^*-module $\mathscr{X}$. Then*

$$\overline{\mathscr{R}(P + PQ)} = \overline{\mathscr{R}(P + PQP)}.$$

Proof. The desired result follows directly from Theorem 3.1. □

In the context of Hilbert C^*-modules, orthogonal projections play a crucial role in understanding the structure and relationships between subspaces. The following theorems illustrate properties of orthogonal projections and their interactions, particularly under the condition of invertibility of certain operators.

Theorem 3.3. *Let P, Q be orthogonal projections on a Hilbert C^*-module $\mathscr{X}$ such that $\|PQ\| < 1$. Then $\mathcal{R}(PQ) \cap \mathcal{R}(QP) = \{0\}$.*

Proof. Let $x \in \mathcal{R}(PQ) \cap \mathcal{R}(QP)$. Then there exist $u, v \in \mathscr{X}$ with

$$x = PQu \quad \text{and} \quad x = QPv.$$

From $x = PQu$ we get $Px = P(PQu) = PQu = x$. Similarly from $x = QPv$ we get $Qx = Q(QPv) = QPv = x$. Thus x is fixed by both P and Q, i.e. $x \in \mathcal{R}(P) \cap \mathcal{R}(Q)$.

But if there were a nonzero vector x with $Px = Qx = x$, then

$$\|PQ\| \geq \frac{\|PQx\|}{\|x\|} = \frac{\|P(Qx)\|}{\|x\|} = \frac{\|Px\|}{\|x\|} = 1,$$

contradicting $\|PQ\| < 1$. Hence no such nonzero x exists, and $\mathcal{R}(PQ) \cap \mathcal{R}(QP) = \{0\}$. □

Theorem 3.4. *Let P and Q be orthogonal projections, and assume that $I - PQ$ and $P + Q - PQ$ are invertible. Then*

$$\mathscr{R}(P(I - Q)) = \mathscr{R}(P - PQP) = \mathscr{R}(P).$$

Proof. Since $I - PQ$ and $P + Q - PQ$ are invertible, we have $(I - PQ)(\mathscr{X}) = (P + Q - PQ)(\mathscr{X}) = \mathscr{X}$. Applying P to both sides yields

$$P((I - PQ)(\mathscr{X})) = P((P + Q - PQ)(\mathscr{X})) = P(\mathscr{X}).$$

Thus, the equality above implies $\mathscr{R}(P(I - Q)) = \mathscr{R}(P)$.

Moreover, since $I - PQ$ is invertible, applying Lemma 1.1 allows us to conclude that $I - QP$ is also invertible. Therefore, $(I - QP)(\mathscr{X}) = \mathscr{X}$. Applying P to both sides yields

$$P((I - QP)(\mathscr{X})) = P(\mathscr{X}).$$

Combining these results, we conclude that $\mathscr{R}(P(I - Q)) = \mathscr{R}(P - PQP) = \mathscr{R}(P)$. □

Theorem 3.5. *Let $\mathscr{X}$ be a Hilbert $\mathscr{A}$-module, and let $P, Q \in \mathscr{L}(\mathscr{X})$ be orthogonal projections. Then*

$$\overline{\mathscr{R}(I-PQ)} = \mathscr{R}(I-P) \oplus \overline{\mathscr{R}(P(I-Q))},$$

Moreover, if $I - PQ$ has dense range in $\mathscr{X}$, then

$$\overline{\mathscr{R}(P(I-Q))} = \mathscr{R}(P).$$

Proof. First, we note the operator identity

$$I - PQ = (I-P) + P(I-Q).$$

Since P is an orthogonal projection, we have $(I-P)P = 0$. For all $x, y \in \mathscr{X}$,

$$\langle (I-P)x,\ P(I-Q)y\rangle = \langle x,\ (I-P)P(I-Q)y\rangle = 0,$$

Hence the ranges of $I-P$ and $P(I-Q)$ are orthogonal.

Because the sum of two orthogonal subsets is orthogonal, the closure of their sum equals the orthogonal direct sum of their closures:

$$\overline{\mathscr{R}(I-PQ)} = \overline{\mathscr{R}((I-P)+P(I-Q))} = \mathscr{R}(I-P) \oplus \overline{\mathscr{R}(P(I-Q))}. \tag{3.1}$$

Now suppose that $\overline{\mathscr{R}(I-PQ)} = \mathscr{X}$.

Then (3.1) gives

$$\mathscr{X} = \mathscr{R}(I-P) \oplus \overline{\mathscr{R}(P(I-Q))}. \tag{3.2}$$

In a Hilbert C^*-module, for any orthogonal projection P we have the orthogonal decomposition

$$\mathscr{X} = \mathscr{R}(I-P) \oplus \mathscr{R}(P). \tag{3.3}$$

Comparing equalities (3.2) with (3.3) and using the uniqueness of orthogonal complement submodule, we conclude

$$\overline{\mathscr{R}(P(I-Q))} = \mathscr{R}(P).$$

□

Theorem 3.6. *Let P and Q be orthogonal projections, and assume that $I - PQ$ is invertible. Then*

$$\mathscr{R}(P+PQ) = \mathscr{R}(P-PQ) = \mathscr{R}(P).$$

Proof. For the first equality, observe that $P + PQ = P(I+Q)$. Since $I + Q$ is invertible. Consequently,

$$\mathscr{R}(P+PQ) = \mathscr{R}(P(I+Q)) = \mathscr{R}(P).$$

For the second equality, note that $P - PQ = P(I-PQ)$. Since $I - PQ$ is invertible, right-multiplying by $(I-PQ)^{-1}$ yields

$$(P-PQ)(I-PQ)^{-1} = P.$$

Hence, for every $y \in \mathscr{R}(P)$, we have

$$y = Py = (P - PQ)(I - PQ)^{-1}y \in \mathscr{R}(P - PQ),$$

which shows $\mathscr{R}(P) \subseteq \mathscr{R}(P - PQ)$. The reverse inclusion $\mathscr{R}(P - PQ) \subseteq \mathscr{R}(P)$ is immediate because $P - PQ = P(I - Q)$. Therefore, $\mathscr{R}(P - PQ) = \mathscr{R}(P)$. □

The following theorem shows that if an orthogonal projection P preserves the range of A, then $PA = A$.

Theorem 3.7. *Let $P, A \in \mathscr{L}(\mathscr{X})$ be operators such that P is a orthogonal projection. If*

$$\mathscr{R}(PA) = \mathscr{R}(A),$$

then $PA = A$.

Proof. Since $\mathscr{R}(PA) = \mathscr{R}(A)$, multiply both sides on the left by $I - P$:

$$\mathscr{R}((I - P)PA) = \mathscr{R}((1 - P)A).$$

Because P is projection, $(I - P)P = 0$, so

$$\mathscr{R}((I - P)A) = \{0\}.$$

This implies $(I - P)A = 0$, and hence $PA = A$. □

Remark 3.8. Let $A, B \in \mathscr{L}(\mathscr{X})$, and assume that B has dense range; that is, $\overline{\mathscr{R}(B)} = \mathscr{R}(I)$. By applying Lemma 1.3 and multiplying on the left by A, we have $\overline{\mathscr{R}(AB)} = \overline{\mathscr{R}(A)}$.

Theorem 3.9. *Let P and Q be orthogonal projections. Then, for every positive linear combination of P and Q, we have*

$$\overline{\mathscr{R}(aP + bQ)} = \overline{\mathscr{R}(P) + \mathscr{R}(Q)}.$$

Proof. For every $\alpha, \beta \in \mathbb{C}$, we consider the operator

$$A = \begin{pmatrix} 0 & 0 \\ \alpha P & \beta Q \end{pmatrix} \in \mathscr{L}(\mathscr{X} \oplus \mathscr{X}).$$

It is evident that

$$\mathscr{R}(A) = \{0\} \oplus \mathscr{R}(\alpha P) + \mathscr{R}(\beta Q) = \{0\} \oplus \mathscr{R}(P) + \mathscr{R}(Q).$$

Furthermore, we observe that

$$\mathscr{R}(AA^*) = \{0\} \oplus \mathscr{R}(|\alpha|^2 P + |\beta|^2 Q) = \{0\} \oplus \mathscr{R}(aP + bQ),$$

where $a = |\alpha|^2$ and $b = |\beta|^2$. The desired result follows directly from Lemma 1.2. □

Acknowledgement. The authors would like to sincerely thank the anonymous referee for carefully reading the paper and for useful comments.

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