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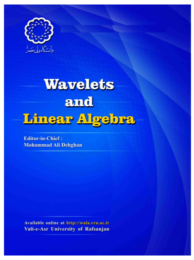

Solving Tensor Equation Systems under the Semi-Tensor Product: A t-Product Approach

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Abstract

This paper addresses the solvability of the tensor system $\mathcal{A} \ltimes \mathcal{X} = \mathcal{B}, \mathcal{X} \ltimes \mathcal{C} = \mathcal{D}$ using the semi-tensor product(STP) framework. Initially, we examine the associated tensor-vector system under the STP operation, deriving necessary and sufficient conditions for the existence of solutions and propose explicit solution methods. Furthermore, we introduce suitable tensor partitioning techniques that enable the reduction of the tensor-vector system to a linear system formulated via the t-product. This approach facilitates a comprehensive study of the solvability of the tensor system when the unknown tensor $\mathcal{X}$ is a vector, matrix or a higher-order tensor under the STP. We apply the STP framework to model and control Boolean gene regulatory networks with pinning control inputs. Several illustrative examples are provided to demonstrate the applicability and effectiveness of the proposed approaches. Our study includes the tensor equation $\mathcal{A} \ltimes \mathcal{X} = \mathcal{B}$ and the matrix system $\mathbf{A} \ltimes \mathbf{X} = \mathbf{B}$, $\mathbf{X} \ltimes \mathbf{C} = \mathbf{D}$ as special cases.

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1. Introduction

The study of systems of equations, particularly

$$\begin{cases} \mathbf{A}X = \mathbf{B}, \\ X\mathbf{C} = \mathbf{D}, \end{cases} \tag{1.1}$$

not only has important theoretical significance but has also been extensively applied in system identification, control theory, image processing, optimization, and robotics. For example, the representation of the state-space model of a dynamic system by the equations $\mathbf{A}X = \mathbf{B}$, and $X\mathbf{C} = \mathbf{D}$ are investigated by Zhou et al.[**?**], Vaidyanathan [**?**] has examined the role of this system of equations in signal processing. Moreover, this matrix system has important role in signal processing and computer vision[**?**]. For a comprehensive review of the solutions to this system of equations, see Wang et al. [**?**]. Given the inherent dimensional constraints of conventional matrix multiplication, Cheng [**?**] proposed the STP of matrices as a novel approach to address linearized problems within nonlinear systems. This method effectively overcomes the dimensional limitations associated with standard matrix multiplication, enabling more flexible and generalized computations. This product has been widely used in many fields, such as physics in nonlinear systems, graphs [**?**], networked evolutionary games [**?**], dynamical games [**?**], cryptographic algorithms [**?**], the analysis and control of Boolean networks [**? ?**], networked non-cooperative games [**?**]. Consequently, many scholars have investigated solutions to various special matrix equations in the context of the STP [**? ? ? ? ? ?**].

In contemporary research and industry, numerous disciplines require the acquisition and utilization of vast quantities of data, including fields such as image and video processing, healthcare, and engineering. Often, these data are inherently multidimensional, exemplified by the storage and analysis of color images and video sequences. Traditional two-dimensional matrix representations prove insufficient for effectively capturing and processing such complex data structures. Consequently, tensorshigher-order generalizations of matriceshave been introduced as a powerful framework for representing and analyzing multidimensional datasets. Yang et al. considered the results of the system (**??**) over dual split quaternion tensors [**?**].

Chen et al. [**?**] defined a STP for third-order tensors. Based on this definition, they present a novel tensor decomposition strategy, provide a specific algorithm, and generalize the tensor singular value decomposition (SVD) using the STP. Although tensors have been widely applied in various fields owing to their multidimensional characteristics, the study of system (**??**) within the tensor framework has been relatively scarce. Hence, this paper aims to examine the system from a tensorial perspective, particularly with respect to the STP, to open new directions for research.

In this paper, we discuss the solvability of the following system of tensor equations under STP:

$$\begin{cases} \mathcal{A} \ltimes \mathcal{X} = \mathcal{B}, \\ \mathcal{X} \ltimes \mathcal{C} = \mathcal{D}, \end{cases} \tag{1.2}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are given, and $\mathcal{X}$ is an unknown tensor, $\ltimes$ represents the STP of tensor and will be described in detail in Section 2. The special case $\mathcal{C} = \mathcal{D} = 0$ was also studied by the author [**?**].

Our paper is organized as follows. In Section 2, we introduce some fundamental definitions and properties. In Section 3, we study the solution of the tensor equation system (**??**) by investigating the tensor-vector equation system in two cases, including permissibility conditions of the tensors, necessary and sufficient conditions, and concrete solving methods. Based on this, the solvability of the tensor-matrix equation system (**??**) under the STP is further examined in Section 4, where similar results are also presented. In Section 5, we investigate the solvability of the tensor-tensor equation system (**??**) under the STP. In Section 6, we apply the STP framework to model and control Boolean gene regulatory networks with pinning control inputs. Finally, several examples are provided in each section to illustrate the effectiveness of our results.

2. Preliminaries

In this section, we present some commonly used symbols, definitions related to matrices and tensors, and their relevant properties. Symbols $\mathbb{R}(\mathbb{C})$, $\mathbb{R}^n(\mathbb{C}^n)$ and $\mathbb{R}^{m\times n}(\mathbb{C}^{m\times n})$ represent the real(complex) number field, the set of real(complex) column vectors with n elements and the set of $m \times n$ matrices over the real(complex) numbers, respectively. Moreover, we denote $\mathbb{R}^n_+ := \{X \in \mathbb{R}^n : X \geq 0\}$, where $X \geq 0$ means $x_i \geq 0$ for $i = 1, 2, \ldots, n$. We use boldface uppercase letters $(\mathbf{A}, \mathbf{B}, \ldots)$ to denote matrices, italicized uppercase case letters $(A, B, \ldots)$ to denote vectors, lower case letters $(x, y, \ldots)$ to denote scalars. This notation applies consistently to substructures. For example, the entry at row i and column j of matrix $\mathbf{A}$ is written as a_{ij}, and the i-th column vector of $\mathbf{A}$ is denoted as A_i. For positive integers m, n, the least common multiple and greatest common divisor are denoted by $\mathrm{lcm}\{m, n\}$ and $\gcd\{m, n\}$, respectively.

A tensor is a multidimensional array of data. The number of indices of a tensor is called modes or ways. Notice that a scalar can be regarded as a zero mode tensor, first mode tensors are vectors and matrices are second mode tensor. The order of a tensor is the dimensional of the array needed to represent it, also known as ways or modes. For a given N-mode (or N-order) tensor $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3 \ldots \times n_N}$, the notation $a_{i_1\ldots i_N}$ (with $1 \leq i_j \leq n_j$ and $j = 1, \ldots N$) stands for the element $(i_1, \ldots, i_N)$ of the tensor $\mathcal{A}$.

Corresponding to a given tensor $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3 \ldots \times n_N}$, the notation

$$\mathcal{A}_{\underbrace{:: \ldots :}_{N-1} k} \text{ for } k = 1, 2, \ldots, n_N,$$

denotes a tensor in $\mathbb{R}^{n_1\times n_2\times n_3 \ldots \times n_{N-1}}$ which is obtained by fixing the last index and is called frontal slice. Fibers are the higher-order analogue of matrix rows and columns. A fiber is defined by fixing all the indexes except one.

In this paper, a tensor is of third-order, i.e., N = 3, that will be denoted by the calligraphic script letters, say $\mathcal{A} = \left(a_{ijk}\right)_{i,j,k=1}^{n_1,n_2,n_3}$. Using $\mathcal{A}_{:jk}$, $\mathcal{A}_{i:k}$ and $\mathcal{A}_{ij:}$ denote mode-1, mode-2, and mode-3 fibers, respectively; see Figure **??**. The notations $\mathcal{A}_{i::}$, $\mathcal{A}_{:j:}$ and $\mathcal{A}_{::k}$ stand for the i-th horizontal, j-th lateral, and k-th frontal slices of $\mathcal{A}$, respectively; see Figure **??**. Each slice is a matrix.

A new concept is proposed for multiplying third-order tensors, based on viewing a tensor as a stack of frontal slices. In fact, if $\mathcal{A} \in \mathbb{R}^{m\times n\times r}$ is a third-order tensor, then it has r frontal slices, and each frontal slice is an $m\times n$ matrix. These r frontal slices of $\mathcal{A}$ can be represented by $\mathcal{A}_{::1}$, $\mathcal{A}_{::2}, \ldots, \mathcal{A}_{::r}$.

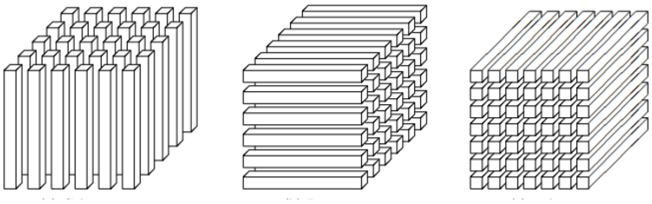

Figure 1: Left to right: a mode-1, mode-2 and mode-3 fiber of third-order tensor.

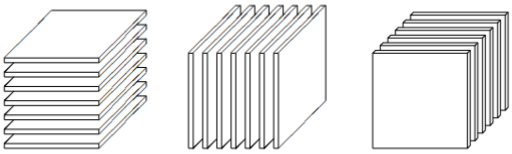

Figure 2: Left to right: a horizontal, lateral and frontal slice of third-order tensor.

In [**?**], the **fold**, **unfold** and **bcirc** operators are defined. Suppose $\mathcal{A} \in \mathbb{R}^{m\times n\times r}$ is a third-order tensor, and its frontal slices are denoted by $\mathcal{A}_{::1}, \mathcal{A}_{::2}, \ldots, \mathcal{A}_{::r}$. Then **unfold** is defined as

$$\text{unfold}(\mathcal{A}) = \begin{bmatrix} \mathcal{A}_{::1} \\ \mathcal{A}_{::2} \\ \vdots \\ \mathcal{A}_{::r} \end{bmatrix} \in \mathbb{C}^{mr\times n}.$$

We can easily see that unfold($\mathcal{A}$) is an $mr \times n$ matrix. Another operator **fold** is the inverse of **unfold** which is defined as

$$\text{fold}(\text{unfold}(\mathcal{A})) = \text{fold}\left(\begin{bmatrix} \mathcal{A}_{::1} \\ \mathcal{A}_{::2} \\ \vdots \\ \mathcal{A}_{::r} \end{bmatrix}\right) = \mathcal{A}.$$

By using the **unfold** operator, we can create an $mr \times nr$ block-circulant matrix denoted by **bcirc**($\mathcal{A}$) with the frontal slices of $\mathcal{A}$. It is given by

$$\text{bcirc}(\mathcal{A}) := \begin{bmatrix} \mathcal{A}_{::1} & \mathcal{A}_{::r} & \ldots & \mathcal{A}_{::2} \\ \mathcal{A}_{::2} & \mathcal{A}_{::1} & \ldots & \mathcal{A}_{::3} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{A}_{::r} & \mathcal{A}_{::r-1} & \cdots & \mathcal{A}_{::1} \end{bmatrix}.$$

Kilmer et al. [**?**] presented a new tensor multiplication called t-product. This tensor multiplication can be described in the following.

Definition 2.1. (t-product [**? ?**]) Let $\mathcal{A}$ be an $m \times n \times r$ third-order tensor and $\mathcal{B}$ be an $n \times s \times r$ third-order tensor, then the product of $\mathcal{A} * \mathcal{B}$ is an $m \times s \times r$ tensor can be represented as

$$\mathcal{A} * \mathcal{B} = \text{ fold } [\text{bcirc}(\mathcal{A}) \cdot \text{unfold}(\mathcal{B})],$$

where $*$ denotes the new tensor multiplication of two third-order tensors called t-product, and . denotes the general matrix product.

The t-product is indeed used for analyzing and computing with third-order tensors, as it extends matrix operations to tensors, providing a framework similar to matrix multiplication but for higher-order tensors.

The Discrete Fourier Transformation (DFT) plays a very important role in the definition of the t-product of tensors. Using the knowledge of Fourier transform, the block-circulant matrix can be transformed into a block-diagonal matrix. Suppose that $\mathbf{F}_r$ is the $r \times r$ Fourier matrix and $\mathbf{F}_r^H$ is the conjugate transpose of $\mathbf{F}_r$. Then

$$(\mathbf{F}_r \otimes I_p) \cdot \mathrm{bcirc}(\mathrm{unfold}(\mathcal{A})) \cdot (\mathbf{F}_r^H \otimes I_q) = \begin{bmatrix} \bar{\mathbf{A}}_1 & & & \\ & \bar{\mathbf{A}}_2 & & \\ & & \ddots & \\ & & & \bar{\mathbf{A}}_r \end{bmatrix}.$$

Let

$$\bar{\mathbf{A}} := \begin{bmatrix} \bar{\mathbf{A}}_1 & & & \\ & \bar{\mathbf{A}}_2 & & \\ & & \ddots & \\ & & & \bar{\mathbf{A}}_r \end{bmatrix}.$$

Then $\bar{\mathbf{A}}$ is a complex $pr \times qr$ block-diagonal matrix. Let $\bar{\mathbf{A}}_i$ be the i-th frontal slice of $\bar{\mathcal{A}}$, where $\bar{\mathcal{A}} \in \mathbb{C}^{p \times q \times r}$ is the result of discrete Fourier transform (DFT) on $\mathcal{A}$ along the third dimension. The MATLAB command for computing $\bar{\mathcal{A}}$ is

$$\bar{\mathcal{A}} = \mathrm{fft}[\mathcal{A}, [], 3]. \tag{2.1}$$

The t-product between two tensors can be understood as the matrix multiplication in the Fourier domain [**?**]. The t-product result between $\mathcal{A}$ and $\mathcal{B}$ can be obtained by multiplying each pair of the frontal slices of $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ and computing the inverse Fourier transform along the third dimension, where $\bar{\mathcal{B}}$ is the result of DFT on $\mathcal{B}$ along the third dimension, the MATLAB commands is $\bar{\mathcal{B}} = \mathrm{fft}[\mathcal{B}, [], 3]$. The t-product can be alternatively computed in the Fourier domain, as shown in Algorithm 1.

Now we provide some definitions that will be used in the following.

Definition 2.2. (Matricization or transforming a tensor into a matrix) Matricization, also known as unfolding or flattening, is the process of reordering the elements of an N-way array into a matrix. For instance, a $3 \times 5 \times 4$ tensor can be arranged as a 15×4 matrix or a 3×20 matrix, and so on. The mode-n matricization of a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is denoted by $\mathcal{A}^{(n)}$ and arranges the mode-n fibers to be the columns of the resulting matrix. Tensor element $(i_1, i_2, \ldots, i_N)$ maps to matrix element (i_n, j), where

$$j = 1 + \sum_{\substack{k=1 \\ k \neq n}}^{N} (i_k - 1) J_k \quad \text{with} \quad J_k = \prod_{\substack{m=1 \\ m \neq n}}^{k-1} I_m.$$

Algorithm 1 t-product $\mathcal{C} = \mathcal{A} * \mathcal{B}$ in the Fourier domain

Require: $\mathcal{A} \in \mathbb{R}^{m\times n\times r}$, $\mathcal{B} \in \mathbb{R}^{n\times s\times r}$

$\bar{\mathcal{A}} \leftarrow \text{fft}(\mathcal{A}, [], 3)$
$\bar{\mathcal{B}} \leftarrow \text{fft}(\mathcal{B}, [], 3)$
for $k = 1, \ldots, r$ **do**
    $\bar{\mathcal{C}}_{::k} \leftarrow \bar{\mathcal{A}}_{::k} \cdot \bar{\mathcal{B}}_{::k}$
end for
$\mathcal{C} \leftarrow \text{ifft}(\bar{\mathcal{C}}, [], 3)$
return $\mathcal{C}$

The matricization makes the tensor easy to be processed as a matrix. Illustrating this concept with an example makes it easier to understand.

Example 2.3. Consider the tensor $\mathcal{A} \in \mathbb{R}^{3\times4\times2}$, which has two frontal slices $\mathcal{A}_{::1}$ and $\mathcal{A}_{::2}$ as follows:

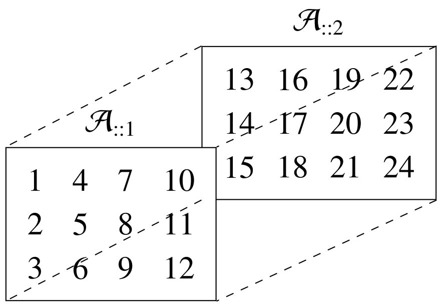

Figure 3: The frontal slices of a third-order tensor $\mathcal{A} \in \mathbb{R}^{3\times4\times2}$.

then the three mode-n unfoldings are

$$\mathcal{A}^{(1)} = \begin{bmatrix} 1 & 4 & 7 & 10 & 13 & 16 & 19 & 22 \\ 2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 \\ 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 \end{bmatrix}, \quad \mathcal{A}^{(2)} = \begin{bmatrix} 1 & 2 & 3 & 13 & 14 & 15 \\ 4 & 5 & 6 & 16 & 17 & 18 \\ 7 & 8 & 9 & 19 & 20 & 21 \\ 10 & 11 & 12 & 22 & 23 & 24 \end{bmatrix},$$

$$\mathcal{A}^{(3)} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \dots & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & \cdots & 21 & 22 & 23 & 24 \end{bmatrix}.$$

Next, we consider the Kronecker product of tensors.

Definition 2.4. [? ?] For two m-order tensors $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{n_1 \times n_2 \times \dots \times n_m}$ and $\mathcal{B} = (b_{i_1 i_2 \dots i_m}) \in \mathbb{R}^{p_1 \times p_2 \times \dots \times p_m}$, the Kronecker product of $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\mathcal{A} \otimes \mathcal{B} := (a_{i_1 i_2 \dots i_m} \mathcal{B}) \in \mathbb{R}^{n_1 p_1 \times n_2 p_2 \times \dots \times n_m p_m},$$

whose every entry is given by

$$(\mathcal{A} \otimes \mathcal{B})_{[i_1 j_1][i_2 j_2] \dots [i_m j_m]} = a_{i_1 i_2 \dots i_m} b_{j_1 j_2 \dots j_m}.$$

Here the index $[i_k j_k]$ is converted into the index $(i_k - 1) p_k + j_k$ and p_k is the dimension of j_k for $1 \leq k \leq m$.

Definition 2.5. [?] (Identity Tensor) The $n \times n \times k$ tensor $\mathcal{I}_{nnk}$ is called identity tensor if its first frontal slice is an $n \times n$ identity matrix and other frontal slices are all zeros. If $k = 1$, then the $n \times n \times 1$ tensor $\mathcal{I}_{nn1}$ is a special identity tensor with the frontal face being the $n \times n$ identity matrix, and we use $\mathcal{I}_n$ to denote it.

A key limitation of the t-product is its strict requirement for matching dimensions between tensors. To circumvent this, Chen et al. [?], developed a STP (STP) for third-order tensors, which relaxes the matching requirement by allow dimensions to be multiples. Building upon this work, the author proposed a generalized definition of the STP that removes all dimensional constraints and enables the t-product of tensors with arbitrary dimensions, as described in detail below:

Definition 2.6. [?] Let $\mathcal{A} = (a_{i_1 i_2 i_3})$ be an $m \times n \times r$ third-order tensor, and $\mathcal{B} = \left(b_{j_1 j_2 j_3}\right)$ be a $p \times q \times s$ third-order tensor. The STP of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \ltimes \mathcal{B}$, is defined as

$$\mathcal{A} \ltimes \mathcal{B} := \left(\mathcal{A} \otimes \mathcal{I}_{(\frac{t}{n})(\frac{t}{n})(\frac{d}{r})}\right) * \left(\mathcal{B} \otimes \mathcal{I}_{(\frac{t}{p})(\frac{t}{p})(\frac{d}{s})}\right) \in C^{(mt/n) \times (qt/p) \times d},$$

where $t = lcm\{n, p\}$ and $d = lcm\{r, s\}$.

When $n = p$ and $r = s$, the STP coincides with the t-product. Moreover, the STP keeps all the major properties of the t-product unchanged. Unlike the t-product, it provides a method to multiply tensors of arbitrary dimensions, thereby serving as a generalization of the t-product.

The Toeplitz tensor is introduced as follows:

Definition 2.7. [?] A third-order tensor $\mathcal{A} = (a_{i_1,i_2,i_3}) \in \mathbb{C}^{n_1\times n_2\times n_3}$ is called a Toeplitz tensor if

$$a_{i_1,i_2,i_3} = g_{i_1-i_2,i_3}, \quad 1 \le i_1 \le n_1, \quad 1 \le i_2 \le n_2, \quad 1 \le i_3 \le n_3,$$

where $\mathcal{G} = (g_{k_1,k_2}) \in \mathbb{C}^{(n_1+n_2-1)\times n_3}$ for $1-n_2 \le k_1 \le n_1-1$ and $1 \le k_2 \le n_3$. We denote $\mathcal{A} := \text{toep}(\mathcal{G})$. In other words, a third-order tensor $\mathcal{A}$ is called Toeplitz tensor if its all frontal slices are a Toeplitz matrix.

Finally, we review some results about the conventional matrix product. Recall that the theory of linear systems over a field operates similarly to that for matrices over the real field. An important result is the following theorem:

Theorem 2.8. *[?] Let $\mathbb{F}$ be a real or complex field, $\mathbf{A} \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^m$. Then the linear system $\mathbf{A}X = B$ has a solution $X \in \mathbb{F}^n$ if and only if* $\text{rank}[\mathbf{A} \quad B] = \text{rank}\,\mathbf{A}$. *The system has a unique solution if and only if* $\text{rank}[\mathbf{A} \quad B] = \text{rank}\,\mathbf{A} = n$. *The system has infinitely many solutions if and only if* $\text{rank}[\mathbf{A} \quad B] = \text{rank}\,\mathbf{A} < n$.

Definition 2.9. [?] Partition a matrix $\mathbf{A} = \left[a_{ij}\right] \in \mathbb{R}^{m\times n}$ according to its columns: $\mathbf{A} = [A_1, A_1, \ldots, A_n]$. The mapping $V_c : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ is

$$V_c(\mathbf{A}) = \left[\; A_1^T \ldots A_n^T \;\right]^T$$

that is, $V_c(\mathbf{A})$ is the vector obtained by stacking the columns of $\mathbf{A}$, left to right.

Clearly, the vector operator $V_c(.)$ is linear and bijective.

Lemma 2.10. *[?] For any matrices $\mathbf{A} \in F^{m\times n}, \mathbf{B} \in F^{n\times p}$, we have*

$$\mathrm{V_c}(\mathbf{AB}) = \left(\mathbf{I}_p \otimes \mathbf{A}\right) \mathrm{V_c}(\mathbf{B}).$$

3. Solvability of the tensor-vector system (??) with *X* be a vector

In this section, we study the solvability of the tensor equation

$$\begin{cases} \overbrace{\mathcal{A} \ltimes X}^{mt/n\times t/p\times r} = \mathcal{B}, \\ \underbrace{X \ltimes C}_{pt_1\times bt_1/a\times r} = \mathcal{D}, \end{cases} \tag{3.1}$$

where $\mathcal{A} \in \mathbb{C}^{m\times n\times r}$, $\mathcal{B} \in \mathbb{C}^{h\times k\times r}$, $C \in \mathbb{C}^{a\times b\times r}$ and $\mathcal{D} \in \mathbb{C}^{l\times d\times r}$ are given, $X \in \mathbb{C}^p$ is an unknown vector, $t = lcm\{n, p\}$ and $t_1 = lcm\{1, a\}$. Firstly, we consider the special case $m = h$ and then study the general form.

3.1. The simple case $m = h$

In this subsection, we consider the solvability of tensor-vector equation (**??**) with $\mathcal{A} \in \mathbb{C}^{m\times n\times r}$, $\mathcal{B} \in \mathbb{C}^{m\times k\times r}$, $C \in \mathbb{C}^{a\times b\times r}$ and $\mathcal{D} \in \mathbb{C}^{l\times d\times r}$. The following results could be easily acquired by STP and figures out the dimension of the solutions.

Lemma 3.1. *If tensor-vector equation (**??**) has a solution $X \in \mathbb{C}^p$, the dimensions of tensors $\mathcal{A}$, $\mathcal{B}$, C and $\mathcal{D}$ should satisfy two conditions:*

(i). $\frac{n}{k}$ and $\frac{l}{a}$ must be positive integers;

(ii). $b = d$ and $p = \frac{l}{a} = \frac{n}{k}$.

Proof. First, for

$$\overbrace{\mathcal{A} \ltimes X}^{mt/n\times t/p\times r} = \mathcal{B}, \tag{3.2}$$

$\frac{mt}{n} = h, \frac{t}{p} = k,$ where $t = lcm\{n, p\}$. Thus $t = n$, and $p = \frac{n}{k}$ is a positive integer. Then, for

$$\underbrace{X \ltimes C}_{pt_1\times bt_1/a\times r} = \mathcal{D}, \tag{3.3}$$

we have that

$$X \ltimes C = \underbrace{(X \otimes \mathcal{I}_{t_1\times t_1\times r}) * \left(C \otimes \mathcal{I}_{\frac{t_1}{a}\times\frac{t_1}{a}\times 1}\right)}_{pt_1\times\frac{bt_1}{a}\times r} = \mathcal{D} \in C^{l\times d\times r},$$

where $t_1 = lcm\{1, a\}$. Therefore, $t_1 = a$, and $pt_1 = l, \frac{bt_1}{a} = d$. It shows that $\frac{l}{a}$ is a positive integer, $p = \frac{l}{a} = \frac{n}{k}$ and $b = d$. Therefore, equation (**??**) has a solution if the two conditions are fulfilled, which completes the proof □

Now we explore the necessary and sufficient conditions for the solvability of Equation (**??**) with case $m = h$ by looking through its partition structure.

Theorem 3.2. *If equation (**??**) has a solution then the following conditions must be satisfied simultaneously.*

(i). $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$ and $\mathcal{B}$ should be linearly dependent in vector space $\mathbb{C}^{m\times k\times r}$, where $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$ are p equal-size blocks of tensor $\mathcal{A}$, belonging to $\mathbb{C}^{m\times k\times r}$ such that

$$\mathcal{A} = \left[\hat{\mathcal{A}}_1\ \hat{\mathcal{A}}_2\ \cdots\ \hat{\mathcal{A}}_p\right].$$

(ii). The elements $d_{t+(i-1)a,j,s} \in \mathcal{D}_i$ could be denoted as:

$$d_{t+(i-1)a,j,s} = \alpha_j c_{t,j,s},$$

where $\mathcal{D}_i \in \mathbb{C}^{a\times d\times r}$ is a block tensor of $\mathcal{D}$,

$$\mathcal{D} = \left[\mathcal{D}_1\ \mathcal{D}_2\ \cdots\ \mathcal{D}_p\right]^T.$$

$c_{t,j,s} \in C, \alpha_j \in \mathbb{C}$ and $i = 1, 2, ..., p$; $j = 1, 2, ..., d$; $t = 1, 2, ..., a$; $s = 1, 2, ..., r$. Actually, $X = [\alpha_1, ..., \alpha_p]^T$.

Furthermore, the solution must be unique unless tensors C and $\mathcal{D}$ are O and $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$ are also linearly dependent based on condition (i)*, where O is a zero tensor.*

Proof. If X is a solution of tensor-vector equation (**??**), then by Lemma **??**, X must belong to $\mathbb{C}^p$, where $p = \frac{n}{k}$. Suppose that

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix}^T \in \mathbb{C}^p.$$

From the definition **??** and the dimension relationship derived in Lemma **??**, Equation (**??**) could be rewritten as

$$\begin{aligned} \mathcal{A} \ltimes X &= \mathcal{A} * (X \otimes \mathcal{I}_{kkr}) \\ &= \begin{bmatrix} \hat{\mathcal{A}}_1 \ \hat{\mathcal{A}}_2 \ \cdots \ \hat{\mathcal{A}}_p \end{bmatrix} * \begin{bmatrix} x_1\mathcal{I}_{kkr} \\ x_2\mathcal{I}_{kkr} \\ \vdots \\ x_p\mathcal{I}_{kkr} \end{bmatrix} \\ &= \sum_{i=1}^{h} x_i \hat{\mathcal{A}}_i = \mathcal{B}, \end{aligned} \tag{3.4}$$

and Equation (**??**) could be described as

$$X \ltimes C = (X \otimes \mathcal{I}_{aar}) * C = \begin{bmatrix} x_1 C \\ x_2 C \\ \vdots \\ x_p C \end{bmatrix} = \begin{bmatrix} \mathcal{D}_1 \\ \mathcal{D}_2 \\ \vdots \\ \mathcal{D}_p \end{bmatrix} = \mathcal{D}. \tag{3.5}$$

Thus, the solvability of Equation (**??**) implies that, the set $\{\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p, \mathcal{B}\}$ is linearly dependent and the elements $d_{t+(i-1)a,j,s} \in \mathcal{D}_i$ must be as follows:

$$d_{t+(i-1)a,j,s} = \alpha_j c_{t,j,s},$$

also as a result, the solution must be unique unless tensors C and $\mathcal{D}$ are O and $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$ are also linearly dependent based on condition (i). □

Remark 3.3. Theorem **??** represents a significant generalization of [**?** , Lemma 3.2] by extending the fundamental principles of linear dependence from matrix spaces to higher-order tensor spaces. While [**?** , Lemma 3.2] addresses matrix-vector systems in $\mathbb{C}^{m\times k}$, our result elevates these concepts to tensor-vector systems in $\mathbb{C}^{m\times q\times r}$. This extension establishes conditions for solution existence and uniqueness in higher-dimensional settings. The theorem thus bridges classical matrix theory with modern tensor analysis, providing a powerful framework for solving a broader class of linear systems encountered in multi-dimensional data analysis.

In Theorem **??** a necessary condition for the solvability of system (3.1) was obtained. We now provide an example to illustrate that this condition is indeed necessary; that is, not only does its violation imply the absence of a solution, but there also exist cases where the condition holds and the system admits a solution. The following example clarifies this point.

Example 3.4. Consider the tensors $\mathcal{A} \in \mathbb{R}^{2\times4\times2}$, $\mathcal{B} \in \mathbb{R}^{2\times2\times2}$, $C \in \mathbb{R}^{2\times2\times2}$, and $\mathcal{D} \in \mathbb{R}^{4\times2\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{::1} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{A}_{::2} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{B}_{::1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \mathcal{B}_{::2} = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix},$$

$$C_{::1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad C_{::2} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \mathcal{D}_{::1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{D}_{::2} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \\ 1 & -1 \\ -1 & -2 \end{bmatrix}.$$

Here, $p = \frac{n}{k} = \frac{l}{a} = 2$. Then, $X \in \mathbb{C}^2$, and $\hat{\mathcal{A}}_1$, $\hat{\mathcal{A}}_2$ with $\mathcal{B}$ are linearly dependent, where the frontal slices of $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$ are given as follows:

$$(\hat{\mathcal{A}}_1)_{::1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad (\hat{\mathcal{A}}_1)_{::2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (\hat{\mathcal{A}}_2)_{::1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (\hat{\mathcal{A}}_2)_{::2} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

furthermore, $\hat{\mathcal{D}}_1$, $\hat{\mathcal{D}}_2$ with C are also linearly dependent, where the frontal slices of $\hat{\mathcal{D}}_1$ and $\hat{\mathcal{D}}_2$ are given as follows:

$$(\hat{\mathcal{D}}_1)_{::1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad (\hat{\mathcal{D}}_1)_{::2} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, (\hat{\mathcal{D}}_2)_{::1} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad (\hat{\mathcal{D}}_2)_{::2} = \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix}.$$

By Theorem **??**, the given tensors are permissible, and it is easy to verify that $X = [1, -1]^T$ is a unique solution of (**??**).

As the following example shows, the conditions in Theorem **??** is just necessary.

Example 3.5. Consider the tensor $\mathcal{D} \in \mathbb{R}^{4\times2\times2}$, with two frontal slices as follows;

$$\mathcal{D}_{::1} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 2 & -4 \\ 0 & 2 \end{bmatrix}, \quad \mathcal{D}_{::2} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ -2 & 2 \\ 2 & 4 \end{bmatrix},$$

and tensors $\mathcal{A}$, $\mathcal{B}$ and C are the same as those in Example **??**. It is not hard to verify that the two conditions hold, but there is no solution.

Next, we will look through the solving process of tensor-vector Equation (**??**). It is easy to know that Equations (**??**) and (**??**) are equivalent to the following linear system

$$\begin{cases} \sum_{i=1}^{p} x_i V_c\left(\mathcal{A}_i^{(1)}\right) = V_c\left(\mathcal{B}^{(1)}\right), \\ x_i V_c\left(C^{(1)}\right) = V_c\left(\mathcal{D}_i^{(1)}\right), \quad i = 1, 2, ..., p, \end{cases} \tag{3.6}$$

where $\mathcal{A}_i^{(1)}$, $\mathcal{B}^{(1)}$, $C^{(1)}$ and $\mathcal{D}_i^{(1)}$ are the 1-mode matricization of $\hat{\mathcal{A}}_i$, $\mathcal{B}$, C and $\mathcal{D}_i$ respectively. Then an equivalent form of tensor-vector equation (**??**) is acquired.

Theorem 3.6. *The tensor-vector system in (**??**), formulated with the STP, is equivalent to the following matrix-vector system formulated with the conventional matrix product;*

$$\begin{cases} \bar{\mathbf{A}}X = V_c\left(\mathcal{B}^{(1)}\right), \\ x_i V_c\left(C^{(1)}\right) = V_c\left(\mathcal{D}_i^{(1)}\right), \quad i = 1, 2, ..., p, \end{cases} \tag{3.7}$$

where

$$\bar{\mathbf{A}} = \left[\begin{array}{cccc} V_c\left(\mathcal{A}_1^{(1)}\right) & V_c\left(\mathcal{A}_2^{(1)}\right) & \cdots & V_c\left(\mathcal{A}_h^{(1)}\right) \end{array}\right].$$

3.2. The general case

In this subsection, we examine the solvability of the tensor-vector equation (**??**) by considering two distinct cases: $m \neq h$ with $l = p$ and $m \neq h$ with $l \neq p$. We will only consider the case where $l \neq p$, the other case is similar. First, we present the following lemma, which establishes a necessary condition for the solvability of the matrix-vector equation (**??**) and determines the dimension of its solutions.

Lemma 3.7. *If tensor-vector equation (**??**) has a solution, the sizes of tensors $\mathcal{A}$, $\mathcal{B}$, C and $\mathcal{D}$ must satisfy the following two conditions:*

(i). $\frac{h}{m}$ *and* $\frac{n}{k}$ *must be positive integers. Furthermore,* $\gcd\left\{k, \frac{h}{m}\right\} = 1$.

(ii). $\frac{l}{a}$ *must be positive integer and* $\frac{l}{a} \neq \frac{n}{k}$. *Moreover, the solution must belong to* $\mathbb{C}^p$, *where* $p = \frac{l}{a} = \frac{nh}{mk}$ *and* $b = d$.

Proof. Taking a similar line to Lemma **??**, we can obtain that

$$\frac{mg}{n} = h, \ \frac{g}{p} = k, \ l = pf, \ d = \frac{bf}{a}, \ \text{ where } f = lcm\{1, a\} \text{ and } g = lcm\{n, p\}.$$

Then, $f = a$, $\frac{g}{n} = \frac{h}{m}$ and $p = \frac{l}{f} = \frac{l}{a} = \frac{g}{k} = \frac{nh}{mk}$. Thus, $\frac{h}{m}$ and $\frac{l}{a}$ are positive integers. Moreover,

$$g = \frac{nh}{m} = lcm\{n, p\} = lcm\left\{n, \frac{nh}{mk}\right\},$$

where $\frac{n}{k}$ is a positive integer, since it is a divisor of $\frac{nh}{k}$. Simultaneously,

$$\frac{g}{n} = \frac{h}{m}, \text{ and } \frac{g}{\frac{nh}{mk}} = k.$$

Consequently, $\frac{h}{m}$ is coprime to k. Suppose $\frac{l}{a} = \frac{n}{k}$ and $p = \frac{h}{m} \cdot \frac{n}{k} = \frac{l}{a}$. It follows that $\frac{h}{m} = 1$, which contradicts the assumption that $m \neq h$. Therefore, the proof is complete. □

The conditions stated in Lemma **??** are necessary for the tensor-vector equation (**??**). We refer to these as permissible conditions for the tensor-vector equation. The tensors $\mathcal{A}$, $\mathcal{B}$, C and $\mathcal{D}$ are considered permissible if these conditions are satisfied. Additionally, we provide several examples to illustrate these results.

Example 3.8. Consider the tensors $\mathcal{A} \in \mathbb{R}^{2\times3\times2}$, $\mathcal{B} \in \mathbb{R}^{4\times3\times2}$, $C \in \mathbb{R}^{3\times2\times2}$, and $\mathcal{D} \in \mathbb{R}^{6\times2\times2}$, each having two frontal slices as follows;

$$\mathcal{A}_{::1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \mathcal{A}_{::2} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathcal{B}_{::1} = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}, \mathcal{B}_{::2} = \begin{bmatrix} -1 & 0 & 2 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix},$$

$$C_{::1} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & -1 \end{bmatrix}, C_{::2} = \begin{bmatrix} -1 & 0 \\ 2 & -1 \\ 3 & -1 \end{bmatrix}, \quad \mathcal{D}_{::1} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 2 & 1 \\ 1 & -1 \\ 0 & 2 \\ -2 & -1 \end{bmatrix}, \mathcal{D}_{::2} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ -3 & 1 \\ -1 & 0 \\ 2 & -1 \\ 3 & -1 \end{bmatrix}.$$

Then, by Lemma **??**, the given tensors are permissible, and it is easy to verify that $X = [1, -1]^T$ is a solution for tensor-vector equation (**??**).

The following example shows that the conditions in Lemma **??** are only necessary.

Example 3.9. Consider the tensor $\mathcal{D} \in \mathbb{R}^{6\times2\times2}$, with two frontal slices as follows;

$$\mathcal{D}_{::1} = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 2 & 1 \\ 1 & 4 \\ 2 & -1 \\ -2 & -1 \end{bmatrix}, \mathcal{D}_{::2} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 4 & 1 \\ -1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix},$$

and tensors $\mathcal{A}$, $\mathcal{B}$ and C are the same as those in Example **??**. By Lemma **??**, the given tensors are permissible, but we can verify that the equation has no solution. Thus, conditions in Lemma **??** are just necessary.

Next, we will further investigate the solvability of Equation (**??**). We assume that the given tensors in the tensor-vector equation are permissible, as established by Lemma **??**. Under this assumption, we can derive the following necessary conditions for solvability.

Theorem 3.10. *If a solution exists for the tensor-vector Equation (**??**), then the tensor $\mathcal{B}$ must be a block Toeplitz tensor. Additionally, the entries $d_{t+(i-1)a,j,s}$ belonging to $\mathcal{D}_i$ must equal $\alpha_j c_{t,j,s}$, where $c_{t,j,s} \in C$, $\alpha_i \in \mathbb{C}$, and $\mathcal{D}_i \in \mathbb{C}^{a\times b\times r}$ is a block of $\mathcal{D}$ for $i = 1, \ldots, p; t = 1, \ldots, a; j = 1, \ldots, b;$ and*

$s = 1, \ldots, r$. Furthermore, tensor B must have the following block form:

$$\mathcal{B} = \left[\begin{array}{c} \mathcal{B}_{1::} \\ \vdots \\ \mathcal{B}_{\frac{h}{m}::} \\ \hline \vdots \\ \hline \mathcal{B}_{h-\frac{h}{m}+1::} \\ \vdots \\ \mathcal{B}_{h::} \end{array}\right] = \begin{bmatrix} \mathrm{Block}_1(\mathcal{B}) \\ \vdots \\ Block\ _m(\mathcal{B}) \end{bmatrix},$$

where $\mathcal{B}_{1::}$, $j = 1, \ldots, h$ are horizontal slices of $\mathcal{B}$ and

$$\mathrm{Block}_s(\mathcal{B}) = \begin{bmatrix} \mathcal{B}_{(s-1)\frac{h}{m}+1::} \\ \vdots \\ \mathcal{B}_{s\frac{h}{m}::} \end{bmatrix}, \quad s = 1, \cdots, m,$$

are Toeplitz tensors.

Proof. By Lemma **??**, if the tensor-vector equation (**??**) has asolution, that solution must belong to $\mathbb{C}^p$, where $p = \frac{nh}{mk}$. Suppose that X is a solution of tensor-vector equation (**??**). Then $X \in \mathbb{C}^{\frac{nh}{mk}}$. Assume hat $k = l_1 \cdot \frac{h}{m} + l_2$. Then for $s = 1, 2, \ldots, m$, we have that

$$\mathcal{A}_{s::} \ltimes X = x_1\mathcal{A}^1 + x_2\mathcal{A}^2 + \cdots + x_{\frac{h}{m}}\mathcal{A}^{\frac{h}{m}} + x_{\frac{h}{m}+1}\mathcal{A}^{\frac{h}{m}+1} + \cdots + x_p\mathcal{A}^p = Block_s(\mathcal{B}) \in \mathbb{C}^{\frac{h}{m}\times k\times r},$$

where the frontal slices of $\mathcal{A}^1, \mathcal{A}^2, \ldots, \mathcal{A}^p$ are as follows:

$$\mathcal{A}^1_{::i} = \begin{bmatrix} a_{s,1,i} & 0 & \cdots & 0 & a_{s,2,i} & 0 & \cdots & 0 & \cdots & a_{s,(l_1+1),i} & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & a_{s,1,i} & 0 & \cdots & 0 & a_{s,2,i} & 0 & \cdots & 0 & \cdots & a_{s,(l_1+1),i} \\ 0 & 0 & 0 & a_{s,1,i} & 0 & \cdots & 0 & a_{s,2,i} & 0 & \cdots & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & a_{s,1,i} & 0 & \cdots & 0 & a_{s,2,i} & 0 & \cdots \end{bmatrix},$$

$$\mathcal{A}^2_{::i} = \begin{bmatrix} 0 & \cdots & a_{s,(1_1+2),i} & 0 & \cdots & 0 & a_{s,([\frac{2k}{h/m}]+1),i} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & a_{s,(1_1+2),i} & 0 & \cdots & 0 & a_{s,([\frac{2k}{h/m}]+1),i} \\ a_{s,(l_1+1),i} & 0 & 0 & 0 & 0 & a_{s,(1_1+2),i} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & a_{s,(l_1+1),i} & 0 & \cdots & 0 & a_{s,(1_1+2),i} & \cdots & \cdots \end{bmatrix},$$

$$\vdots$$

$$\mathcal{A}^{\frac{h}{m}}_{::i} = \begin{bmatrix} 0 & \cdots & 0 & a_{s,k,i} & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & a_{s,k,i} & \cdots & 0 & 0 \\ a_{s,(k-l_1),i} & 0 & 0 & 0 & 0 & 0 & a_{s,k,i} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & a_{s,(k-l_1),i} & 0 & \cdots & 0 & 0 & \cdots & a_{s,k,i} \end{bmatrix},$$

$$\mathcal{A}^{\frac{h}{m}+1}_{::i} = \left[\begin{matrix} a_{s,(k+1),i} & 0 & \cdots & 0 & a_{s,(k+l_1),i} & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & a_{s,(k+1),i} & 0 & \cdots & 0 & a_{s,(k+l_1),i} \\ 0 & 0 & 0 & a_{s,(k+1),i} & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & a_{s,(k+1),i} & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & \cdots & a_{s,(k+l_1+1),i} & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \cdots & a_{s,(k+l_1+1),i} \\ a_{s,(k+l_1),i} & 0 & \cdots & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \cdots & 0 & a_{s,(k+l_1),i} & 0 & \cdots \end{matrix}\right],$$

$$\vdots$$

$$\mathcal{A}^{p}_{::i} = \begin{bmatrix} 0 & \cdots & 0 & a_{s,n,i} & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & a_{s,n,i} & \cdots & 0 & 0 \\ a_{s(n-l_1)i} & 0 & 0 & 0 & 0 & 0 & a_{s,n,i} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & a_{s(n-l_1)i} & 0 & \cdots & 0 & 0 & \cdots & a_{s,n,i} \end{bmatrix}.$$

It is easy to see that $Block_s(\mathcal{B})$ is a Toeplitz tensor. Then, by Equation (**??**) we have

$$Row_i(X) \ltimes C = x_i \times C = \mathcal{D}_i \in \mathbb{C}^{a\times b\times r}.$$

Thus, $d_{t+(i-1)a,j,s} = x_i c_{t,j,s} = \alpha_i c_{t,j,s}$. The proof is completed. □

Remark 3.11. Theorem **??** generalizes the concept presented in [**?** , Theorem 3.2] by extending the block Toeplitz structure from matrices to tensors. While [**?** , Theorem 3.2] focuses on splitting matrix B into Toeplitz blocks in a matrix space, Theorem **??** introduces a higher-order structure where tensor $\mathcal{B}$ is composed of block Toeplitz tensors in a multi-dimensional tensor space. This theorem provides a systematic framework for analyzing tensor-vector equations. The hierarchical arrangement of horizontal slices and blocks in this theorem enables efficient representation and manipulation of tensor data, which is crucial for applications in multidimensional signal processing, machine learning, and scientific computing.

Recalling Example **??**, we observe that the tensor $\mathcal{B}$, for which the corresponding equation admits a solution, indeed satisfies the condition stated in Theorem **??**.

Next we look through the solvability of the equation. By Theorem**??** and its proof, the following equations are established:

$$x_1\mathcal{E}^1 + x_{\frac{h}{m}+1}\mathcal{E}^{\frac{h}{m}+1} + \cdots + x_{(\frac{n}{k}-1)\frac{h}{m}+1}\mathcal{E}^{(\frac{n}{k}-1)\frac{h}{m}+1} = \mathcal{B}^1,$$

$$x_2\mathcal{E}^2 + x_{\frac{h}{m}+2}\mathcal{E}^{\frac{h}{m}+2} + \cdots + x_{(\frac{n}{k}-1)\frac{h}{m}+2}\mathcal{E}^{(\frac{n}{k}-1)\frac{h}{m}+2} = \mathcal{B}^2,$$

$$\vdots$$

$$x_{\frac{h}{m}}\mathcal{E}^{\frac{h}{m}} + x_{2\frac{h}{m}}\mathcal{E}^{2\frac{h}{m}} + \cdots + x_p\mathcal{E}^p = \mathcal{B}^{\frac{h}{m}},$$

$$x_t\mathcal{C} = \mathcal{D}^t \quad t = 1, 2, \ldots, p,$$

where the frontal slices of $\mathcal{E}^1, \mathcal{E}^2, \cdots, \mathcal{E}^p, \mathcal{B}^1, \cdots, \mathcal{B}^{\frac{h}{m}}$ are as follows:

$$\mathcal{E}^1_{::i} = \begin{bmatrix} a_{1,1,i} & a_{1,2,i} & \cdots & a_{1,l_1,i} & a_{1,l_1+1,i} \\ a_{2,1,i} & a_{2,2,i} & \cdots & a_{2,l_1,i} & a_{2,l_1+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,1,i} & a_{m,2,i} & \cdots & a_{m,l_1,i} & a_{m,l_1+1,i} \end{bmatrix},$$

$$\mathcal{E}^{\frac{h}{m}+1}_{::i} = \begin{bmatrix} a_{1,k+1,i} & a_{1,k+2,i} & \cdots & a_{1,k+l_1,i} & a_{1,k+l_1+1,i} \\ a_{2,k+1,i} & a_{2,k+2,i} & \cdots & a_{2,k+l_1,i} & a_{2,k+l_1+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,k+1,i} & a_{m,k+2,i} & \cdots & a_{m,k+l_1,i} & a_{m,k+l_1+1,i} \end{bmatrix},$$

$$\vdots$$

$$\mathcal{E}^{(\frac{n}{k}-1)\frac{h}{m}+1}_{::i} = \begin{bmatrix} a_{1,n-k+1,i} & a_{1,n-k+2,i} & \cdots & a_{1,n-k+l_1,i} & a_{1,n-k+l_1+1,i} \\ a_{2,n-k+1,i} & a_{2,n-k+2,i} & \cdots & a_{2,n-k+l_1,i} & a_{2,n-k+l_1+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,n-k+1,i} & a_{m,n-k+2,i} & \cdots & a_{m,n-k+l_1,i} & a_{m,n-k+l_1+1,i} \end{bmatrix},$$

$$\mathcal{B}^1_{::i} = \begin{bmatrix} b_{1,1,i} & b_{1,\frac{h}{m}+1,i} & \cdots & b_{1,(l_1-1)\frac{h}{m}+1,i} & b_{1,l_1\frac{h}{m}+1,i} \\ b_{\frac{h}{m}+1,1,i} & b_{\frac{h}{m}+1,\frac{h}{m}+1,i} & \cdots & b_{\frac{h}{m}+1,(l_1-1)\frac{h}{m}+1,i} & b_{\frac{h}{m}+1,l_1\frac{h}{m}+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{h-\frac{h}{m}+1,1,i} & b_{h-\frac{h}{m}+1,\frac{h}{m}+1,i} & \cdots & b_{h-\frac{h}{m}+1,(l_1-1)\frac{h}{m}+1,i} & b_{h-\frac{h}{m}+1,l_1\frac{h}{m}+1,i} \end{bmatrix},$$

$$\mathcal{E}^2_{::i} = \begin{bmatrix} a_{1,l_1+1,i} & a_{1,l_1+2,i} & \cdots & a_{1,[\frac{2k}{h/m}],i} & a_{1,[\frac{2k}{h/m}]+1,i} \\ a_{2,l_1+1,i} & a_{2,l_1+2,i} & \cdots & a_{2,[\frac{2k}{h/m}],i} & a_{2,[\frac{2k}{h/m}]+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,l_1+1,i} & a_{m,l_1+2,i} & \cdots & a_{m,[\frac{2k}{h/m}],i} & a_{m,[\frac{2k}{h/m}]+1,i} \end{bmatrix},$$

$$\mathcal{E}_{::i}^{\frac{h}{m}+2} = \begin{bmatrix} a_{1,k+l_1+1,i} & a_{1,k+l_1+2,i} & \cdots & a_{1,k+[\frac{2k}{h/m}],i} & a_{1,k+[\frac{2k}{h/m}]+1,i} \\ a_{2,k+l_1+1,i} & a_{2,k+l_1+2,i} & \cdots & a_{2,k+[\frac{2k}{h/m}],i} & a_{2,k+[\frac{2k}{h/m}]+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,k+l_1+1,i} & a_{m,k+l_1+2,i} & \cdots & a_{m,k+[\frac{2k}{h/m}],i} & a_{m,k+[\frac{2k}{h/m}]+1,i} \end{bmatrix},$$

$$\vdots$$

$$\mathcal{E}_{::i}^{(\frac{n}{k}-1)\frac{h}{m}+1} = \begin{bmatrix} a_{1,n-k+l_1+1,i} & a_{1,n-k+l_1+2,i} & \cdots & a_{1,n-k+[\frac{2k}{h/m}],i} & a_{1,n-k+[\frac{2k}{h/m}]+1,i} \\ a_{2,n-k+l_1+1,i} & a_{2,n-k+l_1+2,i} & \cdots & a_{2,n-k+[\frac{2k}{h/m}],i} & a_{2,n-k+[\frac{2k}{h/m}]+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,n-k+l_1+1,i} & a_{m,n-k+l_1+2,i} & \cdots & a_{m,n-k+[\frac{2k}{h/m}],i} & a_{mn-k+[\frac{2k}{h/m}]+1,i} \end{bmatrix},$$

$$\mathcal{B}_{::i}^{2} = \begin{bmatrix} b_{l_2+1,1,i} & b_{1,\frac{h}{m}-l_2+1,i} & \cdots & \cdots & b_{1,([\frac{2k}{h/m}]-l_1-1)\frac{h}{m}-l_2+1,i} & b_{1,([\frac{2k}{h/m}]-l_1)\frac{h}{m}-l_2+1,i} \\ b_{\frac{h}{m}+l_2+1,1,i} & b_{\frac{h}{m}+1,\frac{h}{m}-l_2+1,i} & \cdots & \cdots & b_{\frac{h}{m}+1,([\frac{2k}{h/m}]-l_1-1)\frac{h}{m}-l_2+1,i} & b_{\frac{h}{m}+1([\frac{2k}{h/m}]-l_1-1)\frac{h}{m}-l_2,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{h-\frac{h}{m}+l_2+1,1,i} & b_{h-\frac{h}{m}+1,\frac{h}{m}-l_2+1,i} & \cdots & \cdots & b_{h-\frac{h}{m}+1,([\frac{2k}{h/m}]-l_1-1)\frac{h}{m}-l_2+1,i} & b_{h-\frac{h}{m}+1,([\frac{2k}{h/m}]-l_1)\frac{h}{m}-l_2+1,i} \end{bmatrix},$$

$$\mathcal{E}_{::i}^{\frac{h}{m}} = \begin{bmatrix} a_{1,k-l_1,i} & a_{1,k-l_1+1,i} & \cdots & a_{1,k-1,i} & a_{1,k,i} \\ a_{2,k-l_1,i} & a_{2,k-l_1+1,i} & \cdots & a_{2,k-1,i} & a_{2,k,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,k-l_1,i} & a_{m,k-l_1+1,i} & \cdots & a_{m,k-1,i} & a_{m,k,i} \end{bmatrix},$$

$$\mathcal{E}_{::i}^{2\frac{h}{m}} = \begin{bmatrix} a_{1,2k-l_1,i} & a_{1,2k-l_1+1,i} & \cdots & a_{1,2k-1,i} & a_{1,2k,i} \\ a_{2,2k-l_1,i} & a_{2,2k-l_1+1,i} & \cdots & a_{2,2k-1,i} & a_{2,2k,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,2k-l_1,i} & a_{m,2k-l_1+1,i} & \cdots & a_{m,2k-1,i} & a_{m,2k,i} \end{bmatrix},$$

$$\vdots$$

$$\mathcal{E}_{::i}^{p} = \begin{bmatrix} a_{1,n-l_1,i} & a_{1,n-l_1+1,i} & \cdots & a_{1,n-1,i} & a_{1,n,i} \\ a_{2,n-l_1,i} & a_{2,n-l_1+1,i} & \cdots & a_{2,n-1,i} & a_{2,n,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m,n-l_1,i} & a_{m,n-l_1+1,i} & \cdots & a_{m,n-1,i} & a_{m,n,i}, \end{bmatrix},$$

$$\mathcal{B}_{::i}^{\frac{h}{m}} = \begin{bmatrix} b_{\frac{h}{m}-l_2+1,1,i} & b_{1,l_2+1,i} & \cdots & b_{1,(l_1-2)\frac{h}{m}]+l_2+1,i)} & b_{1,k-\frac{h}{m}+1,i} \\ b_{2\frac{h}{m}-l_2+1,1,i} & b_{\frac{h}{m}+1,l_2+1,i} & \cdots & b_{\frac{h}{m}+1,(l_1-2)\frac{h}{m}]+l_2+1,i)} & b_{\frac{h}{m}+1,k-\frac{h}{m}+1,i} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{h-l_2+1,1,i} & b_{h-\frac{h}{m}+1,l_2+1,i} & \cdots & b_{h-\frac{h}{m}+1,(l_1-2)\frac{h}{m}+l_2+1,i)} & b_{h-\frac{h}{m}+1,k-\frac{h}{m}+1,i} \end{bmatrix},$$

and $\mathcal{D}^t \in \mathbb{C}^{a\times b\times r}$, $t = 1, \ldots, p$, is a block of $\mathcal{D}$, with frontal slices as follows:

$$\mathcal{D}^t_{::s} = \begin{bmatrix} d_{(t-1)a+1,1,s} & d_{(t-1)a+1,2,s} & \cdots & d_{(t-1)a+1,b,s} \\ d_{(t-1)a+2,1,s} & d_{(t-1)a+2,2,s} & \cdots & d_{(t-1)a+2,b,s} \\ \cdots & \cdots & \cdots & \cdots \\ d_{ta,1,s} & d_{ta,2,s} & \cdots & d_{ta,b,s} \end{bmatrix}.$$

For convenience, take $z.k = l^z_1.\frac{h}{m} + l^z_2, z = 1, \cdots, \frac{h}{m}$, we can simplify them as follows:

$$\begin{cases} x_1 \begin{bmatrix} \mathcal{A}'_1 & \mathcal{A}'_2 & \cdots & \mathcal{A}'_{l^1_1} & \mathcal{A}'_{l^1_1+1} \end{bmatrix} + x_{\frac{h}{m}+1} \begin{bmatrix} \mathcal{A}'_{k+1} & \mathcal{A}'_{k+2} & \cdots & \mathcal{A}'_{k+l^1_1} & \mathcal{A}'_{k+l^1_1+1} \end{bmatrix} \\ \qquad + \cdots + x_{(\frac{n}{k}-1)\frac{h}{m}+1} \begin{bmatrix} \mathcal{A}'_{n-k+1} & \mathcal{A}'_{n-k+2} & \cdots & \mathcal{A}'_{n-k+l^1_1} & \mathcal{A}'_{n-k+l^1_1+1} \end{bmatrix} \\ \qquad = \begin{bmatrix} \tilde{B}_1 & \tilde{B}_{\frac{h}{m}+1} & \cdots & \tilde{B}_{(l^1_1-1)\frac{h}{m}+1} & \tilde{B}_{l^1_1\frac{h}{m}+1} \end{bmatrix}, \\ x_2 \begin{bmatrix} \mathcal{A}'_{l^1_1+1} & \mathcal{A}'_{l^1_1+2} & \cdots & \mathcal{A}'_{l^2_1} & \mathcal{A}'_{l^2_1+1} \end{bmatrix} + x_{\frac{h}{m}+2} \begin{bmatrix} \mathcal{A}'_{k+l^1_1+1} & \mathcal{A}'_{k+l^1_1+2} \cdots & \mathcal{A}'_{k+l^2_1} & \mathcal{A}'_{k+l^2_1+1} \end{bmatrix} \\ \qquad + \cdots + x_{(\frac{n}{k}-1)\frac{h}{m}+2} \begin{bmatrix} \mathcal{A}'_{n-k+l^1_1+1} & \mathcal{A}'_{n-k+l^1_1+2} & \cdots & \mathcal{A}'_{n-k+l^2_1} & \mathcal{A}'_{n-k+l^2_1+1} \end{bmatrix} \\ \qquad = \begin{bmatrix} \tilde{B}_{k+l^1_2} & \tilde{B}_{\frac{h}{m}-l^1_2+1} & \cdots & \tilde{B}_{(l^2_1-l^1_1-1)\frac{h}{m}-l^1_2+1} & \tilde{B}_{(l^2_1-l^1_1)\frac{h}{m}-l^1_2+1} \end{bmatrix}, \\ \vdots \\ x_{\frac{h}{m}} \begin{bmatrix} \mathcal{A}'_{k-l^1_1} & \mathcal{A}'_{k-l^1_1+1} & \cdots & \mathcal{A}'_{k-1} & \mathcal{A}'_k \end{bmatrix} + x_{2\frac{h}{m}} \begin{bmatrix} A_{2k-l^1_1} & \mathcal{A}'_{2k-l^1_1+1} & \cdots & \mathcal{A}'_{2k-1} & \mathcal{A}'_{2k} \end{bmatrix} \\ \qquad + \cdots + x_p \begin{bmatrix} \mathcal{A}'_{n-l^1_1} & \mathcal{A}'_{n-l^1_1+1} & \cdots & \mathcal{A}'_{n-1} & \mathcal{A}'_n \end{bmatrix} \\ \qquad = \begin{bmatrix} \tilde{B}_{k+\frac{h}{m}+l^1_2} & \tilde{B}_{l^1_2+1} & \cdots & \tilde{B}_{(l^1_1-2)\frac{h}{m}-l^1_2+1} & \tilde{B}_{k-\frac{h}{m}+1} \end{bmatrix}, \\ x_1 \begin{bmatrix} \mathcal{C}_{1::} & \mathcal{C}_{2::} & \ldots & \mathcal{C}_{a::} \end{bmatrix}^T = \begin{bmatrix} \mathcal{D}_{1::} & \mathcal{D}_{2::} & \ldots & \mathcal{D}_{a::} \end{bmatrix}^T, \\ x_2 \begin{bmatrix} \mathcal{C}_{1::} & \mathcal{C}_{2::} & \ldots & \mathcal{C}_{a::} \end{bmatrix}^T = \begin{bmatrix} \mathcal{D}_{a+1::} & \mathcal{D}_{a+2::} & \ldots & \mathcal{D}_{2a::} \end{bmatrix}^T, \\ \vdots \\ x_p \begin{bmatrix} \mathcal{C}_{1::} & \mathcal{C}_{2::} & \ldots & \mathcal{C}_{a::} \end{bmatrix}^T = \begin{bmatrix} \mathcal{D}_{(p-1)a+1::} & \mathcal{D}_{(p-1)a+2::} & \ldots & \mathcal{D}_{l::} \end{bmatrix}^T, \end{cases}$$

where $\mathcal{A}'_j := \mathcal{A}_{:j:}$ is the j-th lateral slice of $\mathcal{A}$, $\tilde{B}_j = \tilde{\mathcal{B}}_{:j:}$ is j-th lateral slice of tensor $\tilde{\mathcal{B}}$, and for $s = 1, 2, \cdots, r$, $\tilde{\mathcal{B}}_{::s}$ the s-th frontal slice of $\tilde{\mathcal{B}}$ is as follows:

$$\tilde{\mathcal{B}}_{::s} = \begin{bmatrix} b_{1,1,s} & b_{1,2,s} & \cdots & b_{1,k,s} & b_{2,1,s} & b_{3,1,s} & \cdots & b_{\frac{h}{m},1,s} \\ b_{\frac{h}{m}+1,1,s} & b_{\frac{h}{m}+1,2,s} & \cdots & b_{\frac{h}{m}+1,k,s} & b_{\frac{h}{m}+2,1,s} & b_{\frac{h}{m}+3,1,s} & \cdots & b_{2\frac{h}{m},1,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{h-\frac{h}{m}+1,1,s} & b_{h-\frac{h}{m}+1,2,s} & \cdots & b_{h-\frac{h}{m}+1,s} & b_{h-\frac{h}{m}+2,1,s} & b_{h-\frac{h}{m}+3,1,s} & \cdots & b_{h,1,s} \end{bmatrix}.$$

Moreover, $\mathcal{C}_{i::}$ and $\mathcal{D}_{i::}$ are the i-th horizontal slices of $\mathcal{C}$ and $\mathcal{D}$, respectively. Then we have the following result.

Theorem 3.12. *The solvability of tensor-vector equation (**??**) with case $m \neq h$ is equivalent to the* *solvability of the following tensor-vector equations*

$$\begin{cases} \left[{}_1\widehat{\mathcal{A}}\,{}_{\frac{h}{m}+1}\widehat{\mathcal{A}}\ldots{}_{(\frac{n}{k}-1)\frac{h}{m}+1}\widehat{\mathcal{A}}\right] \ltimes Y_1 = {}_1\bar{\mathcal{B}}, \\ \left[{}_2\widehat{\mathcal{A}}\,{}_{\frac{h}{m}+2}\widehat{\mathcal{A}}\ldots{}_{(\frac{n}{k}-1)\frac{h}{m}+2}\widehat{\mathcal{A}}\right] \ltimes Y_2 = {}_2\bar{\mathcal{B}}, \\ \vdots \\ \left[{}_{\frac{h}{m}}\widehat{\mathcal{A}}\,{}_{\frac{2h}{m}}\widehat{\mathcal{A}}\ldots{}_p\widehat{\mathcal{A}}\right] \ltimes Y_{\frac{h}{m}} = {}_{\frac{h}{m}}\bar{\mathcal{B}}, \\ x_tC = \mathcal{D}^t, \quad t = 1,2,\ldots,p, \end{cases} \tag{3.8}$$

where ${}_1\widehat{\mathcal{A}}, {}_2\widehat{\mathcal{A}}, \cdots, {}_p\widehat{\mathcal{A}}$ *are blocks of tensor* $\mathcal{A}$*, such that*

$${}_1\widehat{\mathcal{A}} = \left[\mathcal{A}'_1 \quad \ldots \quad \mathcal{A}'_{l_1^1} \quad \mathcal{A}'_{l_1^1+1}\right], \quad {}_2\widehat{\mathcal{A}} = \left[\ \mathcal{A}'_{l_1^1+1} \quad \mathcal{A}'_{l_1^1+2} \quad \cdots \quad \mathcal{A}'_{l_1^2} \quad \mathcal{A}'_{l_1^2+1}\ \right], \ldots,$$

$${}_p\widehat{\mathcal{A}} = \left[\mathcal{A}'_{n-l_1^1} \quad \mathcal{A}'_{n-l_1^1+1} \quad \ldots \quad \mathcal{A}'_n\right], \quad \mathcal{A} = \left[{}_1\widehat{\mathcal{A}} \quad {}_2\widehat{\mathcal{A}} \cdots {}_p\widehat{\mathcal{A}}\right].$$

Moreover,

$$\begin{cases} {}_1\bar{\mathcal{B}} = \left[\ \widetilde{B}_1\widetilde{B}_{\frac{h}{m}+1} \quad \cdots \quad \widetilde{B}_{(l_1^1-1)\frac{h}{m}+1} \quad \widetilde{B}_{l_1^1\frac{h}{m}+1}\ \right], \\ {}_2\bar{\mathcal{B}} = \left[\ \widetilde{B}_{k+l_2^1} \quad \widetilde{B}_{\frac{h}{m}-l_2^1+1} \quad \cdots \quad \widetilde{B}_{(l_2^1-l_1^1-1)\frac{h}{m}-l_2^1+1}\widetilde{B}_{(l_1^2-l_1^1)\frac{h}{m}-l_2^1+1}\ \right], \\ \vdots \\ {}_{\frac{h}{m}}\bar{\mathcal{B}} = \left[\ \widetilde{B}_{k+\frac{h}{m}-l_1^1} \quad \widetilde{B}_{l_2^1+1} \quad \ldots \quad \widetilde{B}_{(l_1^1-2)+l_2^1+1}\widetilde{B}_{k-\frac{h}{m}+1}\ \right] \end{cases}$$

and

$$Y = \left[\ Y_1 \quad Y_2 \quad \ldots \quad Y_r \quad \ldots \quad Y_{\frac{h}{m}}\ \right]^T,$$

where

$$Y_r = \left[\ y_{r1} \quad y_{r2} \quad \ldots \quad y_{r\frac{n}{k}}\ \right]^T, \quad r = 1,2\ldots,\frac{h}{m}.$$

Then,

$$X = V_c(Y),$$

*is a solution of tensor-vector equation (**??**).*

Moreover, a necessary and sufficient condition for the solvability of tensor-vector Equation(**??**) is obtained.

Corollary 3.13. *Tensor-vector Equation(**??**) has a solution if and only if the following conditions could be satisfied simultaneously:*

(i). ${}_j\widehat{\mathcal{A}}, {}_{\frac{p}{m}+j}\widehat{\mathcal{A}}, \ldots, {}_{(\frac{n}{k}-1)\frac{h}{m}+j}\widehat{\mathcal{A}}$ *and* ${}_j\bar{\mathcal{B}}$ *are linearly dependent,* $j = 1,2\ldots,\frac{h}{m}$.

(ii). *The elements* $d_{t+(i-1)a,j,s} \in \mathcal{D}_i$ *could be denoted as:*

$$d_{t+(i-1)a,j,s} = \alpha_j c_{t,j,s}$$

where $\mathcal{D}_i \in \mathbb{C}^{a\times d\times r}$ is a block tensor of $\mathcal{D}$,

$$\mathcal{D} = \begin{bmatrix}\mathcal{D}_1 \ \mathcal{D}_2 \ \cdots \ \mathcal{D}_p\end{bmatrix}^T.$$

$c_{t,j,s} \in C, \alpha_j \in \mathbb{C}$ and $i = 1,2,...,p$; $j = 1,2,...,d$; $t = 1,2,...,a$; $s = 1,2,...,r$. Actually, $X = [\alpha_1,...,\alpha_p]^T$.
Furthermore, the solution must be unique unless tensors C and $\mathcal{D}$ are O and $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$ are also linearly dependent.

We now present two numerical examples to illustrate the application of Corollary **??** in determining the solvability of equation (**??**). In the first example, all the conditions of the theorem are satisfied, and we observe that the system admits a unique solution. In the second example, one of the conditions (linear dependence) is violated, leading to no solution. These examples also demonstrate concretely how the blocks ${}_j\widehat{\mathcal{A}}$ and ${}_j\bar{\mathcal{B}}$ are computed and how they relate to the tensors C and $\mathcal{D}$ (the colors used in the tensor representations correspond to these blocks).

Example 3.14. Consider the tensors $\mathcal{A} \in \mathbb{R}^{2\times4\times2}$, $\mathcal{B} \in \mathbb{R}^{6\times4\times2}$, $C \in \mathbb{R}^{2\times3\times2}$, and $\mathcal{D} \in \mathbb{R}^{6\times3\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{::1} = \begin{bmatrix} -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \mathcal{A}_{::2} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},$$

$$\mathcal{B}_{::1} = \left[\begin{array}{cccc} -1 & 0 & -1 & 2 \\ -2 & -1 & 0 & -1 \\ 2 & -2 & -1 & 0 \\ \hline 0 & -2 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{array}\right], \quad \mathcal{B}_{::2} = \left[\begin{array}{cccc} 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 \\ \hline 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 2 & 0 & 1 & 2 \end{array}\right],$$

$$C_{::1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \quad C_{::2} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -3 \end{bmatrix},$$

$$\mathcal{D}_{::1} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & -1 & -2 \\ 2 & -4 & -2 \\ 0 & 2 & 4 \end{bmatrix}, \mathcal{D}_{::2} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & -3 \\ -2 & 1 & -1 \\ -1 & 0 & 3 \\ 4 & -2 & 2 \\ 2 & 0 & -6 \end{bmatrix}.$$

Here $\frac{h}{m} = 3$, $\frac{n}{k} = 1$, $k = 4$, $h = 6$ and $k = 4 = 1.3 + 1$, $p = \frac{nh}{km} = 3$. Then we have:

$${}_1\widehat{\mathcal{A}}_{::1} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad {}_1\widehat{\mathcal{A}}_{::2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad {}_2\widehat{\mathcal{A}}_{::1} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$${}_2\widehat{\mathcal{A}}_{::2} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad {}_3\widehat{\mathcal{A}}_{::1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad {}_3\widehat{\mathcal{A}}_{::2} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$\tilde{\mathcal{B}}_{::1} = \begin{bmatrix} -1 & 0 & -1 & 2 & -2 & 2 \\ 0 & -2 & 0 & 1 & -1 & 0 \end{bmatrix}, \quad \tilde{\mathcal{B}}_{::2} = \begin{bmatrix} 0 & 0 & -1 & -1 & 1 & 2 \\ 1 & 2 & -1 & 0 & 0 & 2 \end{bmatrix},$$

$$_1\bar{\mathcal{B}}_{::1} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad _1\bar{\mathcal{B}}_{::2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad _2\bar{\mathcal{B}}_{::1} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix},$$

$$_2\bar{\mathcal{B}}_{::2} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \quad _3\bar{\mathcal{B}}_{::1} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad _3\bar{\mathcal{B}}_{::2} = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix},$$

hence we can calculate that $_j\widehat{\mathcal{A}}$ and $_j\bar{\mathcal{B}}$, are linearly dependent, with $j = 1, 2, 3$, and $\mathcal{D}^1 = C$, $\mathcal{D}^2 = -C$, $\mathcal{D}^3 = 2C$. Solving (**??**), we have $Y_1 = 1, Y_2 = -1$ and $Y_3 = 2$ is a solution of the equations. Thus $X = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}^T$ is a solution of equation (**??**) and by Corollary **??** the solution is unique.

Example 3.15. Consider the tensors $\mathcal{A} \in \mathbb{R}^{2\times4\times2}$, $\mathcal{B} \in \mathbb{R}^{6\times4\times2}$, $C \in \mathbb{R}^{2\times3\times2}$, and $\mathcal{D} \in \mathbb{R}^{6\times3\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{::1} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 1 & -1 \end{bmatrix}, \quad \mathcal{A}_{::2} = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{B}_{::1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 1 & -1 \end{bmatrix}, \quad \mathcal{B}_{::2} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 6 & 0 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix} . C_{::1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix},$$

$$C_{::2} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \end{bmatrix}, \mathcal{D}_{::1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ -1 & -2 & 1 \\ 0 & 1 & -2 \\ 2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}, \mathcal{D}_{::2} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ -2 & 1 & -1 \\ -1 & -2 & 3 \\ 4 & -2 & 2 \\ 2 & 4 & -6 \end{bmatrix}.$$

Here $\frac{h}{m} = 3$, $\frac{n}{k} = 1$, $k = 4$, $h = 6$ and $k = 4 = 1.3 + 1$, $p = \frac{nh}{km} = 3$. Then we have:

$$_1\widehat{\mathcal{A}}_{::1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad _1\widehat{\mathcal{A}}_{::2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad _2\widehat{\mathcal{A}}_{::1} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix},$$

$$_2\widehat{\mathcal{A}}_{::2} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad _3\widehat{\mathcal{A}}_{::1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}, \quad _3\widehat{\mathcal{A}}_{::2} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\tilde{\mathcal{B}}_{::1} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 6 \\ 1 & -1 & 0 & 1 & 0 & 2 \end{bmatrix}, \quad \tilde{\mathcal{B}}_{::2} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 6 \\ -1 & 1 & 0 & -1 & 0 & 1 \end{bmatrix},$$

$$_1\bar{\mathcal{B}}_{::1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad _1\bar{\mathcal{B}}_{::2} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad _2\bar{\mathcal{B}}_{::1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$ {}_2\bar{\mathcal{B}}_{::2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad {}_3\bar{\mathcal{B}}_{::1} = \begin{bmatrix} 6 & 0 \\ 2 & -1 \end{bmatrix}, \quad {}_3\bar{\mathcal{B}}_{::2} = \begin{bmatrix} 6 & 0 \\ 1 & 1 \end{bmatrix}. $$

Hence $\mathcal{D}^1 = C$, $\mathcal{D}^2 = -C$, $\mathcal{D}^3 = 2C$ and ${}_j\widehat{\mathcal{A}}$, ${}_j\bar{\mathcal{B}}$ are not linearly dependent, $j = 1, 2, 3$. By Corollary **??** the equation **??** has no solution.

4. Solvability of the tensor-matrix system (??)

In this section, we study the solvability of the tensor-matrix equation

$$ \begin{cases} \overbrace{\mathcal{A} \times \mathbf{X}}^{mt/n \times qt/p \times r} = \mathcal{B} \\ \underbrace{\mathbf{X} \times C}_{pt_1/q \times bt_1/a \times r} = \mathcal{D}, \end{cases} \tag{4.1} $$

where $\mathcal{A} \in \mathbb{C}^{m \times n \times r}, \mathcal{B} \in \mathbb{C}^{h \times k \times r}, C \in \mathbb{C}^{a \times b \times r}$ and $\mathcal{D} \in \mathbb{C}^{l \times d \times r}$ are are known, and $\mathbf{X} \in \mathbb{C}^{p \times q}$ is an unknown matrix. $t = lcm\{n, p\}$ and $t_1 = lcm\{q, a\}$.

In this section, we carry the same line as the last section, with the simple case $m = h$ firstly studied and followed by the general case.

4.1. *The simple case $m = h$ with $l = p$*

In this subsection, we consider the solvability of tensor-matrix Equation (**??**) with $\mathcal{A} \in \mathbb{C}^{m \times n \times r}$, $\mathcal{B} \in \mathbb{C}^{m \times k \times r}$. Using STP, we could obtain the following results about the dimension necessary condition if Equation (**??**) has a solution.

Lemma 4.1. *If the tensor-matrix Equation (**??**) with $m = h$ and $l = p$ has a solution with size $p \times q$, then the dimensions of the tensors $\mathcal{A}, \mathcal{B}, C, \mathcal{D}$ and the integer q must satisfy*

- *(i). $\frac{n}{l}$ and $\frac{d}{b}$ must be positive integers;*
- *(ii). $\frac{ad}{b}|k$, $\frac{n}{l}|k$ and $q = \frac{ad}{b} = \frac{lk}{n}$.*

Proof. Suppose that $\mathbf{X} \in \mathbb{C}^{p \times q}$ is a solution of tensor-matrix Equation (**??**), By Definition **??**, we have that

$$ \begin{cases} \overbrace{\mathcal{A} \times \mathbf{X}}^{mt/n \times qt/p \times r} = \mathcal{B} \in \mathbb{C}^{h \times k \times r}, \\ \\ \underbrace{\mathbf{X} \times C}_{pt_1/q \times bt_1/a \times r} = \mathcal{D} \in \mathbb{C}^{l \times d \times r}, \end{cases} \qquad \text{where } t = lcm\{n, p\} \text{ and } t_1 = lcm\{q, a\}. $$

Then, $\frac{mt}{n} = h$, $\frac{qt}{p} = k$, $\frac{pt_1}{q} = l$ and $\frac{bt_1}{a} = d$. Since $m = h$ with $l = p$, we obtain $t = n$, $t_1 = q$ and $\frac{t_1}{a} = \frac{q}{a} = \frac{d}{b}$, $\frac{t}{p} = \frac{n}{l} = \frac{k}{q}$. Thus, $p = l$, $q = \frac{ad}{b}$; and $\frac{d}{b}, \frac{n}{l}$ are positive integers. Moreover, $q = \frac{ad}{b}$, divides k and $\frac{n}{l}$ is also divisible by k, which completes the proof. □

Now, we will investigate the solvability of the tensor-matrix equation (**??**).

Theorem 4.2. *The tensor-matrix equation (**??**) has a solution $\mathbf{X} \in \mathbb{C}^{p\times q}$ if and only if*

(i). $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$ *with* $\hat{\mathcal{B}}_j$ *are linearly dependent in vector space* $\mathbb{C}^{m\times\frac{n}{l}\times r}$, $j = 1, 2, ..., q$, *where* $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$ *are* p *equal-size blocks of tensor* $\mathcal{A}$ *such that* $\mathcal{A} = \begin{bmatrix}\hat{\mathcal{A}}_1 & \hat{\mathcal{A}}_2 & \cdots & \hat{\mathcal{A}}_p\end{bmatrix}$, *and* $\hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2, \cdots, \hat{\mathcal{B}}_q$ *are* q *equal-size blocks of tensor* $\mathcal{B}$ *such that* $\mathcal{B} = \begin{bmatrix}\hat{\mathcal{B}}_1 & \hat{\mathcal{B}}_2 & \cdots & \hat{\mathcal{B}}_q\end{bmatrix}$.

(ii). $\mathcal{D}$ *is a tensor combination of* $\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_a$ *with respect to t-product; i.e.,*

$$\mathcal{D} = \sum_{i=1}^{a} \Lambda_i * \hat{C}_i, \ \textit{where}\ \Lambda_i \in \mathbb{C}^{p\times\frac{d}{b}\times r},\ \hat{C}_i = C_{i::} \otimes \mathcal{I}_{(\frac{d}{b})\times(\frac{d}{b})\times 1}.$$

Proof. By Definition **??**, Equation (**??**) can be rewritten as

$$\begin{aligned}\mathcal{A} \ltimes \mathbf{X} &= \mathcal{A} * (\mathbf{X} \otimes \mathcal{I}_{\alpha\alpha r}) = \begin{bmatrix}\hat{\mathcal{A}}_1 & \hat{\mathcal{A}}_2 & \cdots & \hat{\mathcal{A}}_p\end{bmatrix} * \begin{bmatrix}X_1 \otimes \mathcal{I}_{\alpha\alpha r} & X_2 \otimes \mathcal{I}_{\alpha\alpha r} \cdots X_q \otimes \mathcal{I}_{\alpha\alpha r}\end{bmatrix} \\ &= \left[\sum_{i=1}^{p} x_{i1}\hat{\mathcal{A}}_i \quad \sum_{i=1}^{p} x_{i2}\hat{\mathcal{A}}_i \ \cdots \ \sum_{i=1}^{p} x_{iq}\hat{\mathcal{A}}_i\right] = \begin{bmatrix}\hat{\mathcal{B}}_1 & \hat{\mathcal{B}}_2 & \cdots & \hat{\mathcal{B}}_q\end{bmatrix},\end{aligned} \tag{4.2}$$

$$\mathbf{X} \ltimes C = \sum_{i=1}^{a} \bar{\mathbf{X}}_i \ltimes \hat{C}_i = \sum_{i=1}^{a} \hat{\mathcal{X}}_i * \hat{C}_i = \mathcal{D}, \tag{4.3}$$

where $\alpha = \frac{n}{l}$, X_j is the j-th column of $\mathbf{X}$, $\bar{\mathbf{X}}_i = \begin{bmatrix}X_{\frac{(i-1)d}{b}+1} & X_{\frac{(i-1)d}{b}+2} \dots X_{\frac{(id)}{b}}\end{bmatrix} \in \mathbb{C}^{p\times\frac{d}{b}}$, $\hat{\mathcal{X}}_i = \bar{\mathbf{X}}_i \otimes \mathcal{I}_{(1)\times(1)\times r}$ and $C_{i::}$ is the i-th horizontal slice of tensor C, $i = 1, 2, \dots, a$. Thus, the solvability of Equation (**??**) implies that, for each j, we can write $\hat{\mathcal{B}}_j$ in terms of a linear combination of $\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p$, or equivalently, the set $\{\hat{\mathcal{A}}_1, \hat{\mathcal{A}}_2, \cdots, \hat{\mathcal{A}}_p, \hat{\mathcal{B}}_j\}$ is linearly dependent and $\mathcal{D}$ is a tensor combination of $\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_a$ with respect to t-product. The proof of the converse is straightforward. □

To derive the solutions for the tensor equation (**??**), we will reformulate Equations (**??**) and (**??**) into the following form:

Theorem 4.3. *The tensor-matrix equation (**??**) is equivalent to the following matrix-vector equation with conventional matrix product:*

$$\begin{cases} \left(\mathbf{I}_q \otimes \bar{\mathbf{A}}\right) V_c(\mathbf{X}) = V_c(\bar{\mathbf{B}}), \\ \left(\left[\hat{C}_1^{(1)}\right]^T \otimes \mathbf{I}_p \quad \left[\hat{C}_2^{(1)}\right]^T \otimes \mathbf{I}_p \quad \dots \quad \left[\hat{C}_a^{(1)}\right]^T \otimes \mathbf{I}_p\right) V_c(\mathbf{X}) = V_c(\mathcal{D}^{(1)}), \quad (\bigstar) \end{cases} \tag{4.4}$$

where

$$\bar{\mathbf{A}} = \left[V_c\left(\mathcal{A}_1^{(1)}\right) \quad \cdots \quad V_c\left(\mathcal{A}_p^{(1)}\right)\right], \quad \bar{\mathbf{B}} = \left[V_c\left(\mathcal{B}_1^{(1)}\right) \quad \cdots \quad V_c\left(\mathcal{B}_q^{(1)}\right)\right],$$

and $\mathcal{A}_i^{(1)}$, $\mathcal{B}_j^{(1)}$, $\hat{C}_t^{(1)}$ *and* $\mathcal{D}^{(1)}$ *are the* 1*-mode matricization of* $\hat{\mathcal{A}}_i$, $\hat{\mathcal{B}}_j$, $\hat{C}_t$ *and* $\mathcal{D}$ *respectively,* $i = 1, 2, ..., p$, $j = 1, 2, ..., q$, $t = 1, 2, ..., a$.

Proof. By considering each block tensor in the equation (**??**), we obtain

$$\sum_{i=1}^{p} x_{ij}\hat{\mathcal{A}}_i = \hat{\mathcal{B}}_j. \tag{4.5}$$

For each j, we apply the 1-mode matricization and vector operators to Equation (**??**), respectively. Thus, obtain the following equation

$$\begin{aligned} V_c\left(\hat{\mathcal{B}}_j^{(1)}\right) &= x_{1j}V_c\left(\mathcal{A}_1^{(1)}\right) + x_{2j}V_c\left(\mathcal{A}_2^{(1)}\right) + \cdots + x_{pj}V_c\left(\mathcal{A}_p^{(1)}\right) \\ &= \left[\begin{array}{ccc} V_c\left(\mathcal{A}_1^{(1)}\right) & \cdots & V_c\left(\mathcal{A}_p^{(1)}\right) \end{array}\right] \left[\begin{array}{c} x_{1j} \\ \vdots \\ x_{pj} \end{array}\right]. \end{aligned} \tag{4.6}$$

Let us denote

$$\begin{aligned} \bar{\mathbf{A}} &= \left[\begin{array}{ccc} V_c\left(\mathcal{A}_1^{(1)}\right) & \cdots & V_c\left(\mathcal{A}_p^{(1)}\right) \end{array}\right], \\ \bar{\mathbf{B}} &= \left[\begin{array}{ccc} V_c\left(\mathcal{B}_1^{(1)}\right) & \cdots & V_c\left(\mathcal{B}_q^{(1)}\right) \end{array}\right]. \end{aligned}$$

From (**??**), we can put all j equations to get

$$\bar{\mathbf{A}}\mathbf{X} = \mathbf{B}.$$

Now taking the vector operator and utilizing Lemma **??** yields

$$V_c\left(\bar{\mathbf{B}}\right) = V_c\left(\bar{\mathbf{A}}\mathbf{X}\right) = \left(\mathbf{I}_q \otimes \bar{\mathbf{A}}\right) V_c\left(\mathbf{X}\right).$$

Similarly, we can conclude (★). □

4.2. The general form $m = h$ with $l \neq p$

The following lemma gives a necessary condition for solvability of tensor Equation (**??**) with case $m = h$ with $l \neq p$ and figures out the dimension of the solutions.

Lemma 4.4. *If tensor-matrix equation (**??**) has a solution with size $p \times q$, then the dimensions of tensors $\mathcal{A}$, $\mathcal{B}$, C, $\mathcal{D}$ and the integers p, q should satisfy*

(i). $\frac{d}{b}$ *must be a positive integer;*

(ii). $p = \frac{l}{\beta} = \frac{n}{\alpha}, q = \frac{ad}{b\beta} = \frac{k}{\alpha}$, *where α and $\beta \neq 1$ represent the common divisor of n with k, and l with a, respectively. Moreover,* $\gcd\left\{\beta, \frac{d}{b}\right\} = 1$.

Proof. The results presented above can be established using an argument similar to that in Lemma **??**; therefore, we omit the proof. □

Remark 4.5. Assume that $\alpha_i, i = 1, \ldots, s$, is a common divisor of n and k,$\beta_j \neq 1, j = 1, \ldots, t$, is a common factor of l and $\frac{ad}{b}$. Thus, Equation (**??**) may have solutions with sizes $p_u \times q_u$, $p_u = \frac{n}{\alpha_u} = \frac{l}{\beta_u}$, and $q_u = \frac{k}{\alpha_u} = \frac{ad}{b\beta_u}$. We call them permissible sizes, and there are some properties of different permissible sizes.

(i). It is assumed that $\mathbf{X}^{p_1 \times q_1}$ and $\mathbf{X}^{p_2 \times q_2}$ are two different permissible sizes of solutions, and $\frac{q_2}{q_1} = \frac{p_2}{p_1} \in \mathbb{Z} > 1$. Hence, $\mathbf{X}^{p_2 \times q_2} = \mathbf{X}^{p_1 \times q_1} \otimes I_{\frac{q_2}{q_1}}$. Conversely, if Equation (**??**) has a unique solution $\mathbf{X}^{p_2 \times q_2}$, the solution $\mathbf{X}^{p_1 \times q_1}$, if it exists, must be unique.

(ii). Let $\tilde{\alpha} = \gcd\{n, k\}, \widetilde{\beta} = \gcd\{l, a\}$. If $\tilde{p} = \frac{n}{\tilde{\alpha}} = \frac{l}{\beta}, \tilde{q} = \frac{k}{\tilde{\alpha}} = \frac{ad}{b\tilde{\beta}}$, and if there is a solution with minimum size $\tilde{p} \times \tilde{q}$ for Equation (**??**), then it has a solution for every permissible size without $\beta = 1$.

In the following example, we illustrate the application of Lemma **??** and Remark **??** in determining the permissible sizes of solutions for equation (**??**). Using Lemma **??**, we first compute the possible (permissible) sizes of solutions. We then observe that a solution of the smaller size (1×3) exists, and according to Remark **??**, a solution of the larger size (2×6) can be obtained via the Kronecker product of the smaller solution with the identity matrix $(\mathbf{I}_2)$. This example confirms the relationship between solutions of different permissible sizes and demonstrates how solutions of larger dimensions can be constructed from a minimal-sized solution.

Example 4.6. Consider the tensors $\mathcal{A} \in \mathbb{R}^{3\times2\times2}$, $\mathcal{B} \in \mathbb{R}^{3\times6\times2}$, $\mathcal{C} \in \mathbb{R}^{4\times2\times2}$, and $\mathcal{D} \in \mathbb{R}^{4\times6\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{::1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad \mathcal{A}_{::2} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{B}_{::1} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -2 & 0 \end{bmatrix},$$

$$\mathcal{B}_{::2} = \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{C}_{::1} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{C}_{::2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\mathcal{D}_{::1} = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & -2 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{D}_{::2} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -2 & -1 & 0 \\ 0 & -1 & 0 & 0 & -2 & -1 \\ 1 & 0 & -1 & 0 & 0 & -2 \end{bmatrix}.$$

By Lemma **??**, the admissible sizes of solutions of the Equation (**??**) are 1×3 and 2×6. It is easy to verify that $\mathbf{X}_1 = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$, is a solution of equation (**??**). Therefore, $\mathbf{X}_2 = \mathbf{X}_1 \otimes \mathbf{I}_2$ is also a solution for Equation (**??**).

Now, we will examine the conditions for the solvability of the tensor Equation (**??**) assuming that all conditions of Lemma **??** hold true. First, we will discuss the minimum admissible size $p \times q$, noting that the solutions for other admissible sizes will follow a similar approach. A necessary condition for the solvability of Equation (**??**) is that the tensor $\mathcal{D}$ is composed of block Toeplitz tensors. For

$$\overbrace{\mathcal{A} \times \mathbf{X}}^{mt/n \times qt/p \times r} = \mathcal{B}, \tag{4.7}$$

we have

$$\mathcal{A} \ltimes \mathbf{X} = \begin{bmatrix} \hat{\mathcal{A}}_1 & \hat{\mathcal{A}}_2 & \cdots & \hat{\mathcal{A}}_p \end{bmatrix} * \begin{bmatrix} X_1 \otimes \mathcal{I}_{\alpha\alpha r} & X_2 \otimes \mathcal{I}_{\alpha\alpha r} \cdots X_q \otimes \mathcal{I}_{\alpha\alpha r} \end{bmatrix} = \begin{bmatrix} \hat{\mathcal{B}}_1 \, \hat{\mathcal{B}}_2 \, \cdots \, \hat{\mathcal{B}}_q \end{bmatrix}, \tag{4.8}$$

and for

$$\underbrace{\mathbf{X} \times \mathcal{C}}_{pt_1/q \times bt_1/a \times r} = \mathcal{D}, \tag{4.9}$$

if $\mathbf{X}^{p\times q}$ be a solution of this equation, where $p = \frac{l}{\beta}, q = \frac{ad}{b\beta}$. As $\frac{d}{b}, \beta$ are coprime. Let $\beta = l_3.\frac{d}{b} + l_4$. Then for $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, b$, we have that

$$\begin{aligned}
\mathrm{Row}_i(\mathbf{X}) \ltimes \mathcal{C}_{:\mathrm{j}:} &= \left(\mathrm{Row}_i(\mathbf{X}) \otimes I_\beta\right) * \left(\mathcal{C}_{:\mathrm{j}:} \otimes I_{\frac{d}{b}}\right) \\
&= x_{i1}\mathcal{C}^1 + x_{i2}\mathcal{C}^2 + \cdots + x_{i\frac{d}{b}}\mathcal{C}^{\frac{d}{b}} + x_{i,\frac{d}{b}+1}\mathcal{C}^{\frac{d}{b}+1} + \cdots + x_{iq}\mathcal{C}^q \\
&= Block_{ij}(\mathcal{D}) \in \mathbb{C}^{\beta \times \frac{d}{b} \times r}.
\end{aligned} \tag{4.10}$$

where the frontal slices of $\mathcal{C}^1, \mathcal{C}^2, \ldots \mathcal{C}^q$ are as follows:

$$C^1_{::k} = \begin{bmatrix} c_{1,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{1,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{1,j,k} \\ & & \vdots & & \\ c_{l_3,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{l_3,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{l_3,j,k} \\ c_{l_3+1,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{l_3+1,j,k} & 0 & \cdots \end{bmatrix}, \quad C^2_{::k} = \begin{bmatrix} \cdots & 0 & c_{l_3+1,j,k} & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & c_{l_3+1,j,k} \\ c_{l_3+2,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{l_3+2,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{l_3+2,j,k} \\ & & \vdots & & \\ c_{l_{\frac{2\beta}{d/b+1}},j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{l_{\frac{2\beta}{d/b+1}},j,k} & 0 & 0 \end{bmatrix},$$

$$C^{\frac{d}{b}}_{::k} = \begin{bmatrix} \cdots & 0 & c_{\beta-l_3,j,k} & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & c_{\beta-l_3,j,k} \\ c_{\beta-l_3+1,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{\beta-l_3+1,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{\beta-l_3+1,j,k} \\ & & \vdots & & \\ c_{\beta,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{\beta,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{\beta,j,k} \end{bmatrix},$$

$$C^{\frac{d}{b}+1}_{::k} = \begin{bmatrix} c_{\beta+1,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{\beta+1,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{\beta+1,j,k} \\ & & \vdots & & \\ c_{\beta+l_3,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{\beta+l_3,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{\beta+l_3,j,k} \\ c_{\beta+l_3+1,j} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{\beta+l_3+1,j,k} & 0 & \cdots \end{bmatrix},$$

$$C^{q}_{::k} = \begin{bmatrix} \cdots & 0 & c_{a-l_3,j,k} & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & c_{a-l_3,j,k} \\ c_{a-l_3+1,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{a-l_3+1,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{a-l_3+1,j,k} \\ & & \vdots & & \\ c_{a,j,k} & 0 & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & c_{a,j,k} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & c_{a,j,k} \end{bmatrix}.$$

It is easy to see that $Block_s(\mathcal{D}_{ij})$ is a Toeplitz tensor.

Next, we will explore the solvability conditions of Equation (**??**) using the process outlined above. Consequently, by taking $j.\beta = l_3^j.\frac{d}{b} + l_4^j$, we can easily establish the following equations:

$$x_{11}\begin{bmatrix} C_{1::} \\ C_{2::} \\ \vdots \\ C_{l_3^1+1::} \end{bmatrix} + x_{1,\frac{d}{b}+1}\begin{bmatrix} C_{\beta+1::} \\ C_{\beta+2::} \\ \vdots \\ C_{\beta+l_3^1+1::} \end{bmatrix} + \cdots + x_{1,q-\frac{d}{b}+1}\begin{bmatrix} C_{a-\beta+1::} \\ C_{a-\beta+2::} \\ \vdots \\ C_{a-\beta+l_3^1+1::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{1::} \\ \bar{\mathcal{D}}_{\frac{d}{b}+1::} \\ \vdots \\ \bar{\mathcal{D}}_{l_3^1\frac{d}{b}+1::} \end{bmatrix},$$

$$x_{12}\begin{bmatrix} C_{l_3^1+1::} \\ C_{l_3^1+2::} \\ \vdots \\ C_{l_3^2+1::} \end{bmatrix} + x_{1,\frac{d}{b}+1}\begin{bmatrix} C_{\beta+l_3^1+1::} \\ C_{\beta+l_3^1+2::} \\ \vdots \\ C_{\beta+l_3^2+1::} \end{bmatrix} + \cdots + x_{1,q-\frac{d}{b}+1}\begin{bmatrix} C_{a-\beta+l_3^1+1::} \\ C_{a-\beta+l_3^1+2::} \\ \vdots \\ C_{a-\beta+l_3^2+1::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{\beta+l_4^1::} \\ \bar{\mathcal{D}}_{\frac{d}{b}-l_4^1+1::} \\ \vdots \\ \bar{\mathcal{D}}_{\left(l_3^2-l_3^1\right)\frac{d}{b}-l_4^1+1::} \end{bmatrix},$$

$$\vdots$$

$$x_{1,\frac{d}{b}}\begin{bmatrix} C_{\beta-l_3^1::} \\ C_{\beta-l_3^1+2::} \\ \vdots \\ C_{\beta::} \end{bmatrix} + d_{1,\frac{2d}{b}}\begin{bmatrix} C_{2\beta-l_3^1::} \\ C_{2\beta-l_3^1+2::} \\ \vdots \\ C_{2\beta::} \end{bmatrix} + \cdots + x_{1,q}\begin{bmatrix} C_{a-l_3^1::} \\ C_{a-l_3^1+2::} \\ \vdots \\ C_{a::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{\beta+\frac{d}{b}-l_4^1::} \\ \bar{\mathcal{D}}_{l_4^1+1::} \\ \vdots \\ \bar{\mathcal{D}}_{\beta-\frac{d}{b}+1::} \end{bmatrix},$$

$$x_{21}\begin{bmatrix} C_{1::} \\ C_{2::} \\ \vdots \\ C_{l_3^1+1::} \end{bmatrix} + x_{2,\frac{d}{b}+1}\begin{bmatrix} C_{\beta+1::} \\ C_{\beta+2::} \\ \vdots \\ C_{\beta+l_3^1+1::} \end{bmatrix} + \cdots + x_{2,q-\frac{d}{b}+1}\begin{bmatrix} C_{a-\beta+1::} \\ C_{a-\beta+2::} \\ \vdots \\ C_{a-\beta+l_3^1+1::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{\beta+\frac{d}{b}::} \\ \bar{\mathcal{D}}_{\beta+2\frac{d}{b}::} \\ \vdots \\ \bar{\mathcal{D}}_{\beta+\left(l_3^1+1\right)\frac{d}{b}::} \end{bmatrix},$$

$$x_{22}\begin{bmatrix} C_{l_3^1+1::} \\ C_{l_3^1+2::} \\ \vdots \\ C_{l_3^2+1::} \end{bmatrix} + x_{2,\frac{d}{b}+2}\begin{bmatrix} C_{\beta+l_3^1+1::} \\ C_{\beta+l_3^1+2::} \\ \vdots \\ C_{\beta+l_3^2+1::} \end{bmatrix} + \cdots + x_{2,q-\frac{d}{b}+2}\begin{bmatrix} C_{a-\beta+l_3^1+1::} \\ C_{a-\beta+l_3^1+2::} \\ \vdots \\ C_{a-\beta+l_3^2+1::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{2\beta+l_4^1+\frac{d}{b}-1::} \\ \bar{\mathcal{D}}_{\beta+2\frac{d}{b}-l_4^1::} \\ \vdots \\ \bar{\mathcal{D}}_{\beta+\left(l_3^2-l_3^1+1\right)\frac{d}{b}-l_4^1::} \end{bmatrix},$$

$$\vdots$$

$$x_{2,\frac{d}{b}}\begin{bmatrix} C_{\beta-l_3^1::} \\ C_{\beta-l_3^1+2::} \\ \vdots \\ C_{\beta::} \end{bmatrix} + d_{2,\frac{2d}{b}}\begin{bmatrix} C_{2\beta-l_3^1::} \\ C_{2\beta-l_3^1+2::} \\ \vdots \\ C_{2\beta::} \end{bmatrix} + \cdots + x_{2,q}\begin{bmatrix} C_{a-l_3^1::} \\ C_{a-l_3^1+2::} \\ \vdots \\ C_{a::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{2\beta+2\frac{d}{b}-l_4^1-1::} \\ \bar{\mathcal{D}}_{2\beta+\frac{d}{b}+l_4^1::} \\ \vdots \\ \bar{\mathcal{D}}_{2\beta::} \end{bmatrix},$$

$$\vdots$$

$$x_{p,1}\begin{bmatrix} C_{1::} \\ C_{2::} \\ \vdots \\ C_{l_3^1+1::} \end{bmatrix} + \cdots + x_{p,\frac{d}{b}+1}\begin{bmatrix} C_{a-\beta+1::} \\ C_{a-\beta+2::} \\ \vdots \\ C_{a-\beta+l_3^1+1::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+1::} \\ \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+\frac{d}{b}+1::} \\ \vdots \\ \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+l_3^1\frac{d}{b}+1::} \end{bmatrix},$$

$$x_{p2}\begin{bmatrix} C_{l_3^1+1::} \\ C_{l_3^1+2::} \\ \vdots \\ C_{l_3^2+1::} \end{bmatrix} + \cdots + x_{p,q-\frac{d}{b}+2}\begin{bmatrix} C_{a-\beta+l_3^1+1::} \\ C_{a-\beta+l_3^1+2::} \\ \vdots \\ C_{a-\beta+l_3^2+1::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+\tilde{\beta}+l_4^1::} \\ \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+\frac{d}{b}-l_4^1+1::} \\ \vdots \\ \bar{\mathcal{D}}_{(\rho-1)\left(\beta+\frac{d}{b}-1\right)+\left(l_3^2-l_3^1\right)\frac{d}{b}-l_{41}+1::} \end{bmatrix},$$

$$\vdots$$

$$x_{p,\frac{d}{b}}\begin{bmatrix} C_{\beta-l_3^1::} \\ C_{\beta-l_3^1+2::} \\ \vdots \\ C_{\beta::} \end{bmatrix} + \cdots + x_{p,q}\begin{bmatrix} C_{a-l_3^1::} \\ C_{a-l_3^1+2::} \\ \vdots \\ C_{a::} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+\beta+\frac{d}{b}-l_4^1::} \\ \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+l_4^1+1::} \\ \vdots \\ \bar{\mathcal{D}}_{(p-1)\left(\beta+\frac{d}{b}-1\right)+\beta-\frac{d}{b}+1::} \end{bmatrix},$$

where $\bar{\mathcal{D}}$ is a tensor with a frontal slices as follow:

$$\bar{\mathcal{D}}_{::k} = \begin{bmatrix} d_{1,1,k} & d_{1,\frac{d}{b}+1,k} & \cdots & d_{1,d-\frac{d}{b}+1,k} \\ d_{2,1,k} & d_{2,\frac{d}{b}+1,k} & \cdots & d_{2,d-\frac{d}{b}+1,k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{\beta,1,k} & d_{\beta,\frac{d}{b}+1,k} & \cdots & d_{\beta,d-\frac{d}{b}+1,k} \\ d_{12k} & d_{1,\frac{d}{b}+2,k} & \cdots & d_{1,d-\frac{d}{b}+2,k} \\ d_{13k} & d_{1,\frac{d}{b}+3,k} & \cdots & d_{1,d-\frac{d}{b}+3,k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,\frac{d}{b},k} & d_{1,2\frac{d}{b},k} & \cdots & d_{1,d,k} \\ \vdots & \vdots & \vdots & \vdots \\ d_{l-\beta+1,1,k} & d_{l-\beta+1,\frac{d}{b}+1,k} & \cdots & d_{l-\beta+1,d-\frac{d}{b}+1,k} \\ d_{l-\beta+2,1,k} & d_{l-\beta+2,\frac{d}{b}+1,k} & \cdots & d_{l-\beta+2,d-\frac{d}{b}+1,k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{l,1} & d_{l,\frac{d}{b}+1} & \cdots & d_{l,d-\frac{d}{b}+1} \\ d_{l-\beta+1,2,k} & d_{l-\beta+1,\frac{d}{b}+2,k} & \cdots & d_{l-\beta+1,d-\frac{d}{b}+2,k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{l-\beta+1,\frac{d}{b},k} & d_{l-\beta+1,2\frac{d}{b},k} & \cdots & d_{l-\beta+1,d,k} \end{bmatrix},$$

and $C_{i::}$, is the i-th horizontal slice of tensor C and $\bar{\mathcal{D}}_{s::}$, is the s-th horizontal slice of tensor $\bar{\mathcal{D}}$.

Theorem 4.7. *The solvability of the tensor-matrix Equation (**??**) is equivalent to the following tensor-vector equations:*

$$\begin{cases} Y_{i,1} \ltimes \left[\tilde{C}_1\tilde{C}_{d/b+1}\cdots\tilde{C}_{(a/\widetilde{\beta}-1)d/b+1}\right]^T = \tilde{\mathcal{D}}_{i,1}, \\ Y_{i,2} \ltimes \left[\tilde{C}_2\tilde{C}_{d/b+2}\ldots\tilde{C}_{(a/\widetilde{\beta}-1)d/b+2}\right]^T = \tilde{\mathcal{D}}_{i,2}, \\ \ldots \\ Y_{i,d/b} \ltimes \left[\tilde{C}_{d/b}\tilde{C}_{2d/b}\ldots\tilde{C}_q\right]^T = \tilde{\mathcal{D}}_{i,d/b}, \\ \left(\mathbf{I}_q \otimes \bar{\mathbf{A}}\right) V_c(\mathbf{X}) = V_c(\bar{\mathbf{B}}), \end{cases} \tag{4.11}$$

where

$$\begin{cases} C = [\overbrace{C_{1::} \dots C_{l_3^1::} C_{l_3^1+1::}}^{\tilde{C}_1} \dots C_{l_3^2::} C_{l_3^2+1::} \dots \overbrace{C_{a-l_3^1::} C_{a-l_3^1+1::} \dots C_{a::}}^{\tilde{C}_q}]^T, \\ \tilde{\mathcal{D}}_{i,1} = \left[\bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+1,:,:} \bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+\frac{d}{b}+1,:,:} \cdots \bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+l_3^1\frac{d}{b}+1,:,:}\right]^T, \\ \tilde{\mathcal{D}}_{i,2} = \left[\bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+\beta+l_4^1,:,:} \bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+\frac{d}{b}-l_4^1+1,:,:} \cdots \bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+\left(l_3^2-l_3^1\right)\frac{d}{b}-l_{41}+1,:,:}\right]^T, \\ \vdots \\ \tilde{\mathcal{D}}_{i,d/b} = \left[\bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+\beta+\frac{d}{b}-l_4^1,:,:} \bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+l_4^1+1,:,:} \cdots \bar{\mathcal{D}}_{(i-1)\left(\beta+\frac{d}{b}-1\right)+\beta-\frac{d}{b}+1,:,:}\right]^T, \\ \bar{\mathbf{A}} = \left[V_c\left(\mathcal{A}_1^{(1)}\right) \quad \cdots \quad V_c\left(\mathcal{A}_p^{(1)}\right)\right], \quad \bar{\mathbf{B}} = \left[V_c\left(\mathcal{B}_1^{(1)}\right) \quad \cdots \quad V_c\left(\mathcal{B}_q^{(1)}\right)\right], \end{cases}$$

$\mathcal{A}_i^{(1)}$*, and* $\mathcal{B}_j^{(1)}$*are the* 1*-mode matricization of* $\hat{\mathfrak{A}}_i$ *and* $\hat{\mathfrak{B}}_j$ *respectively,* $i = 1, 2, ..., p$, $j = 1, 2, ..., q$. $\bar{\mathfrak{D}}_{i::}$ *and* $C_{i::}$ *are the horizontal slices of* $\bar{\mathfrak{D}}$ *and* C *respectively. Vector* Y_{ij} *could be given by* $Y_{ij} = \begin{bmatrix} y_{ij1} & y_{ij2} & \dots & y_{ij(a/\beta)} \end{bmatrix}$, $i = 1 \cdots p$, $j = 1 \dots d/b$, *then the matrix* $\mathbf{X}$ *is as follows:*

$$\mathbf{X} = \begin{bmatrix} y_{111} & y_{121} & \cdots & y_{1(\frac{d}{b})1} & y_{112} & y_{122} \cdots & y_{1(\frac{d}{b})2} & \cdots & y_{11(\frac{a}{\beta})} & y_{12(\frac{a}{\beta})} \cdots & y_{1(\frac{a}{\beta})(\frac{a}{\beta})} \\ y_{211} & y_{221} & \cdots & y_{2(\frac{d}{b})1} & y_{212} & y_{222} \cdots & y_{2(\frac{d}{b})2} & \cdots & y_{21(\frac{a}{\beta})} & y_{22(\frac{a}{\beta})} \cdots & y_{2(\frac{a}{\beta})(\frac{a}{\beta})} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & & \\ y_{p11} & y_{p21} & \cdots & y_{p(\frac{d}{b})1} & y_{p12} & y_{p22} \cdots & y_{p(\frac{d}{b})2} & \cdots & y_{p1(\frac{a}{\beta})} & y_{p2(\frac{a}{\beta})} \cdots & y_{p(\frac{a}{\beta})(\frac{a}{\beta})} \end{bmatrix}.$$

The following example illustrates the application of Theorem **??** to solve the tensor-matrix equation (**??**) for the permissible size 1×3. By constructing the required blocks $\bar{\mathcal{D}}$, $\tilde{C}_j$ and the matrices $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$, we solve the equivalent tensor-vector equations to obtain the solution vector Y and consequently the matrix solution. The solution for the larger permissible size is then obtained via the Kronecker product confirming Remark **??**.

Example 4.8. Consider the tensors $\mathcal{A} \in \mathbb{R}^{3\times2\times2}$, $\mathcal{B} \in \mathbb{R}^{3\times6\times2}$, $C \in \mathbb{R}^{4\times2\times2}$, and $\mathcal{D} \in \mathbb{R}^{4\times6\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{::1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad \mathcal{A}_{::2} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{B}_{::1} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & -2 & 0 \end{bmatrix},$$

$$\mathcal{B}_{::2} = \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 1 & 2 & 0 & 0 & -1 & -2 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad C_{::1} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 2 \\ -1 & 0 \end{bmatrix}, \quad C_{::2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\mathcal{D}_{::1} = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & -2 \\ 1 & 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 3 \\ 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{D}_{::2} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -2 & -1 & 0 \\ 0 & -1 & 0 & 0 & -2 & -1 \\ 1 & 0 & -1 & 0 & 0 & -2 \end{bmatrix}.$$

By Lemma **??**, the admissible sizes of solutions of the Equation (**??**) are 1×3 and 2×6. First, for admissible size 1×3, we have $\bar{\mathcal{D}}$, $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ as follows;

$$\bar{\mathcal{D}}_{::1} = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 0 & 0 \\ 2 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad \bar{\mathcal{D}}_{::2} = \begin{bmatrix} 0 & -1 \\ -1 & -2 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad (\tilde{C}_1)_{::1} = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, \quad (\tilde{C}_1)_{::2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$(\tilde{C}_2)_{::1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad (\tilde{C}_2)_{::2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\tilde{C}_3)_{::1} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, \quad (\tilde{C}_3)_{::2} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

$$\bar{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & -1 \end{bmatrix}^T,$$

$$\bar{\mathbf{B}} = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & 0 & 2 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 1 & -1 & 0 & -2 & -1 & 0 & 0 & -2 & 1 \end{bmatrix}^T.$$

Hence, $Y_{11} = 1, Y_{12} = 0, Y_{13} = -1$, and then, $\mathbf{X_1} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$ is a solution for Equation(**??**), and the permissible size 2×6 is similar to the case 1×3, which we can show that $\mathbf{X_2} = \mathbf{X_1} \otimes \mathbf{I}_2$ is also a solution for Equation (**??**).

In the next example, we consider a tensor-matrix equation for which Lemma **??** yields only one permissible size, namely 2×6. Applying Theorem **??** we compute the necessary blocks and solve the corresponding tensor-vector equations. The resulting solution $\mathbf{X}$ is shown to be unique, demonstrating the case where no smaller-sized solution exists and the solution is uniquely determined.

Example 4.9. Consider the tensors $\mathcal{A} \in \mathbb{R}^{3\times2\times2}$, $\mathcal{B} \in \mathbb{R}^{3\times6\times2}$, $C \in \mathbb{R}^{9\times4\times2}$, and $\mathcal{D} \in \mathbb{R}^{6\times8\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{::1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{A}_{::2} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & -1 \end{bmatrix}, \quad \mathcal{B}_{::1} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 3 \\ 2 & 1 & 3 & 2 & -2 & 7 \\ 0 & -1 & 1 & -2 & 0 & -1 \end{bmatrix},$$

$$\mathcal{B}_{::2} = \begin{bmatrix} 1 & 2 & 0 & 4 & -1 & 5 \\ 0 & 3 & -3 & 6 & 0 & 3 \\ 2 & -1 & 5 & -2 & -2 & 5 \end{bmatrix}, \quad C_{::1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad C_{::2} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ -1 & 1 & 1 & 0 \end{bmatrix},$$

$$\mathcal{D}_{::1} = \begin{bmatrix} 0 & 0 & 4 & 3 & -1 & 0 & 1 & -3 \\ -3 & 0 & 0 & 4 & 3 & -1 & 0 & 1 \\ 0 & -3 & 0 & 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & -2 & 2 & 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 1 & 0 \end{bmatrix}, \quad \mathcal{D}_{::2} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 6 & 0 & -3 \\ -3 & -1 & 3 & 1 & 3 & 0 & 0 & 0 \\ 0 & -3 & 2 & 3 & -1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 3 & 0 & -2 \\ -1 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 & 1 & 0 \end{bmatrix}.$$

By Lemma (**??**), the only admissible size of solution is 2×6, and

$$\bar{\mathcal{D}}_{::1} = \begin{bmatrix} 0 & 4 & -1 & 1 \\ -3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -3 \\ 1 & -2 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & -2 \end{bmatrix}, \quad \bar{\mathcal{D}}_{::2} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 3 & 3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 6 & -3 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 2 & 3 & -2 \end{bmatrix}, \quad (\tilde{C}_1)_{::1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$(\tilde{C}_1)_{::2} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad (\tilde{C}_2)_{::1} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad (\tilde{C}_2)_{::2} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix},$$

$$(\tilde{C}_3)_{::1} = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (\tilde{C}_3)_{::2} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad (\tilde{C}_4)_{::1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$(\tilde{C}_4)_{::2} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (\tilde{C}_5)_{::1} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad (\tilde{C}_5)_{::2} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

$$(\tilde{C}_6)_{::1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad (\tilde{C}_6)_{::2} = \begin{bmatrix} 0 & 0 & 2 & -1 \\ -1 & 1 & 1 & 0 \end{bmatrix}, \quad (\tilde{\mathcal{D}}_{1,1})_{::1} = \begin{bmatrix} 0 & 4 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$(\tilde{\mathcal{D}}_{1,1})_{::2} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}, \quad (\tilde{\mathcal{D}}_{1,2})_{::1} = \begin{bmatrix} 0 & 3 & 0 & -3 \\ -3 & 0 & 3 & 0 \end{bmatrix}, \quad (\tilde{\mathcal{D}}_{1,2})_{::2} = \begin{bmatrix} 0 & 0 & 6 & -3 \\ -3 & 3 & 3 & 0 \end{bmatrix}.$$

$$(\tilde{\mathcal{D}}_{2,1})_{::1} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (\tilde{\mathcal{D}}_{2,1})_{::2} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad (\tilde{\mathcal{D}}_{2,2})_{::1} = \begin{bmatrix} 0 & 2 & 0 & -2 \\ 2 & 0 & 2 & 0 \end{bmatrix},$$

$$(\tilde{\mathcal{D}}_{2,2})_{::2} = \begin{bmatrix} 0 & 2 & 3 & -2 \\ -1 & 0 & 2 & 0 \end{bmatrix}.$$

It follows that the permissible solution size 2×6, is

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 3 \\ 0 & 1 & -1 & 2 & 0 & 1 \end{bmatrix},$$

and the uniqueness of $\mathbf{X}$ is readily apparent.

Now we give a necessary condition for the solvability of the Equation (**??**).

Theorem 4.10. *If Equation (**??**) has a solution $\mathbf{X}^{p\times q}$, tensors $\mathcal{B}$ and $\mathcal{D}$ are composed of block tensors, where each block of $\mathcal{B}$ is a common block and the block for tensor $\mathcal{D}$ is a Toeplitz tensor. Tensors $\mathcal{B}$ and $\mathcal{D}$ are limited to the following form:*

$$\mathcal{B} = \begin{bmatrix} \mathcal{B}_{1::} \\ \vdots \\ \mathcal{B}_{h::} \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} Block_{\,11}(\mathcal{D}) & \cdots & Block_{\,1b}(\mathcal{D}) \\ \vdots & \ddots & \vdots \\ \mathrm{Block}_{p1}(\mathcal{D}) & \cdots & Block_{\,mb}(\mathcal{D}) \end{bmatrix},$$

*where $\mathcal{B}_{t::}$, is the t-th horizontal slice of $\mathcal{D}, t = 1, \ldots, h$; and $\mathrm{Block}_{ij}(\mathcal{D}) \in C^{\beta\times d/b\times r}$ is a Toeplitz tensor, $i = 1, \ldots, p, j = 1, \ldots, b$. Moreover, the solvability of Equation (**??**) is equivalent to the one of*

$$\begin{cases} \mathcal{A}_{1::} \ltimes \mathbf{X} = \mathcal{B}_{1::}, \\ \vdots \\ \mathcal{A}_{h::} \ltimes \mathbf{X} = \mathcal{B}_{h::}, \\ \mathbf{X} \ltimes C_{:1:} = \begin{bmatrix} \mathrm{Block}(\mathcal{D})_{11} \\ \mathrm{Block}(\mathcal{D})_{21} \\ \vdots \\ \mathrm{Block}(\mathcal{D})_{p1} \end{bmatrix}, \\ \vdots \\ \mathbf{X} \ltimes C_{:b:} = \begin{bmatrix} \mathrm{Block}(\mathcal{D})_{1b} \\ \mathrm{Block}(\mathcal{D})_{2b} \\ \vdots \\ \mathrm{Block}(\mathcal{D})_{pb} \end{bmatrix}, \end{cases}$$

where $\mathcal{A}_{t::}$ and $\mathcal{B}_{t::}$, $t = 1, \ldots, h$, are the t-th horizontal slices of $\mathcal{A}$ and $\mathcal{B}$, respectivly. And $C_{:k:}$ is the k-th lateral slice of C; $\mathrm{Block}(\mathcal{D})_{ij} \in \mathbb{C}^{\beta\times\frac{d}{b}\times r}$ is a Toeplitz tensor; $i = 1, \ldots, p$; $j = 1, \ldots, q$.

Proof. The proof is analogous to that of Theorem **??**; thus, we omit it. □

4.3. The general form $m \neq h$ with $l = p$

In this subsection, we delve into the solvability of the tensor Equation (**??**), where the tensors $\mathcal{A}, \mathcal{B}, C$ and $\mathcal{D}$ are invariant. Building on the framework established in Lemma **??**, we will outline the necessary conditions for the existence of a solution to Equation (**??**) and specify the dimensions of such solutions.

Lemma 4.11. *The necessary conditions for the existence of a solution $\mathbf{X} \in \mathbb{C}^{p\times q}$ of Equation (**??**) are as follows:*

(i). Both $\frac{h}{m}$ and $\frac{d}{b}$ must be positive integers;

(ii). The dimensions of the solution matrix are given by $p = l = \frac{n}{\alpha}\frac{h}{m}, q = \frac{ad}{b} = \frac{k}{\alpha}$*, where* α *is a common factor of n and k. Moreover,* $\gcd\left\{\alpha, \frac{h}{m}\right\} = 1$*, under these conditions, the solution is unique.*

Proof. The proof follows a similar approach to that of Lemma **??**; therefore, we omit it for brevity. □

We continue our investigation into the solvability of Equation (**??**). To solve this equation, we first need to determine the permissible dimensions $p \times q$. Therefore, we proceed by calculating these dimensions;

$$\mathcal{A} \ltimes \mathbf{X} = \mathcal{A} \ltimes \left[X_1, \ldots, X_q\right] = \left[\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_q\right],$$

where X_j is the j-th column of $\mathbf{X}$, $\mathcal{B}_j \in \mathbb{C}^{h\times\alpha\times r}$ is a block of $\mathcal{B}$, $j = 1, \ldots, q$. Therefore, it is equivalent to solving q tensor-vector equations under STP, and by reiterating subsection **??**, we obtain

$$\mathcal{A} \ltimes X_j = \mathcal{B}_j = \begin{bmatrix} \text{Block}(\mathcal{B})_{1j} \\ \text{Block}(\mathcal{B})_{2j} \\ \vdots \\ \text{Block}(\mathcal{B})_{mj} \end{bmatrix}, j = 1, \ldots, q,$$

where Block $(\mathcal{B})_{mj}$ is a Toeplitz tensor. Similarly, Equation (**??**) could be further described as:

$$\mathbf{X} \ltimes C = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} \ltimes C = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} \left(C \otimes I_{\frac{d}{b}}\right) = \begin{bmatrix} \mathcal{D}_{1::} \\ \vdots \\ \mathcal{D}_{p::} \end{bmatrix},$$

which is equivalent to solving p matrix-vector equations under the STP, where Y_i is the i-th row of $\mathbf{X}$ and $\mathcal{D}_{i::}, i = 1, \ldots, p$ is the i-th horizontal slice of $\mathcal{D}$.

$$Y_i \ltimes C = \mathcal{D}_i, \quad i = 1, \ldots, p. \tag{4.12}$$

Thus, we have:

Theorem 4.12. *If Equation (**??**) has a solution* $\mathbf{X}^{p\times q}$*, tensors* $\mathcal{B}$ *and* $\mathcal{D}$ *are composed of block tensors, where each block of* $\mathcal{B}$ *is a Toeplitz tensor and the block for tensor* $\mathcal{D}$ *is a common block. Tensors* $\mathcal{B}$ *and* $\mathcal{D}$ *are limited to the following form:*

$$\mathcal{B} = \begin{bmatrix} Block_{\ 11}(\mathcal{B}) & \cdots & Block_{\ 1q}(\mathcal{B}) \\ \vdots & \ddots & \vdots \\ \text{Block}_{m1}(\mathcal{B}) & \cdots & Block_{\ mq}(\mathcal{B}) \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \mathcal{D}_{1::} \\ \vdots \\ \mathcal{D}_{p::} \end{bmatrix}$$

where $\text{Block}_{sj}(\mathcal{B}) \in C^{h/m\times k/q\times r}$ *is a Toeplitz tensor,* $s = 1, \ldots, m, j = 1, \ldots, q$*, and* $\mathcal{D}_{i::}$*, is the i-th horizontal slice of* $\mathcal{D}, i = 1, \ldots, p$*. Moreover, the solvability of Equation (**??**) is equivalent to the* *one of*

$$\begin{cases} \mathcal{A} \ltimes X_1 = \mathcal{B}_1 = \begin{bmatrix} \mathrm{Block}(\mathcal{B})_{11} \\ \mathrm{Block}(\mathcal{B})_{21} \\ \vdots \\ \mathrm{Block}(\mathcal{B})_{m1} \end{bmatrix} \in \mathbb{C}^{h\times\alpha\times r}, \\ \vdots \\ \mathcal{A} \ltimes X_q = \mathcal{B}_q = \begin{bmatrix} \mathrm{Block}(\mathcal{B})_{1q} \\ \mathrm{Block}(\mathcal{B})_{2q} \\ \vdots \\ \mathrm{Block}(\mathcal{B})_{mq} \end{bmatrix} \in \mathbb{C}^{h\times\alpha\times r}, \\ Y_1 \ltimes C = \mathcal{D}_{1::} \in \mathbb{C}^{1\times d\times r}, \\ \vdots \\ Y_p \ltimes C = \mathcal{D}_{p::} \in \mathbb{C}^{1\times d\times r}, \end{cases}$$

where X_j is the j-th column of $\mathbf{X}$, $\mathrm{Block}(\mathcal{B})_{sj} \in \mathbb{C}^{\frac{h}{m}\times\alpha\times r}$ *is a Toeplitz matrix; Y_i is the i-th row of* $\mathbf{X}$ *and $\mathcal{D}_{i::}$, is the i-th horizontal slice of $\mathcal{D}$, $s = 1,\ldots,m$; $j = 1,\ldots,q$; $i = 1,\ldots,p$.*

Proof. The proof is analogous to that of Theorem **??**; thus, we omit it. □

The following example verifies the correctness of Theorem **??** for the tensor-matrix equation endowed with a special tensor structure.

Example 4.13. Consider the tensors $\mathcal{A} \in \mathbb{R}^{2\times4\times2}$, $\mathcal{B} \in \mathbb{R}^{6\times8\times2}$, $C \in \mathbb{R}^{2\times3\times2}$, and $\mathcal{D} \in \mathbb{R}^{3\times3\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{1::} = \begin{bmatrix} 2 & 0 & 1 & 2 \\ -1 & 1 & 3 & 0 \end{bmatrix}, \quad \mathcal{A}_{2::} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix},$$

$$\mathcal{B}_{1::} = \begin{bmatrix} 2 & 2 & 0 & 0 & -2 & 4 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & -2 & 4 & 1 \\ 1 & 0 & 2 & 2 & 2 & 0 & -2 & 4 \\ -1 & 0 & 0 & 1 & 1 & 0 & 3 & -1 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 3 \\ 3 & 0 & -1 & 0 & 6 & 1 & 1 & 0 \end{bmatrix},$$

$$\mathcal{B}_{2::} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & -2 & 2 \\ 2 & 0 & 1 & -1 & 4 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 \end{bmatrix}, \quad C_{::1} = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix},$$

$$C_{::2} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}, \mathcal{D}_{::1} = \begin{bmatrix} 3 & -2 & 2 \\ -1 & 2 & -1 \\ 3 & 2 & 1 \end{bmatrix}, \quad \mathcal{D}_{::2} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 2 & 3 & 7 \end{bmatrix}.$$

$\frac{h}{m} = 3, \frac{d}{b} = 1$, and $\alpha = 4$. Thus, $p = 3 = \frac{4}{4} \cdot 3, q = \frac{8}{4} = 2 \cdot \frac{3}{3}$, and

$$X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

is a unique solution.

4.4. The general form $m \neq h$ with $l \neq p$

The solvability of the tensor equation (**??**) is investigated under the conditions $m \neq h$ and $p \neq l$, where the tensors $\mathcal{A} \in \mathbb{C}^{m\times n\times r}, \mathcal{B} \in \mathbb{C}^{h\times k\times r}, \mathcal{C} \in \mathbb{C}^{a\times b\times r}$ and $\mathcal{D} \in \mathbb{C}^{l\times d\times r}$ are given. A necessary condition for solvability is derived based on Definition **??**.

Lemma 4.14. *The necessary condition for the existence of a solution to tensor Equation (**??**) is that the ratios $\frac{h}{m}$ and $\frac{d}{b}$ are positive integers. Furthermore, the parameters p and q are given by*

$$p = \frac{l}{\beta} = \frac{n}{\alpha} \cdot \frac{h}{m}, \quad \textit{and} \quad q = \frac{k}{\alpha} = \frac{a}{\beta} \cdot \frac{d}{b},$$

where α and $\beta \neq 1$ denote the greatest common divisors of n with k, and a with l, respectively. In addition, the following coprimality conditions must be satisfied:

$$\gcd\left\{\alpha, \frac{h}{m}\right\} = 1, \quad \textit{and} \quad \gcd\left\{\beta, \frac{d}{b}\right\} = 1.$$

Proof. By Definition **??**,

$$\frac{mt}{n} = h, \frac{qt}{p} = k, \frac{ps}{q} = l, \frac{bs}{a} = d \quad \text{where } t = lcm\{n, p\}, s = lcm\{q, a\}$$

Then, $\frac{t}{n} = \frac{h}{m}, \frac{s}{a} = \frac{d}{b}$ are positive integers, and $\frac{t}{p} = \frac{nh/m}{q} = \alpha, \frac{s}{q} = \frac{l}{p} = \beta$, thus, $p = \frac{l}{\beta} = \frac{n}{\alpha} \cdot \frac{h}{m}, q = \frac{k}{\alpha} = \frac{a}{\beta} \cdot \frac{d}{b}$. Moreover, if $\beta = 1$, then $l = p$, which does not satisfy the condition. Furthermore, we have

$$\frac{t}{p} = \alpha, \frac{t}{n} = \frac{h}{m}; \quad \frac{s}{q} = \beta, \frac{s}{a} = \frac{d}{b}.$$

Consequently, $\gcd\left\{\alpha, \frac{h}{m}\right\} = 1$ and $\gcd\left\{\beta, \frac{d}{b}\right\} = 1$, which completes the proof. □

Remark 4.15. Assume that the solution size $p \times q$ satisfies the condition specified in Lemma **??**, thereby qualifying as an admissible dimension. In analogy with Remark **??**, suppose that the ratios $\frac{q_2}{q_1} = \frac{p_2}{p_1} > 1$ hold, and let $\mathbf{X} \in \mathbb{C}^{p_1\times q_1}$ be a solution to the tensor equation (**??**). Then, the matrix $\mathbf{X}_1 = \mathbf{X} \otimes I_{\frac{q_2}{q_1}}$ also constitutes a solution, where $\mathbf{X}_1 \in \mathbb{C}^{p_2\times q_2}$.

Furthermore, if the solution $\mathbf{X}_1$ with the admissible dimension $p_2 \times q_2$ is unique, it follows that the solution $\mathbf{X}$ with dimension $p_1 \times q_1$ must also be unique.

Summarizing the above discussion, we can derive a necessary condition for the solvability of the tensor equation (**??**). For the subsequent analysis, we assume that m divides h and b divides d.

Theorem 4.16. *Suppose the tensor equation (**??**) admits a solution* $\mathbf{X} \in \mathbb{C}^{p\times q}$*. Then, it is necessary that the tensors* $\mathcal{B}$ *and* $\mathcal{D}$ *possess a block-partitioned structure, where each block corresponds to a Toeplitz tensor. More precisely, the tensors* $\mathcal{B}$ *and* $\mathcal{D}$ *must be decomposable into partitioned tensors of the following forms:*

$$\mathcal{B} = \begin{bmatrix} \mathrm{Block}_{11}(\mathcal{B}) & \cdots & \mathrm{Block}_{1q}(\mathcal{B}) \\ \vdots & \cdots & \vdots \\ \mathrm{Block}_{m1}(\mathcal{B}) & \cdots & \mathrm{Block}_{mq}(\mathcal{B}) \end{bmatrix}, \mathcal{D} = \begin{bmatrix} \mathrm{Block}_{11}(\mathcal{D}) & \cdots & \mathrm{Block}_{1b}(\mathcal{D}) \\ \vdots & \cdots & \vdots \\ \mathrm{Block}_{p1}(\mathcal{D}) & \cdots & \mathrm{Block}_{pb}(\mathcal{D}) \end{bmatrix},$$

where $\mathrm{Block}_{sj}(\mathcal{B}) \in \mathbb{C}^{\frac{h}{m}\times\frac{k}{q}\times r}$ *and* $\mathrm{Block}_{it}(\mathcal{D}) \in C^{\beta\times\frac{d}{b}\times r}$ *are Toeplitz tensors, with* $s = 1,\ldots,m$*,* $j = 1,\ldots,q; i = 1,\ldots,p, t = 1,\ldots,b$*.*

Proof. Consider the tensor equation (**??**) expressed as

$$\mathcal{A} \ltimes \mathbf{X} = \mathcal{A} \ltimes [X_1,\ldots,X_q] = [\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_q],$$

where X_j denotes the j-th column of $\mathbf{X}$, and $\mathcal{B}_j \in C^{h\times\alpha\times r}$ corresponds to the j-th block of $\mathcal{B}$, for $j = 1,\ldots,q$. This formulation is equivalent to solving q tensor-vector equations under the STP framework. By applying the approach outlined in Section **??** to each of these tensor-vector equations, we obtain the following result:

$$\mathcal{A} \ltimes X_j = \mathcal{B}_j = \begin{bmatrix} \mathrm{Block}(\mathcal{B})_{1j} \\ \mathrm{Block}(\mathcal{B})_{2j} \\ \vdots \\ \mathrm{Block}(\mathcal{B})_{mj} \end{bmatrix}, j = 1,\ldots,q,$$

where $\mathrm{Block}(\mathcal{B})_{1j},\ldots,\mathrm{Block}(\mathcal{B})_{mj}$ are Toeplitz tensors, and Equation (**??**) can be rewritten as:

$$\mathbf{X} \ltimes C = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix} \ltimes C = \mathcal{D} = \begin{bmatrix} \mathcal{D}_1 \\ \vdots \\ \mathcal{D}_p \end{bmatrix} \quad i = 1,\ldots,p,$$

where Y_i is the i-th row of $\mathbf{X}$ and $\mathcal{D}_i \in \mathbb{C}^{\beta\times d\times r}$ is a block of $\mathcal{D}$. Then, repeating Equation (**??**) we have

$$Y_i \ltimes C = \left(Y_i \otimes I_\beta\right)\left(C \otimes I_{\frac{d}{b}}\right) = C_i = [\mathrm{Block}_{i1}(\mathcal{D}),\ldots,\mathrm{Block}_{ib}(\mathcal{D})]$$

where $\mathrm{Block}_{it}(\mathcal{D})$ is a Toeplitz tensor, $i = 1,\ldots, p; t = 1,\ldots,b$. The proof is completed. □

Theorem 4.17. *The solvability of the tensor Equation (**??**) is equivalent to the one of*

$$\begin{cases} \mathcal{A} \ltimes X_1 = \mathcal{B}_1 = \begin{bmatrix} \text{Block}(\mathcal{B})_{11} \\ \text{Block}(\mathcal{B})_{21} \\ \vdots \\ \text{Block}(\mathcal{B})_{m1} \end{bmatrix} \in \mathbb{C}^{h\times\alpha\times r}, \\ \vdots \\ \mathcal{A} \ltimes X_q = \mathcal{B}_q = \begin{bmatrix} \text{Block}(\mathcal{B})_{1q} \\ \text{Block}(\mathcal{B})_{2q} \\ \vdots \\ \text{Block}(\mathcal{B})_{mq}{}^{A} \end{bmatrix} \in \mathbb{C}^{h\times\alpha\times r}, \\ Y_1 \ltimes C = \mathcal{D}_1 = [\text{Block}_{11}(\mathcal{D}), \dots, \text{Block}_{1b}(\mathcal{D})] \in \mathbb{C}^{\beta\times d\times r}, \\ \vdots \\ Y_p \ltimes C = \mathcal{D}_p = \left[\text{Block}_{p1}(\mathcal{D}), \dots, \text{Block}_{pb}(\mathcal{D})\right] \in C^{\beta\times d\times r}, \end{cases}$$

where X_j is the j-th column of $\mathbf{X}$, *$Block(\mathcal{B})_{rj} \in \mathbb{C}^{\frac{h}{m}\times\alpha\times r}$ is a Toeplitz tensor; Y_i is the i-th row of* $\mathbf{X}$ *and Block $(\mathcal{D})_{is} \in \mathbb{C}^{\beta\times\frac{d}{b}\times r}$ is also a Toeplitz tensor, $r = 1,\dots,m$, $j = 1,\dots,q$; $s = 1,\dots,b$, $i = 1,\dots,p$.*

Proof. The proof is analogous to that of Theorem **??**; thus, we omit it. □

The following example clearly shows an application of this theorem and confirms the uniqueness of the solution.

Example 4.18. Consider the tensors $\mathcal{A} \in \mathbb{R}^{2\times4\times2}$, $\mathcal{B} \in \mathbb{R}^{6\times12\times2}$, $C \in \mathbb{R}^{2\times3\times2}$, and $\mathcal{D} \in \mathbb{R}^{6\times9\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{1::} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ -1 & 2 & 0 & 3 \end{bmatrix}, \quad \mathcal{A}_{2::} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix},$$

$$\mathcal{B}_{1::} = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 & 2 & 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 2 & 2 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & -2 & 1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ -1 & -3 & 0 & 2 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & -1 & -3 & 0 & 4 & 0 & 3 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 0 & 4 & 0 & 3 & 0 & 2 & 1 & 0 \end{bmatrix},$$

$$\mathcal{B}_{2::} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 1 & 4 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 2 & 0 & 1 & 4 & 1 & 0 & 0 & 2 \\ -2 & 0 & 0 & -1 & 2 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\mathcal{C}_{::1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathcal{C}_{::2} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

$$\mathcal{D}_{::1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 2 & 0 & -1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 & 0 & -1 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 & -2 & 0 \end{bmatrix},$$

$$\mathcal{D}_{::2} = \begin{bmatrix} 3 & 0 & 0 & 2 & -1 & 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 0 & 2 & -1 & 0 & 1 & 1 \\ 0 & 0 & 6 & 0 & 1 & 4 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 & -2 & 0 & -1 \\ -3 & 0 & 3 & -2 & 0 & 2 & -1 & 0 & 1 \\ 0 & -3 & 0 & 1 & -2 & 0 & -1 & -1 & 0 \end{bmatrix}.$$

$\alpha = 4$ and $\beta = 2$, thus $p = \frac{6}{2} = \frac{4}{4} \cdot \frac{6}{2}$ and $q = \frac{2}{2} \cdot \frac{9}{3} = \frac{12}{4}$, so the permissible size is 3×3. It can be easily demonstrated that

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

is the unique solution.

5. Solvability of the tensor-tensor system (??)

In this section, we study the solvability of the tensor-tensor equation

$$\begin{cases} \overbrace{\mathcal{A} \ltimes \mathcal{X}}^{mt/n \times qt/p \times r} = \mathcal{B} \\ \underbrace{\mathcal{X} \ltimes \mathcal{C}}_{pt_1/q \times bt_1/a \times r} = \mathcal{D}, \end{cases} \tag{5.1}$$

where $\mathcal{A} \in \mathbb{C}^{m \times n \times r}$, $\mathcal{B} \in \mathbb{C}^{h \times k \times s}$, $\mathcal{C} \in \mathbb{C}^{a \times b \times u}$ and $\mathcal{D} \in \mathbb{C}^{l \times d \times v}$ are are known, and $\mathcal{X} \in \mathbb{C}^{p \times q \times f}$ is an unknown tensor. $t = lcm\{n, p\}$ and $t_1 = lcm\{q, a\}$. The problem is to find a third order tensor $\mathcal{X}$ satisfying tensor equation (**??**). We have the following theorem.

Theorem 5.1. *If the tensor equation (**??**) has a solution, then both $\frac{h}{m}$ and $\frac{d}{b}$ must be positive integers. Moreover, if a tensor of size $p \times q \times f$ is a solution to equation (**??**), then the following relations hold:*

$$p = \frac{nh}{\alpha m} = \frac{l}{\beta}, \quad q = \frac{ad}{b\beta} = \frac{k}{\alpha},$$

where α is a common divisor of n and k, and β is a common divisor of a and l, satisfying

$$\gcd\left\{\alpha, \frac{h}{m}\right\} = 1, \quad \gcd\left\{\beta, \frac{d}{b}\right\} = 1.$$

Additionally, $\frac{s}{r}$ *and* $\frac{v}{u}$ *must be positive integers, and* f *is a common divisor of* s *and* v *such that*

$$\gcd\{r, f\} = s, \quad \gcd\{f, u\} = v.$$

Proof. Suppose that $\mathcal{X}$ is a solution of tensor equation (**??**), and its dimension is $p \times q \times f$, by Definition **??**, we have that

$$\mathcal{B} = \mathcal{A} \ltimes \mathcal{X} = (\mathcal{A} \otimes \mathcal{I}_{(t/n)\times(t/n)\times(w/r)}) * \left(\mathcal{X} \otimes \mathcal{I}_{(t/p)\times(t/p)\times(w/f)}\right) \in \mathbb{C}^{(mt/n)\times(qt/p)\times(w)},$$

where $t = lcm\{n, p\}$ and $w = lcm\{r, f\}$. And

$$\mathcal{D} = \mathcal{X} \ltimes \mathcal{C} = \left(\mathcal{X} \otimes \mathcal{I}_{(t_1/q)\times(t_1/q)\times(z/f)}\right) * (\mathcal{C} \otimes \mathcal{I}_{(t_1/a)\times(t_1/a)\times(z/u)}) \in \mathbb{C}^{(pt_1/q)\times(bt_1/a)\times(z)},$$

where $t_1 = lcm\{q, a\}$ and $z = lcm\{f, u\}$.

On the other hand, it is assumed that $\mathcal{B} \in \mathbb{C}^{h\times k\times s}$ and $\mathcal{D} \in \mathbb{C}^{l\times d\times v}$. Then, $h = mt/n$, $k = qt/p$, $w = s$, $l = pt_1/q$, $d = bt_1/a$, and $z = v$, where $t = lcm\{n, p\}$, $w = lcm\{r, f\}$, $t_1 = lcm\{q, a\}$ and $z = lcm\{f, u\}$. Consequently $t = \frac{nh}{m}$, $t_1 = \frac{ad}{b}$, $\frac{t}{p} = \frac{k}{q}$ and $\frac{t_1}{q} = \frac{l}{p}$. Hence $\frac{h}{m}$ and $\frac{d}{b}$ must be positive integers. Denote

$$\alpha = \frac{nh/m}{p} = \frac{t}{p} = \frac{k}{q}, \qquad \beta = \frac{ad/b}{q} = \frac{t_1}{q} = \frac{l}{p},$$

then we have $p = \frac{nh}{\alpha m} = \frac{l}{\beta}$ and $q = \frac{ad}{\beta b} = \frac{k}{\alpha}$. On the other hand,

$$\frac{nh}{m} = t = lcm\{n, p\} = lcm\left\{n, \frac{nh}{m\alpha}\right\}, \qquad \frac{ad}{b} = t_1 = lcm\{a, q\} = lcm\left\{a, \frac{ad}{b\beta}\right\}.$$

Then $\frac{h}{m}$ must be a factor of $\frac{nh}{m\alpha}$ and $\frac{d}{b}$ must be a factor of $\frac{ad}{b\beta}$. So we have that $\frac{n}{\alpha}$ and $\frac{b}{\beta}$ must be positive integers, Therefore α is a common divisor of n and k, and β is a common divisor of b and l. Furthermore,

$$\frac{nh}{m} = \mathrm{lcm}\left\{n, \frac{nh}{m\alpha}\right\} = \frac{n}{\alpha} \cdot lcm\left\{\alpha, \frac{h}{m}\right\}, \frac{ad}{b} = \mathrm{lcm}\left\{a, \frac{ad}{b\beta}\right\} = \frac{a}{\beta} \cdot lcm\left\{\beta, \frac{d}{b}\right\},$$

Consequently,

$$lcm\left\{\alpha, \frac{h}{m}\right\} = \alpha \cdot \frac{h}{m}, \quad lcm\left\{\beta, \frac{d}{b}\right\} = \beta \cdot \frac{d}{b},$$

thus,

$$\gcd\left\{\alpha, \frac{h}{m}\right\} = 1, \gcd\left\{\beta, \frac{d}{b}\right\} = 1.$$

Since $s = lcm\{r, f\}$ and $v = lcm\{u, f\}$; $\frac{s}{r}$ and $\frac{v}{u}$ must be positive integers and f is a common divisor of s and v. The proof is completed. □

Remark 5.2. Suppose that $\alpha_i, i = 1, \cdots, L$ are all the common divisors of n and k satisfies $\gcd\left\{\alpha_i, \frac{h}{m}\right\} = 1$, $\beta_j, j = 1, \cdots, M$ are all the common divisors of a and l such that $\gcd\left\{\beta_j, \frac{d}{b}\right\} = 1$. Moreover, $f_t, t = 1, \cdots, N$ are all the common divisors of s and v satisfies $\gcd\{r, f\} = s$; $\gcd\{f, u\} = v$. By Theorem **??**, tensor equation (**??**) may have solutions of sizes $p_i \times q_j \times f_t$, where $p_i = \frac{nh}{\alpha_i m}$, $q_i = \frac{ad}{b\beta_j}, i = 1, \cdots, L;\ j = 1, \cdots, M$. We refer to the sizes as permissible, and there are some relationships between the solutions of different permissible sizes.

(i). It is assumed that $p_1\times q_1\times f_1$ and $p_2\times q_2\times f_2$ are two different permissible sizes of solutions, $1<\frac{p_2}{p_1}=\frac{q_2}{q_1}\in\mathbb{Z}$ and $1\leq\frac{f_2}{f_1}\in\mathbb{Z}$. Hence, $\mathcal{X}^{p_2\times q_2\times f_2}=\mathcal{X}^{p_1\times q_1\times f_1}\otimes\mathcal{I}_{(\frac{p_2}{p_1})(\frac{q_2}{q_1})(\frac{f_2}{f_1})}$. Conversely, if equation (**??**) has a unique solution $\mathcal{X}^{p_2\times q_2\times f_2}$, the solution $\mathcal{X}^{p_1\times q_1\times f_1}$, if it exists, must be unique.

(ii). Denote $\bar{\alpha}=\gcd\{n,k\}$, $\bar{\beta}=\gcd\{a,l\}$, $\bar{f}=\min\{f_t,t=1,\cdots,N\}$, $\bar{p}=\frac{nh}{\bar{\alpha}m}$ and $\bar{q}=\frac{ad}{b\bar{\beta}}$, if there is a solution with minimum size $\bar{p}\times\bar{q}\times\bar{f}$ for equation (**??**), then it has a solution for every permissible size.

The following example emphasizes that the admissibility condition on the dimensions according to the Theorem **??** is not sufficient for the existence of a solution, and it is possible that only some of the admissible sizes actually admit a solution. Furthermore, by presenting an explicit solution, it demonstrates the practical application of the theorem in determining the possible dimensions and finding the solution.

Example 5.3. Consider the tensors $\mathcal{A}\in\mathbb{R}^{2\times4\times2}$, $\mathcal{B}\in\mathbb{R}^{4\times4\times2}$, $\mathcal{C}\in\mathbb{R}^{2\times2\times2}$, and $\mathcal{D}\in\mathbb{R}^{8\times4\times2}$, each having two frontal slices as follows:

$$\mathcal{A}_{::1}=\begin{bmatrix}0&1&1&3\\0&2&0&1\end{bmatrix},\mathcal{A}_{::2}=\begin{bmatrix}1&1&1&0\\0&1&0&2\end{bmatrix},\quad\mathcal{B}_{::1}=\begin{bmatrix}2&1&0&3\\7&4&4&6\\6&0&0&4\\5&4&4&3\end{bmatrix},\quad\mathcal{B}_{::2}=\begin{bmatrix}6&1&1&5\\10&4&5&1\\4&0&-1&3\\4&3&4&4\end{bmatrix}.$$

$$\mathcal{C}_{::1}=\begin{bmatrix}1&0\\0&2\end{bmatrix},\quad\mathcal{C}_{::2}=\begin{bmatrix}0&2\\1&0\end{bmatrix}.\quad\mathcal{D}_{::1}=\begin{bmatrix}0&0&0&-2\\2&1&0&4\\1&0&2&4\\0&1&2&2\\1&1&0&2\\3&1&4&0\\0&1&2&0\\2&0&4&4\end{bmatrix},\quad\mathcal{D}_{::2}=\begin{bmatrix}0&-1&0&0\\0&2&4&2\\1&2&2&0\\1&1&0&2\\0&1&2&2\\2&0&6&2\\1&0&0&2\\2&2&4&0\end{bmatrix}.$$

By Theorem **??**, the admissible sizes of solutions are $8\times4\times1$, $8\times4\times2$. We can verify that the equation has no solution for permissible sizes $8\times4\times1$ and it is easy to verify that $\mathcal{X}$, is a solution for permissible sizes $8\times4\times2$, where the fontal slices of $\mathcal{X}$ are as follows:

$$\mathcal{X}_{::1}=\begin{bmatrix}1&0&0&-1\\0&1&0&1\\1&0&0&2\\0&1&1&0\\0&1&0&1\\2&1&0&0\\0&0&0&0\\1&0&0&2\end{bmatrix},\quad\mathcal{X}_{::2}=\begin{bmatrix}0&0&-1&0\\0&1&2&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\2&0&1&0\\1&0&0&1\\2&0&1&0\end{bmatrix}.$$

6. Application to Pinning Control of a Boolean Network

In this section, we demonstrate the practical relevance of our theoretical results through the pinning control of a Boolean gene regulatory network. Pinning control refers to the strategy of applying control inputs to only a subset of nodes in order to steer the entire network toward a desired state. While the semi-tensor product (STP) of matrices has been widely used for the analysis and control of Boolean networks, the presence of higher-order interactions between control inputs naturally leads to a tensor formulation. In such cases, the resulting system cannot be represented in a structurally transparent manner using only matrix-based STP without substantial dimensional lifting. The tensor-based STP framework developed in this paper provides a direct and natural modeling and solution approach.

6.1. Model Formulation

Consider a simplified Boolean gene regulatory network consisting of three genes G_1, G_2, G_3. Each gene takes values in $\{0, 1\}$, representing inactive and active expression states, respectively. Following the standard STP representation of Boolean networks, each Boolean variable is identified with its canonical vector form:

$$1 \leftrightarrow \delta_2^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad 0 \leftrightarrow \delta_2^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the global state vector is represented in vector form as $\mathbf{x}(t)$, and logical dynamics become multilinear algebraic equations under the STP framework.

In addition to the gene states, the system is influenced by two external Boolean control inputs u_1 and u_2. A key feature of this model is that the control inputs do not act independently on individual genes; rather, their combined action produces a regulatory rule depending on their pairwise interaction. This induces an inherently third-order interaction structure.

The state update equation can therefore be written as

$$\mathbf{x}(t+1) = \mathcal{A} \ltimes \mathbf{x}(t) \ltimes \mathbf{u}(t), \tag{6.1}$$

where $\mathbf{u}(t) = [u_1(t), u_2(t)]^T$, and $\mathcal{A} \in \mathbb{R}^{3\times3\times2}$ is a third-order system tensor. The three modes of $\mathcal{A}$ correspond to: (i) next state, (ii) current state, and (iii) control-input interaction.

The objective of pinning control is to design a state-feedback control law

$$\mathbf{u}(t) = \mathcal{X} \ltimes \mathbf{x}(t),$$

such that the closed-loop system stabilizes to the desired fixed point $[1, 0, 0]^T$, which may represent a prescribed cellular differentiation phenotype.

Substituting the control law into (**??**) yields

$$\mathbf{x}(t+1) = \mathcal{A} \ltimes \mathbf{x}(t) \ltimes (\mathcal{X} \ltimes \mathbf{x}(t)),$$

which is a bilinear tensor expression in $\mathbf{x}(t)$. Rather than attempting to reduce this to a matrix equation through dimensional augmentation, we directly enforce that the closed-loop dynamics coincide with a desired tensor $\mathcal{B}$. This leads to the tensor system

$$\begin{cases} \mathcal{A} \ltimes \mathcal{X} = \mathcal{B}, \\ \mathcal{X} \ltimes \mathcal{C} = \mathcal{D}, \end{cases} \tag{6.2}$$

where the second equation imposes structural constraints on the controller.

Here:

- $\mathcal{A} \in \mathbb{R}^{3\times3\times2}$ is the system tensor;
- $\mathcal{X}$ is the unknown controller tensor;
- $\mathcal{B}$ is the desired closed-loop tensor satisfying $\mathcal{B} \ltimes [1,0,0]^T = [1,0,0]^T$;
- C and $\mathcal{D}$ encode actuation constraints.

Although it is possible in principle to convert such a model into a matrix STP representation through dimensional augmentation, doing so merges distinct interaction modes and significantly increases system size. The tensor formulation preserves the natural multi-mode structure.

Second, the constrained controller design problem is naturally expressed as a coupled tensor-tensor system. Matrix STP methods typically address equations of the form $\mathbf{A} \ltimes X = \mathbf{B}$ with matrix variables. In contrast, the present framework accommodates higher-order controller tensors subject to structural constraints.

In summary, this model illustrates a realistic Boolean control scenario in which the tensor STP framework offers a structurally transparent modeling approach, explicit solvability conditions, and constructive controller design. This demonstrates the applicability and significance of the theoretical results developed in this paper.

Conclusion

In this paper, we explored the solvability of the tensor system $\mathcal{A} \ltimes \mathcal{X} = \mathcal{B}, \mathcal{X} \ltimes C = \mathcal{D}$ under the STP framework. We established necessary and sufficient conditions for solutions, proposed explicit solution methods, and introduced tensor partitioning techniques to reduce the problem to linear systems via the t-product. Our results extend classical matrix equation systems to higher-order tensors, providing a unified approach to multidimensional data problems. We apply the STP framework to model and control Boolean gene regulatory networks with pinning control inputs. The examples demonstrated the effectiveness of our methods. Future work may focus on numerical algorithms and applications in various fields involving tensor computations.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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