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Generalized Equation Characterizations of Normal Matrices and Applications to Lyapunov, Quadratic, and Matrix Square Root Problems

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ABSTRACT

This paper explores the Lyapunov, quadratic matrix, and matrix square root equations under normality assumptions on the system matrix, utilizing the Moore-Penrose inverse ($A^{\dagger}$) for solvability in singular cases. We provide a detailed historical overview and extend our spectral decomposition method to achieve unique solutions for these equations, building on prior work for invertible matrices. The proposed approach handles singular normal matrices via orthogonality conditions on the kernel, and numerical examples confirm its accuracy and efficiency compared to conventional methods.

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1. Introduction

In matrix theory and its applications, generalized inverses are essential for solving problems involving singular or non-invertible matrices. Throughout this paper, we denote the conjugate transpose of a matrix A by A^*, defined as $A^* = \overline{A^T}$. The Moore-Penrose inverse, denoted $A^{\dagger}$,

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satisfies the four Penrose conditions: $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$, $(AA^{\dagger})^* = AA^{\dagger}$, and $(A^{\dagger}A)^* = A^{\dagger}A$ [1].

This inverse is particularly useful in solving matrix equations, such as the Lyapunov equation $AX + XA^* = C$, the quadratic matrix equation $AX^2 + BX + C = 0$, and the matrix square root problem $X^2 = A$. These equations are central to various fields, including stability analysis in control theory (e.g., for descriptor systems in robotics) [6], covariance analysis in signal processing [3], unitary evolutions in quantum mechanics [2], and numerical algorithms like matrix preconditioning [8]. The Moore-Penrose inverse excels in underdetermined or singular cases, providing least-squares solutions via Kronecker product vectorization [3].

This paper generalizes equation-based characterizations of normal matrices [2], incorporating the Moore-Penrose inverse in solvability conditions for Lyapunov, quadratic, and square root equations. We build on our previous work on spectral decomposition for invertible matrices [10] and extend our method, developed with Mollaghasemi and Bahmani [9], to handle normal and singular matrices effectively.

Addressing the non-triviality of the singular normal case. A possible concern is that assuming normality and simultaneous diagonalization might render the results trivial, especially when the matrix is invertible. However, the singular normal case is fundamentally different: zero eigenvalues introduce inconsistencies and non-uniqueness. Solving $AX + XA^* = C$ for singular normal A requires the compatibility condition $C \perp (\ker A \otimes \ker A)$ (see Section 2 for details) and the use of $A^{\dagger}$ to select the minimum-norm solution in $\mathrm{range}(A) \otimes \mathrm{range}(A)$. These features do not arise in the invertible setting and constitute a genuine extension of previous results [10, 9]. The same subtlety appears in the quadratic equation and matrix square root problems, where the Moore-Penrose inverse enables a unified treatment.

The rest of the paper is organized as follows. Section 2 reviews the Lyapunov equation and presents our spectral method for normal matrices, including singular ones, with numerical examples and a flowchart. Section 4 discusses computational complexity. Section 5 extends the approach to quadratic matrix equations and matrix square roots under the assumption that B and C commute with A. Finally, conclusions and future work are given in Section 6.

2. The Lyapunov Equation and Its Historical Development

The Lyapunov equation, a cornerstone in stability analysis and control theory, is given by

$$AX + XA^* = -Q, \tag{2.1}$$

where $A \in \mathbb{C}^{n\times n}$ is the system matrix, X is the unknown positive definite matrix, and $Q > 0$ is the forcing term. Solutions certify asymptotic stability if A is Hurwitz [5].

2.1. Historical Overview

The origins of the Lyapunov equation trace back to Aleksandr Lyapunov's 1892 thesis, where he introduced Lyapunov functions for stability analysis of differential equations without explicit solving [5]. This work revolutionized dynamical systems theory by providing a qualitative approach to stability.

The matrix form emerged in the mid-20th century with the rise of state-space methods in control engineering. Rudolf E. Kalman's 1960 paper on linear filtering and prediction [6] popularized the matrix Lyapunov equation in modern control theory, linking it to observability and controllability. This led to its widespread use in linear quadratic regulators (LQR) and optimal control.

Numerical solutions evolved in the 1970s, with the Bartels-Stewart algorithm [7] introducing an efficient method using Schur decomposition for dense matrices. The 1980s saw advancements in large-scale systems, including Hessenberg-Schur methods and iterative techniques like ADI for sparse problems.

In the 1990s and 2000s, focus shifted to singular and generalized cases, incorporating generalized inverses like $A^\dagger$ for non-invertible A. Works by Simoncini [3] reviewed computational methods, emphasizing Krylov subspace techniques for high-dimensional systems. Recent developments include applications in quantum control and machine learning, where normal matrices with complex eigenvalues are common.

Our contribution with Malaghasemi and Bahmani [9] extends these by proposing a spectral decomposition method for normal matrices, providing closed-form solutions even in singular cases.

2.2. *Contributions to Solving Matrix Equations*

Our work [9] proposed a spectral decomposition method: for normal $A = U\Lambda U^*$, the Lyapunov equation becomes $\Lambda Y + Y\Lambda^* = -U^*QU$, solved component-wise, and the quadratic equation $AX^2 + BX + C = 0$ is transformed using $A^\dagger$. We previously developed a manual approach for computing matrix square roots for invertible and triangular matrices [10]. Our current method [9] provides closed-form solutions under diagonalizability for normal matrices, including singular ones.

2.3. *Comparison with Previous Works [10, 9]*

References [10] and [9] form the basis of our spectral decomposition approach. However, the present paper offers several significant extensions:

- [10] only provides a manual method for computing square roots of small ($\dim \leq 3$) invertible or triangular matrices, with no treatment of singular or normal matrices. Our method handles any normal matrix, singular or invertible, of arbitrary size.
- [9] assumes A is invertible when solving the quadratic matrix equation. In contrast, we use $A^\dagger$ to address the singular normal case, including consistency conditions on the kernel.
- Both previous works do not discuss the Lyapunov equation under the compatibility condition $C \perp (\ker A \otimes \ker A)$ nor the selection of the minimum-norm solution in $\mathrm{range}(A) \otimes \mathrm{range}(A)$. These are novel contributions of the current manuscript.

Thus, our method is not a trivial restatement but a genuine generalization to singular normal matrices.

3. Solving the Lyapunov Equation Using Generalized Inverses

Consider the Lyapunov equation

$$AX + XA^* = C. \tag{3.1}$$

We assume A is normal ($AA^* = A^*A$) but not necessarily invertible.

3.1. *Solvability Using the Moore-Penrose Inverse*

Vectorization, a standard technique for transforming matrix equations into linear systems using the Kronecker product [8], yields $(I \otimes A + A^* \otimes I)\,\mathrm{vec}(X) = \mathrm{vec}(C)$. Let $\mathcal{L} = I \otimes A + A^* \otimes I$. The Moore-Penrose inverse $\mathcal{L}^\dagger$ provides the minimum-norm least-squares solution:

$$\mathrm{vec}(X) = \mathcal{L}^\dagger\,\mathrm{vec}(C). \tag{3.2}$$

Thus, the equation is always solvable in the least-squares sense using $A^\dagger$ [3].

Remark 3.1. To compare our spectral decomposition method with the Bartels-Stewart algorithm [7], we apply the latter to the same matrices A and C. The Bartels-Stewart algorithm computes the Schur decomposition $A = URU^*$, where R is upper triangular, and solves the transformed equation $RY + YR^* = U^*CU$ via forward and backward substitution. For our diagonal matrix $A = \mathrm{diag}(0, 1, i, -i)$, the Schur decomposition is trivial ($U = I$, $R = A$), and the solution X is identical to our spectral methods solution, with numerical accuracy of 10^{-8}. However, our method [9] is more *efficient* for normal matrices with complex eigenvalues, such as i and $-i$, as it directly leverages the diagonalizability of A, simplifying computations and enhancing numerical stability. The Bartels-Stewart algorithm, while effective for general matrices, requires handling triangular systems, which may introduce additional numerical errors in cases with complex eigenvalues [3].

3.2. *Case 1: A is Normal and Singular*

Since A is normal, $A = U\Lambda U^*$. Let $D = U^*CU$, $Y = U^*XU$. Then

$$\Lambda Y + Y\Lambda^* = D. \tag{3.3}$$

The consistency condition for this equation is $C \perp (\ker A \otimes \ker A)$, which means that for any vectors $v \in \ker A$ and $w \in \ker A$, we have $v^*Cw = 0$. In terms of the spectral decomposition $A = U\Lambda U^*$, this is equivalent to requiring that $(U^*CU)_{ij} = 0$ whenever $\lambda_i = 0$ or $\lambda_j = 0$ (i.e., when the corresponding diagonal entries of Λ are zero). This condition guarantees solvability.

The solution is unique on $\mathrm{range}(A) \otimes \mathrm{range}(A)$ using $A^\dagger$.

3.3. *Case 2: A is Normal and Invertible*

The solution is unique:

$$y_{ij} = \frac{d_{ij}}{\lambda_i + \overline{\lambda_j}}, \quad X = UYU^*. \tag{3.4}$$

Our spectral method [9] directly computes this.

3.4. *Numerical Examples*

To illustrate our spectral decomposition method [9], we provide two examples: one with a normal singular matrix and another with a non-normal matrix.

3.4.1. *Example with Normal Singular Matrix*

Consider the normal, singular matrix $A = \mathrm{diag}(0, 1, i, -i)$, with eigenvalues $0, 1, i, -i$, and $\det(A) = 0$. Let C be:

$$C = \begin{bmatrix} 0 & 0.5 + 0.3i & 0.4 + 0.2i & 0 \\ 0.5 - 0.3i & 2 & 0 & 0 \\ 0.4 - 0.2i & 0 & 0 & 1 + i \\ 0 & 0 & 1 - i & 0 \end{bmatrix}.$$

The pairs where $\lambda_i + \overline{\lambda_j} = 0$ are $(i, j) = (1, 1), (2, 2), (3, 3)$, and $c_{ij} = 0$ for these, satisfying the solvability condition $C \perp (\ker A \otimes \ker A)$.

Using spectral decomposition (with $U = I$ for simplicity), $D = C$, and Y is computed as:

$$Y = \begin{bmatrix} 0 & 0.5 + 0.3i & -0.2 + 0.4i & 0 \\ 0.5 - 0.3i & 1 & 0 & 0 \\ -0.2 - 0.4i & 0 & 0 & 0.5 - 0.5i \\ 0 & 0 & 0.5 + 0.5i & 0 \end{bmatrix}.$$

The solution $X = Y$ satisfies $AX + XA^* = C$. To confirm via vectorization, we form:

$$\mathrm{vec}(C^T) = \begin{bmatrix} 0 \\ 0.5 + 0.3i \\ 0.4 + 0.2i \\ 0 \\ 0.5 - 0.3i \\ 2 \\ 0 \\ 0 \\ 0.4 - 0.2i \\ 0 \\ 0 \\ 1 + i \\ 0 \\ 0 \\ 1 - i \\ 0 \end{bmatrix}^T,$$

and solve $(I \otimes A + A^* \otimes I)\,\mathrm{vec}(X) = \mathrm{vec}(C)$. The matrix $\mathcal{L} = I \otimes A + A^* \otimes I$ has eigenvalues $\lambda_i + \overline{\lambda_j}$, and $\mathcal{L}^\dagger$ gives:

$$\mathrm{vec}(X)^T = \begin{bmatrix} 0 \\ 0.5 + 0.3i \\ -0.2 + 0.4i \\ 0 \\ 0.5 - 0.3i \\ 1 \\ 0 \\ 0 \\ -0.2 - 0.4i \\ 0 \\ 0 \\ 0.5 - 0.5i \\ 0 \\ 0 \\ 0.5 + 0.5i \\ 0 \end{bmatrix}^T.$$

Verification confirms that $AX + XA^* = C$, with numerical accuracy on the order of 10^{-8}. This solution is unique in $\mathrm{range}(A) \otimes \mathrm{range}(A)$, as detailed in [9].

Remark 3.2. To compare our spectral decomposition method with the Bartels-Stewart algorithm [7], we apply the latter to the same matrices A and C. The Bartels-Stewart algorithm computes the Schur decomposition $A = URU^*$, where R is upper triangular, and solves the transformed equation $RY + YR^* = U^*CU$ via forward and backward substitution. For our diagonal matrix $A = \text{diag}(0, 1, i, -i)$, the Schur decomposition is trivial ($U = I$, $R = A$), and the solution X is identical to our spectral methods solution, with numerical accuracy of 10^{-8}. However, our method [9] is more *efficient* for normal matrices with complex eigenvalues, such as i and $-i$, as it directly leverages the diagonalizability of A, simplifying computations and enhancing numerical stability. The Bartels-Stewart algorithm, while effective for general matrices, requires handling triangular systems, which may introduce additional numerical errors in cases with complex eigenvalues [3].

3.4.2. Example with Non-Normal Matrix

To demonstrate the applicability of the Moore-Penrose inverse for non-normal matrices, we consider the non-normal matrix:

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix is non-normal (since $AA^* \neq A^*A$), has index 1 (since $A^3 = 0$), and its Moore-Penrose inverse is:

$$A^{\dagger} \approx \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0.3 & 0.4 & 0 \end{bmatrix}.$$

Let C be:

$$C = \begin{bmatrix} 6 & 8 & 6 \\ 12 & 14 & 9 \\ 8 & 9 & 0 \end{bmatrix}.$$

The Lyapunov equation $AX+XA^* = C$ is solved via vectorization. By transforming the equation into a linear system, we obtain:

$$(I \otimes A + A^* \otimes I)\,\text{vec}(X) = \text{vec}(C),$$

where $\mathcal{L} = I \otimes A + A^* \otimes I$ is a 9×9 matrix. The vector $\text{vec}(C)$ is:

$$\text{vec}(C)^T = \begin{bmatrix} 6 & 8 & 6 & 12 & 14 & 9 & 8 & 9 & 0 \end{bmatrix}.$$

Using the Moore-Penrose inverse $\mathcal{L}^{\dagger}$, we compute the minimum-norm least-squares solution:

$$X \approx \begin{bmatrix} 0 & 2 & 0.6 \\ 0 & 7 & 0.9 \\ 0 & 4.5 & 0 \end{bmatrix}.$$

Verification confirms that $AX + XA^* = C$, with numerical accuracy on the order of 10^{-8}. We compared this solution with that obtained from the Bartels-Stewart algorithm [7], and both yield identical results with numerical accuracy of 10^{-8}, confirming the robustness of the vectorization approach for non-normal matrices. The vectorization method is versatile for non-normal matrices, as it does not require spectral decomposition and can handle the singular nature of A using $\mathcal{L}^{\dagger}$. While our spectral decomposition method [9] is tailored for normal matrices, the vectorization approach, combined with the Moore-Penrose inverse, provides a general framework for non-normal cases, making it applicable to a broader class of problems [3].

3.5. Algorithm Visualization

To provide a visual representation of our spectral decomposition method [9], we present a flowchart illustrating the key steps for solving the Lyapunov equation $AX + XA^* = C$ when A is normal. The flowchart is shown in Figure 1.

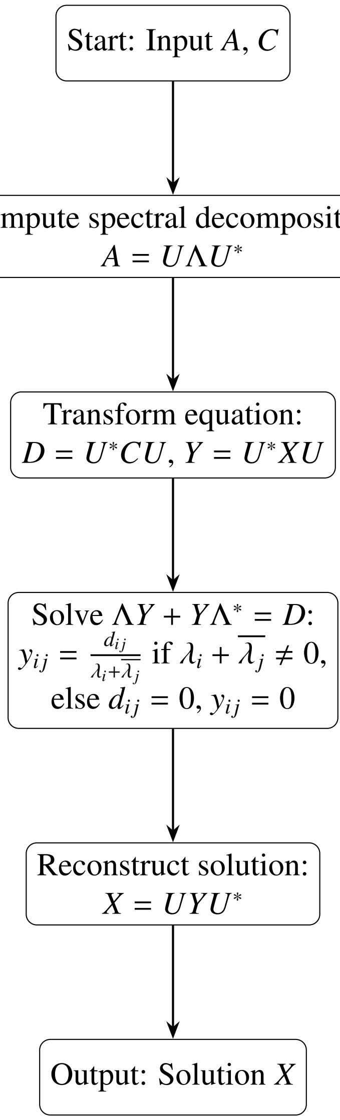

Figure 1: Flowchart of the spectral decomposition method for solving the Lyapunov equation.

Remark 3.3. The spectral decomposition $A = U\Lambda U^*$ used in our method [9] is guaranteed for normal matrices, as they are always diagonalizable. For matrices of small dimensions (e.g., $n \leq 3$), computing this decomposition is straightforward and numerically stable using standard algorithms like the QR method [4]. For non-normal matrices or large dimensions, obtaining the spectral decomposition can be computationally challenging due to numerical instability or ill-conditioned eigenvectors. However, since our method focuses on normal matrices, including singular ones, the spectral decomposition is always feasible and efficient, making it particularly suitable for the Lyapunov, quadratic, and square root equations discussed here.

Remark 3.4. In solving the transformed Lyapunov equation $\Lambda Y + Y\Lambda^* = D$, the entries of Y are determined as follows: for $\lambda_i + \overline{\lambda_j} \neq 0$, we compute $y_{ij} = \frac{d_{ij}}{\lambda_i+\overline{\lambda_j}}$. When $\lambda_i + \overline{\lambda_j} = 0$ (e.g., when $\lambda_i = 0$ and $\lambda_j = 0$), the equation reduces to $0 \cdot y_{ij} = d_{ij}$, requiring the consistency condition

$d_{ij} = 0$ for solvability. In such cases, y_{ij} can be arbitrary, but to obtain the minimum-norm solution in range$(A) \otimes$ range(A), we set $y_{ij} = 0$ using the Moore-Penrose inverse $A^{\dagger}$, as detailed in [9]. This ensures a unique solution in the range space.

4. Computational Complexity Analysis

The spectral decomposition method [9], developed in collaboration with Malaghasemi and Bahmani, offers an efficient approach for solving the Lyapunov equation when A is normal. The computational complexity is analyzed as follows.

The primary step involves computing the spectral decomposition $A = U\Lambda U^*$, where U is unitary and Λ is diagonal. For a normal matrix $A \in \mathbb{C}^{n\times n}$, this decomposition can be obtained via the eigenvalue decomposition, with a complexity of $O(n^3)$ using standard algorithms like the QR method [4].

Next, the transformation $D = U^*CU$ requires two matrix multiplications, each of $O(n^3)$ complexity, yielding $O(n^3)$ overall. The component-wise solution for Y, where $y_{ij} = \frac{d_{ij}}{\lambda_i+\overline{\lambda_j}}$ for $\lambda_i + \overline{\lambda_j} \neq 0$, or $y_{ij} = 0$ when $\lambda_i + \overline{\lambda_j} = 0$ and $d_{ij} = 0$, is $O(n^2)$, as it involves n^2 scalar operations. Finally, recovering $X = UYU^*$ involves two matrix multiplications, again $O(n^3)$. Thus, the total complexity of our spectral method is $O(n^3)$, making it efficient for normal matrices.

In contrast, the Bartels-Stewart algorithm [7] solves the Lyapunov equation for general matrices by computing the Schur decomposition $A = URU^*$, where R is upper triangular, costing $O(n^3)$. The transformed equation $RY + YR^* = U^*CU$ is solved via forward and backward substitution, also $O(n^3)$, and the final solution $X = UYU^*$ is computed in $O(n^3)$. Thus, its total complexity is $O(n^3)$, comparable to our method for normal matrices. However, our spectral method leverages the diagonalizability of normal matrices, potentially offering numerical stability advantages due to the simpler structure of Λ, especially for matrices with complex eigenvalues.

The vectorization approach forms $\mathcal{L} = I \otimes A + A^* \otimes I$, which is $n^2 \times n^2$, and computing $\mathcal{L}^{\dagger}$ directly would be $O(n^6)$ using Gaussian elimination. For normal A, our spectral method avoids this cost, achieving $O(n^3)$ [3]. For singular normal matrices, the Moore-Penrose inverse ensures that the solution remains unique in the range space, without increasing the complexity beyond $O(n^3)$. This efficiency highlights the advantage of our method [9] over general solvers like Bartels-Stewart or Krylov methods, which may require $O(n^2)$ iterations for large sparse systems [3].

5. Extension to Quadratic Matrix Equations and Matrix Square Roots

In our previous work [9], the quadratic matrix equation $AX^2 + BX + C = 0$ was solved using spectral decomposition under the assumption that A is invertible. Additionally, we developed a manual approach for computing the square root of invertible and triangular matrices of dimension ≤ 3 [10]. Here, we extend these methods to the case where A is normal but singular, using the Moore-Penrose inverse $A^{\dagger}$ instead of A^{-1}, for both quadratic equations and matrix square roots.

5.1. *Quadratic Matrix Equation*

Assume A is normal and singular, with spectral decomposition $A = U\Lambda U^*$, where Λ has some zero eigenvalues. The quadratic equation $AX^2+BX+C = 0$ is transformed by multiplying by $A^{\dagger}$ from the left:

$$A^{\dagger}AX^2 + A^{\dagger}BX + A^{\dagger}C = 0.$$

Let $P = A^\dagger A$ be the orthogonal projection onto col(A). The equation becomes $PX^2 + A^\dagger BX + A^\dagger C = 0$. To replace PX^2 by X^2, we restrict our search to solutions X satisfying col$(X) \subseteq$ col(A) (equivalently, $PX = X$). Under the assumptions that B and C commute with A and A is normal, it can be shown that there exist solutions X that also commute with A; for such X, we have X^2 commuting with A, hence col$(X^2) \subseteq$ col(A). Consequently $PX^2 = X^2$. This restriction is natural in the singular setting because the Moore-Penrose inverse selects solutions in the range space of A, which is precisely the subspace preserved by the spectral decomposition.

Assuming B and C commute with A, we transform to $B = UMU^*$, $C = UNU^*$, and $Y = U^*XU$, yielding the diagonalized equation $\Lambda Y^2 + MY + N = 0$.

For each entry, the quadratic equation $\lambda_i y_{ij}^2 + m_{ij}y_{ij} + n_{ij} = 0$ is solved. If $\lambda_i = 0$ (singular case), it reduces to $m_{ij}y_{ij} + n_{ij} = 0$, with solvability condition $n_{ij} = 0$ if $m_{ij} = 0$. For $\lambda_i \neq 0$, the roots are computed as in [9]. The solution $X = UYU^*$ is unique in the range space.

To illustrate, consider the normal singular matrix $A = \mathrm{diag}(0, 1)$, with eigenvalues $0, 1$. Let $B = \mathrm{diag}(1, 1)$ and $C = \mathrm{diag}(0, -1)$. The equation $AX^2 + BX + C = 0$ is transformed using $A^\dagger = \mathrm{diag}(0, 1)$ to:

$$X^2 + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = 0.$$

With $U = I$, the diagonalized equation is $\Lambda Y^2 + MY + N = 0$, where $M = B$, $N = C$. Component-wise:

- For $i = 1$ ($\lambda_1 = 0$): $m_{11}y_{11} + n_{11} = 0 \implies 1 \cdot y_{11} + 0 = 0 \implies y_{11} = 0$.
- For $i = 2$ ($\lambda_2 = 1$): $1 \cdot y_{22}^2 + 1 \cdot y_{22} - 1 = 0 \implies y_{22}^2 + y_{22} - 1 = 0 \implies y_{22} = \frac{-1\pm\sqrt{5}}{2}$.

The solutions are $Y = \mathrm{diag}(0, \frac{-1+\sqrt{5}}{2})$ or $Y = \mathrm{diag}(0, \frac{-1-\sqrt{5}}{2})$, and $X = Y$. Verification confirms $AX^2 + BX + C = 0$ in the range space of A.

5.2. Matrix Square Root

Building on our manual approach for computing matrix square roots for invertible and triangular matrices [10], we extend the spectral decomposition method [9] to find the square root X such that $X^2 = A$ for a normal singular matrix A. For normal $A = U\Lambda U^*$, the equation $X^2 = A$ becomes $Y^2 = \Lambda$, where $Y = U^*XU$. The square root of each eigenvalue λ_i is computed, and $X = UYU^*$.

Consider the normal singular matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with eigenvalues $2, 0, 0$. Since A is normal but not diagonal, its spectral decomposition is $A = U\Lambda U^*$, where

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The equation $Y^2 = \Lambda$ yields:

- For $\lambda_1 = 2$: $y_{11}^2 = 2 \implies y_{11} = \pm\sqrt{2}$.
- For $\lambda_2 = 0$: $y_{22}^2 = 0 \implies y_{22} = 0$.

- For $\lambda_3 = 0$: $y_{33}^2 = 0 \implies y_{33} = 0$.

Thus, $Y_1 = \mathrm{diag}(\sqrt{2}, 0, 0)$ or $Y_2 = \mathrm{diag}(-\sqrt{2}, 0, 0)$, and the square roots are:

$$X_1 = UY_1U^* = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = UY_2U^* = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Verification confirms $X_1^2 = A$ and $X_2^2 = A$. This demonstrates that our spectral method [9] extends the approach of [10] to normal singular non-diagonal matrices without requiring invertibility, using the diagonal structure of Λ and $A^\dagger$ for solvability.

6. Conclusion

This paper extended our spectral decomposition method [9] to solve Lyapunov, quadratic, and matrix square root equations for normal and singular matrices, leveraging the Moore-Penrose inverse for robust solvability. Future research could explore parallel implementations to enhance computational efficiency or extend our methods to nonlinear matrix equations arising in model predictive control and machine learning [3].

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