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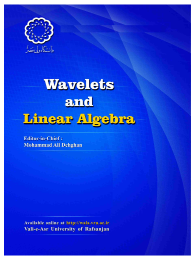

An Operator Bundle admitting no Frames

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ABSTRACT

We investigate the existence of frames in Hilbert C^*-modules via the framework of operator bundles. In particular, we present criteria under which a Hilbert C^*-module does not admit any frames.

1. Introduction and Preliminaries

In [5], we use equivariant functions on the space of irreducible representations of a C^*-algebra A to develop a duality theory for Hilbert C^*-modules. Within this framework, each Hilbert C^*-module corresponds to an operator bundle defined over the set of all non-zero irreducible representations of A.

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Let A be a C^*-algebra and H a sufficiently large Hilbert space such that every cyclic representation $\pi : A \to B(H)$ can be realized on a subspace of H. We denote by Irr(A : H), the set of all non-zero irreducible representations of A on H. Fujimoto [10] provides a representation of a C^*-algebra as a space of $B(H)$-valued functions over its irreducible representations, with continuity and compatibility conditions that generalize the classical GelfandNaimark theorem to the non-commutative setting.

Definition 1.1 ([10])**.** Let A be a C^*-algebra and let $\mathrm{Irr}(A : H)$ denote the set of all non-zero essentially irreducible representations of A on a separable Hilbert space H. For each $\pi \in \mathrm{Irr}(A : H)$, let

$$K_\pi = \overline{\pi(A)H}$$

be its essential subspace, and let p_π denote the orthogonal projection onto K_π.

A map

$$\gamma : \mathrm{Irr}(A : H) \longrightarrow B(H)$$

is called a *uniformly WOT-continuous operator field* if:

1. **Compatibility with partial isometries:** for every partial isometry $u \in B(H)$ such that

$$u^* u \geq p_\pi \quad \text{and} \quad \pi u = u\pi,$$

one has

$$\gamma(\pi)u = u\gamma(\pi).$$

2. **Uniform WOT-continuity:** for any net $\pi_j \to \pi$ in the Fell topology on $\mathrm{Irr}(A : H)$,

$$\gamma(\pi_j) \to \gamma(\pi)$$

in the weak operator topology, uniformly on the unit ball of H.

We denote by

$$A_u(\mathrm{Irr}(A : H), B(H))$$

the set of all such fields, and by

$$A_{u,0}(\mathrm{Irr}(A : H), B(H))$$

the subalgebra of fields vanishing at the limit representation 0, i.e.,

$$\lim_{\pi \to 0} \|\gamma(\pi)\| = 0.$$

Fujimoto's representation theorem establishes a canonical *-isometric isomorphism

$$A \;\cong\; A_{u,0}(\mathrm{Irr}(A : H), B(H)).$$

In [5], we introduced the concept of an operator bundle as follows.

Definition 1.2. Let A be a C^*-algebra, and let $\{K_\pi\}_{\pi\in\mathrm{Irr}(A:H)}$ be a family of Hilbert spaces. We denote by $A_{u,0} - \prod_{\pi\in\mathrm{Irr}(A:H)} B(H, K_\pi)$ the set

$$\{T \in \prod_{\pi\in\mathrm{Irr}(A:H)} B(H, K_\pi) : T^*T \in A_{u,0}(\mathrm{Irr}(A : H), B(H))\}.$$

A pair $(\{K_\pi\}_{\pi\in\mathrm{Irr}(A:H)}, \Gamma)$, where $\Gamma \subset \prod_{\pi\in\mathrm{Irr}(A:H)} B(H, K_\pi)$, is called an *operator bundle* over $\mathrm{Irr}(A : H)$ if it satisfies the following conditions:

1. $\Gamma \subset A_{u,0} - \prod_{\pi\in\mathrm{Irr}(A:H)} B(H, K_\pi)$.
2. For every $\pi \in \mathrm{Irr}(A : H)$ and $h \in H$, the set $\{T(\pi)(h) : T \in \Gamma\}$ spans K_π.
3. Let $T \in A_{u,0} - \prod_{\pi\in\mathrm{Irr}(A:H)} B(H, K_\pi)$. If for every $\pi \in \mathrm{Irr}(A : H)$, every $\epsilon > 0$ and every $h \in H$, there exists some $T' \in \Gamma$ such that

$$\|T(\pi')h - T'(\pi')h\| < \epsilon$$

for every π' in some neighborhood of π, then $T \in \Gamma$.

Every operator bundle Γ over $\mathrm{Irr}(A : H)$ can be equipped with a right Hilbert A-module structure via pointwise operations, with inner product

$$\langle T_1, T_2\rangle = T_1^* \circ T_2.$$

Indeed, this construction yields a duality theory for Hilbert C^*-modules [5].

In this paper, as an application of this construction, we employ operator bundles to study the existence of frames in Hilbert C^*-modules. For further details on frames in Hilbert C^*-modules, the reader is referred to [9, 11, 3].

2. Application of Operator Bundles to the Frame Existence Problem

The concept of frames was extended to Hilbert C^*-modules by Frank and Larson [9]. They posed in [9, Problem 8.1] the natural question: for which C^*-algebras A does every Hilbert A-module admit a frame? H. Li [11] addressed this problem for unital commutative C^*-algebras, showing that in this case, the existence of frames for all Hilbert A-modules characterizes finite-dimensional algebras. Indeed, in this paper Hanfeng Li considers a general form of the SerreSwan theorem [13], which asserts that the category of Hilbert $C(Z)$-modules is equivalent to the category of continuous fields of Hilbert spaces over Z, where Z is a compact Hausdorff space (see also [13]). For reference on continuous fields of Banach spaces, see [7].

Later, a conjecture was proposed in [1] stating that a Hilbert C^*-module over A admits a frame if and only if A is a compact C^*-algebra; that is, A admits a faithful $*$-representation into the algebra of compact operators $K(H)$ on some Hilbert space H. This conjecture was refined in [3], where, using a Hilbert bundle characterization of Hilbert C^*-modules and the approach of [11, Proposition 2.4], it was shown that there exists a Hilbert C^*-module over

$$A := K(\ell^2) \oplus \mathbb{C}I_{\ell^2},$$

the unitization of the algebra of compact operators on the separable Hilbert space ℓ^2, which does not admit any frames [4]. This example provides a concrete realization of a *frame-less Hilbert C^*-module*, giving a definitive negative answer to the general frame existence problem posed by Frank and Larson.

Definition 2.1. Let A be a C^*-algebra and $\mathcal{E}$ a Hilbert A-module. A family $\{x_i\}_{i\in I} \subset \mathcal{E}$, with arbitrary index set I, is called a *standard frame* for $\mathcal{E}$ if there exist constants $C, D > 0$ such that

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle$$

for all $x \in \mathcal{E}$, where the sum converges in the C^*-norm, taking the supremum over all finite subsets of I.

We now apply the operator bundle framework to provide a criterion under which a Hilbert C^*-module does not admit a frame.

Proposition 2.2. *Let A be a separable C^*-algebra, H a separable infinite-dimensional Hilbert space, and $(\{K_\pi\}_{\pi\in \mathrm{Irr}(A:H)}, \Gamma)$ an operator bundle. Suppose there exists $\pi_0 \in \mathrm{Irr}(A : H)$ and a countable subset $W \subset \mathrm{Irr}(A : H)$ with $\pi_0 \in \overline{W} \setminus W$, such that K_π is separable for all $\pi \in W$, while K_{π_0} is non-separable. Then Γ, viewed as a Hilbert A-module, admits no frame.*

Proof. Suppose, for contradiction, that $\{T_i\}_{i\in I}$ is a frame for Γ. Then there exist constants $C, D > 0$ such that for all $T \in \Gamma$,

$$CT^*T \leq \sum_{i\in I} T^*T_iT_i^*T \leq DT^*T.$$

Evaluating at $\pi \in \mathrm{Irr}(A : H)$ and $h \in H$ yields

$$C\|T(\pi)(h)\|^2 \leq \sum_{i\in I} |\langle T(\pi)(h), T_i(\pi)(h)\rangle|^2 \leq D\|T(\pi)(h)\|^2.$$

Since $\{T(\pi)h : T \in \Gamma\} = K_\pi$, the inequality extends to all $y_\pi \in K_\pi$:

$$C\|y_\pi\|^2 \leq \sum_{i\in I} |\langle y_\pi, T_i(\pi)(h)\rangle|^2 \leq D\|y_\pi\|^2.$$

For each $\pi \in W$, K_π is separable with a countable basis E_π, and therefore the set

$$F_{h,\pi} = \{i \in I : T_i(\pi)(h) \neq 0\} = \bigcup_{y_\pi\in E_\pi}\{i \in I : \langle y_\pi, T_i(\pi)(h)\rangle \neq 0\}$$

is countable. Enumerate $W = \{\pi_n : n \in \mathbb{N}\}$ and let $F = \bigcup_{n\in\mathbb{N}} F_{h,\pi_n}$, which remains countable.

Since A and H are separable, for each $i \in F$, the images $\mathrm{Im}(T_i(\pi_0))$ form a countable subset of the non-separable space K_{π_0}. Let $K_0 = \bigcup_{i\in F} \mathrm{Im}(T_i(\pi_0))$, which is countable. Hence, there exists $0 \neq y_{\pi_0} \in K_{\pi_0}$ orthogonal to K_0. Then $T_i^*(\pi_0)y_{\pi_0} = 0$ for all $i \in F$, and $T_i(\pi_0)(h) = 0$ for all $i \in I \setminus F$ and $h \in H$.

Since $\pi_0 \in \overline{W}$ and $\pi \mapsto T_i^*(\pi)T(\pi)$ is wot-continuous, we deduce $\langle y_{\pi_0}, T_i(\pi_0)(h)\rangle = 0$ for all $i \in I$, forcing $y_{\pi_0} = 0$, a contradiction. Therefore, Γ admits no frame. □

Remark 2.3. Let A be a separable C^*-algebra, H a separable infinite-dimensional Hilbert space, and $(\{K_\pi\}_{\pi\in \mathrm{Irr}(A:H)}, \Gamma)$ an operator bundle. By [2, Theorem 2.3], $K_\pi \cong K_{\pi'}$ whenever π and π' are unitarily equivalent. Hence, if all irreducible representations of A are unitarily equivalent, no operator bundle satisfies the conditions of the above theorem. It is known that such a separable C^*-algebra is $*$-isomorphic to $K(\ell^2)$ [12]. This observation refines the conjecture in [1], confirming that if every Hilbert C^*-module over A admits a frame, then A must be a compact C^*-algebra.

Example 2.4 (Operator Bundle over a Commutative Unital C^*-Algebra)**.** Let $A = C([0, 1])$. For any irreducible $*$-representation

$$\pi\colon A \to \mathcal{B}(H),$$

the Hilbert space H is one-dimensional. Consequently, $H \cong \mathbb{C}$ and π is given by evaluation at a point $x \in [0, 1]$, that is, $\pi = \delta_x$, where

$$\delta_x(f) = f(x), \quad (f \in C([0, 1])).$$

Hence, the set of irreducible representations of A is naturally identified with $[0, 1]$.

By [11, Lemma 2.2], there exists a continuous field of Hilbert spaces

$$\{(K_\pi, \Gamma)\}_{\pi\in \mathrm{Irr}(A:H)}$$

with the following property: there exists a countable subset $W \subset \mathrm{Irr}(A; H)$ and a representation $\pi_0 \in \mathrm{Irr}(A; H) \setminus W$ such that K_π is separable for all $\pi \in W$, while K_{π_0} is non-separable.

Since, K_π is isomorphic to $B(\mathbb{C}, K_\pi)$, for all $\pi \in \mathrm{Irr}(A : H)$, this continuous field of Hilbert spaces forms an operator bundle satisfying conditions of Proposition 3.2. However, according to [11, Lemma 3.2] the continuous field of Hilbert spaces (operator bundle)

$$\{(K_\pi, \Gamma)\}_{\pi\in \mathrm{Irr}(A:H)}$$

admits no frames.

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