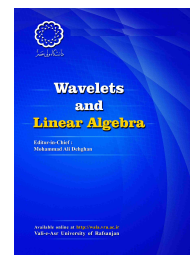


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A Robust Optimization Approach

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ABSTRACT

The theory of constrained best approximation in Hilbert spaces has been systematically developed well over a decade and effective characterizations of best approximations have been known under some qualifications on the constraints. Yet, the existing theory does not explain how to characterize a best approximation in the face of data uncertainty in the constraints, despite the reality that the data of the constraints are often uncertain (that is, they are not known exactly) due to estimation errors, prediction errors or lack of information. This paper explains when the best approximation over uncertain linear constraints in a real Hilbert space is immunized against bounded data uncertainty. This study is done by characterizing the best approximation of the robust counterpart of the uncertain constrained best approximation problem where the constraints are enforced for all possible uncertainties within the prescribed uncertainty sets. We show that under a new robust strong conical hull intersection property (robust strong CHIP) the same kind of effective characterizations of constrained best approximation hold for the robust best approximation that is immunized against bounded data uncertainty. We also establish a strong duality theorem for the robust constrained best approximation problem and its associated dual problem under the robust strong CHIP. Some examples are given to illustrate the obtained results.

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1. Introduction

Solutions to constrained best approximation problems [8, 9, 10, 11, 12, 13, 15] are typically characterized in terms of qualifications on the constraints. Traditional constrained best approximation studies assume perfect information (that is, accurate values for the input quantities, data or parameters), despite the reality that such precise knowledge is rarely available for practical problems. The data of practical problems are often uncertain (that is, they are not known exactly) because of estimation or prediction errors, or lack of information.

Recently, robust optimization approach has been successfully used to find solutions to finite-dimensional constrained optimization problems under uncertainty [3, 4, 5, 6, 14]. In this approach, a robust solution that is immunized against data uncertainty is found by choosing a solution to its robust counterpart, where the constraints are enforced for all possible uncertainties within the prescribed bounded uncertainty sets. It has now emerged as a computationally attractive approach for dealing with optimization problems that are affected by data uncertainty.

The apparent success of this robust optimization approach presents a challenge to the field of constrained best approximation [7, 15] where the optimization takes place over infinite dimensional spaces. The challenge, for instance, in constrained best approximation, is to explain how to characterize and find uncertainty immunized best approximations in the face of data uncertainty in the constraints. This problem provides the stimulus for the present study.

In this paper, we explain when a best approximation over uncertain linear constraints in a real Hilbert space is robust in the sense that it is immunized against bounded data uncertainty. We provide explanations by characterizing a best approximation of the robust counterpart of an uncertain constrained best approximation model problem within the framework of robust optimization and, in some cases, by finding it in terms of the solution of an associated dual problem.

Our problem in the present work is to characterize the robust best approximation of a point $x \in X$ from the intersection of a closed convex subset C of a real Hilbert space X and the set of uncertain linear constraints:

$$D_{(A,b)} := \{z \in X : Az \leq b\},$$

where the data (A, b) are uncertain and they belong to the closed and convex uncertainty set \mathcal{U} of $L(X, Y) \times Y$, where $L(X, Y)$ is the space of all continuous linear mappings from X to Y , Y is a real Hilbert space and Y is equipped with a nonempty closed convex pointed cone S . The robust best approximation x_0 of the point x from the set that is immunized against the data uncertainty is the solution to the robust counterpart of the uncertain best approximation problem, called the robust best approximation problem:

$$\min_{z \in X} \{\|x - z\| : z \in C \cap D_{\mathcal{U}}\}, \quad (1.1)$$

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where the set

$$D_{\mathcal{U}} := \{z \in X : Az \leq b, \forall (A, b) \in \mathcal{U}\}$$

ensures that the linear inequality constraint is realized for all possible uncertainties in the prescribed uncertainty set \mathcal{U} .

The strong CHIP [7, 8] is the key property that characterizes the constrained best approximation in the absence of data uncertainty in the constraints. We show that under a new robust strong conical hull intersection property (robust strong CHIP) the same kind of effective characterizations of constrained best approximation hold for the robust best approximation under bounded data uncertainty. We further establish a strong duality theorem for the robust constrained best approximation problem and its associated dual problem under the robust strong CHIP. Consequently, in the case of the ellipsoidal uncertainty set, we show how a robust best approximation can easily be obtained from the solution of a simple finite-dimensional dual problem.

The outline of the paper is as follows. Section 2 provides preliminary results on convex analysis that will be used later in the paper. Section 3 presents characterizations of the robust best approximation in terms of a robust strong CHIP. In Section 4, we give duality results for robust constrained best approximation problems. By given examples we illustrate our results.

2. Preliminaries

We begin this section by fixing the notations and definitions that will be used later. Throughout the paper, let X and Y be real Hilbert spaces. Assume that Y is equipped with a nonempty closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). We say that

$$x \leq y \text{ or } y \geq x \text{ if and only if } y - x \in S.$$

One can easily check that “ \leq ” is a partial order on Y generated by S , and (Y, \leq) is an ordered Hilbert space. We denote by $\langle \cdot, \cdot \rangle$ the inner product of X or Y , and the induced norm is denoted by $\|\cdot\|$ and is defined by $\|x\| := \sqrt{\langle x, x \rangle}$ for each $x \in X$ (or for each $x \in Y$). Put

$$S^+ := \{\lambda \in Y : \langle y, \lambda \rangle \geq 0, \forall y \in S\},$$

where $S^+ \subseteq Y$ is called the positive dual cone (positive polar cone) of S .

We use $L(X, Y) := \{A : X \rightarrow Y : A \text{ is a continuous linear mapping}\}$ to denote the space of all continuous linear mappings from X to Y . Let \mathcal{U} be a closed and convex uncertainty set of $L(X, Y) \times Y$, and

$$D_{\mathcal{U}} := \{z \in X : Az \leq b, \forall (A, b) \in \mathcal{U}\}.$$

It is clear that $D_{\mathcal{U}}$ is a closed and convex set in X . For a linear mapping $A \in L(X, Y)$, we define the adjoint of A by:

$$A^* : Y \rightarrow X, \langle A^*y, x \rangle := \langle y, Ax \rangle, \forall x \in X, \forall y \in Y. \quad (2.2)$$

For a subset W of X , the weak closure (resp., closure) of W will be denoted by $w\text{-cl } W$ (resp., $\text{cl } W$). We denote by $\text{int } A$ the interior of A , where A is a subset of X . For a nonempty subset D of X , the indicator function $\delta_D : X \rightarrow [0, +\infty]$ is defined by:

$$\delta_D(x) := \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{otherwise.} \end{cases}$$

The support function of D , $\sigma_D : X \rightarrow (-\infty, +\infty]$ is defined by:

$$\sigma_D(u) := \sup_{x \in D} \langle x, u \rangle, \quad (u \in X).$$

Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The epigraph of f , $\text{epi } f$, is defined by:

$$\text{epi } f := \{(x, r) \in X \times \mathbb{R} : x \in \text{dom } f, f(x) \leq r\},$$

where the effective domain of f , $\text{dom } f$, is defined by:

$$\text{dom } f := \{x \in X : f(x) < +\infty\}.$$

The conjugate function of f is defined by $f^* : X \rightarrow \mathbb{R} \cup \{+\infty\}$:

$$f^*(u) := \sup_{x \in \text{dom } f} \{\langle x, u \rangle - f(x)\}, \quad (u \in X).$$

Clearly, f^* is a proper lower semicontinuous convex function and $\alpha \text{epi } f^* = \text{epi}(\alpha f)^*$ for any $\alpha > 0$, where αf is defined by $(\alpha f)(x) := \alpha f(x)$ for each $x \in X$. If $f_1, f_2 : X \rightarrow (-\infty, +\infty]$ are two proper lower semicontinuous convex functions, then we have

$$\text{epi}(f_1 + f_2)^* = w\text{-cl}(\text{epi } f_1^* + \text{epi } f_2^*). \quad (2.3)$$

Furthermore, the closure is superfluous if $(\text{int dom } f_1) \cap \text{dom } f_2 \neq \emptyset$. For more details see [16].

For a nonempty subset W of X , we define the negative dual cone (negative polar cone) of W by:

$$W^\circ := \{u \in X : \langle w, u \rangle \leq 0, \forall w \in W\},$$

and the convex hull (resp., conical hull) of W denoted by $\text{co } W$ (resp., $\text{cone } W$) which is the intersection of all convex sets (resp., convex cones) containing W , and they are defined by:

$$\text{co } W := \left\{ \sum_{i=1}^m \lambda_i w_i : w_i \in W, \lambda_i \geq 0, i = 1, 2, \dots, m, \sum_{i=1}^m \lambda_i = 1, m \in \mathbb{N} \right\},$$

$$\text{cone } W := \left\{ \sum_{i=1}^m \lambda_i w_i : w_i \in W, \lambda_i \geq 0, i = 1, 2, \dots, m, m \in \mathbb{N} \right\}.$$

Lemma 2.1. (cf. [16]) *Let I be an arbitrary index set and let $f_i : X \rightarrow (-\infty, +\infty]$, $i \in I$, be a proper lower semicontinuous convex function. Suppose that there exists $x_0 \in X$ such that $\sup_{i \in I} f_i(x_0) < +\infty$. Then*

$$\text{epi}(\sup_{i \in I} f_i)^* = w\text{-cl}(\text{co} \bigcup_{i \in I} \text{epi} f_i^*),$$

where $\sup_{i \in I} f_i : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i \in I} f_i)(x) = \sup_{i \in I} f_i(x)$ for each $x \in X$.

The nonnegative orthant of \mathbb{R}^n is denoted by \mathbb{R}_+^n and is defined by:

$$\mathbb{R}_+^n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, 2, \dots, n\}.$$

For a proper convex function $f : X \rightarrow (-\infty, +\infty]$, the subdifferential of f at a point $x \in \text{dom}(f)$, denoted by, $\partial f(x)$, and is defined by:

$$\partial f(x) := \{u \in X : f(y) \geq f(x) + \langle y - x, u \rangle, \forall y \in X\},$$

and if $x \notin \text{dom}(f)$, we define $\partial f(x) := \emptyset$. It is well known that $\partial f(x) \neq \emptyset$ for all $x \in X$ if f is a continuous convex function on X . For a nonempty subset W of X and $x \in X$, we define $d(x, W) := \inf_{w \in W} \|x - w\|$. A point $w_0 \in W$ is called a best approximation for a point $x \in X$ (i.e., $w_0 \in P_W(x)$) (see [15]), if

$$d(x, W) = \|x - w_0\|,$$

where $P_W(x)$ is defined by:

$$P_W(x) := \{w \in W : \|x - w\| = d(x, W)\}.$$

3. Robust Best Approximations: Characterizations

In this section, let X and Y be real Hilbert spaces. We introduce a robust version of the normal cone intersection formula, called robust conical hull intersection property (robust strong CHIP, in short). This property is used to provide characterizations for the robust best approximations. Let C be a nonempty closed convex subset of the real Hilbert space X , and

$$D_{\mathcal{U}} := \{x \in X : Ax \leq b, \forall (A, b) \in \mathcal{U}\}, \tag{3.4}$$

which is also a closed convex set in X , where $\mathcal{U} \subset L(X, Y) \times Y$.

Now, let

$$\mathcal{F}(x) := \{(A, b), \lambda \in \mathcal{U} \times S^+ : \langle Ax - b, \lambda \rangle = 0\}, (x \in X). \tag{3.5}$$

Definition 3.1. *The pair $\{C, D_{\mathcal{U}}\}$ is said to have the robust strong conical hull intersection property (robust strong CHIP, in short) at a point $x_0 \in K_{\mathcal{U}} := C \cap D_{\mathcal{U}} \neq \emptyset$, if*

$$(K_{\mathcal{U}} - x_0)^\circ = (C - x_0)^\circ + \bigcup_{((A,b),\lambda) \in \mathcal{F}(x_0)} \{A^*(\lambda)\}. \tag{3.6}$$

Lemma 3.1. *If the pair $\{C, D_{\mathcal{U}}\}$ has the robust strong CHIP at a point $x_0 \in K_{\mathcal{U}}$, then, the pair $\{C, D_{\mathcal{U}}\}$ has the strong conical hull intersection property (strong CHIP, in short) at the point x_0 , i.e.,*

$$(K_{\mathcal{U}} - x_0)^{\circ} = (C - x_0)^{\circ} + (D_{\mathcal{U}} - x_0)^{\circ}.$$

Proof. Let

$$x^* \in \bigcup_{((A,b),\lambda) \in \mathcal{F}(x_0)} \{A^*(\lambda)\}$$

be arbitrary. Thus, there exists $((\bar{A}, \bar{b}), \bar{\lambda}) \in \mathcal{F}(x_0)$ such that $x^* = (\bar{A})^*(\bar{\lambda})$. Now, let $x \in D_{\mathcal{U}}$ be arbitrary. Therefore, in view of (3.4), $Ax \leq b$ for all $(A, b) \in \mathcal{U}$, and so, in particular, $\bar{A}x \leq \bar{b}$. Hence, since $\bar{\lambda} \in S^+$, it follows that

$$\langle \bar{A}x - \bar{b}, \bar{\lambda} \rangle \leq 0, \forall x \in D_{\mathcal{U}}. \tag{3.7}$$

Since $((\bar{A}, \bar{b}), \bar{\lambda}) \in \mathcal{F}(x_0)$, we conclude from (3.5) that $\langle \bar{A}x_0 - \bar{b}, \bar{\lambda} \rangle = 0$. This together with (3.7) and the fact that $x^* = (\bar{A})^*(\bar{\lambda})$ implies that

$$\begin{aligned} \langle x - x_0, x^* \rangle &= \langle \bar{A}x - \bar{A}x_0, \bar{\lambda} \rangle \\ &= \langle \bar{A}x - \bar{b}, \bar{\lambda} \rangle - \langle \bar{A}x_0 - \bar{b}, \bar{\lambda} \rangle \\ &\leq 0, \forall x \in D_{\mathcal{U}}. \end{aligned}$$

Hence, $x^* \in (D_{\mathcal{U}} - x_0)^{\circ}$. Thus, one has

$$\bigcup_{((A,b),\lambda) \in \mathcal{F}(x_0)} \{A^*(\lambda)\} \subseteq (D_{\mathcal{U}} - x_0)^{\circ}. \tag{3.8}$$

Since we always have

$$(C - x_0)^{\circ} + (D_{\mathcal{U}} - x_0)^{\circ} \subseteq (K_{\mathcal{U}} - x_0)^{\circ},$$

the result follows from Definition 3.1. □

It is worth nothing that the converse statement to Lemma 3.1 is not true. See the following example.

Example 3.1. *Let $X := \mathbb{R}^2, Y := \mathbb{R}^2, S := \mathbb{R}_+^2, C := \mathbb{R}^2$ and $\mathcal{U} := \mathcal{V} \times \mathcal{W}$, where*

$$\mathcal{V} := \left\{ A \in \mathbb{R}^{2 \times 2} : A := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in [-1, 1] \right\} \text{ and } \mathcal{W} := \{(0, 0)\}.$$

Let $x_0 = (0, 0)$. Then, $(C - x_0)^{\circ} = \{(0, 0)\}$ and

$$C \cap D_{\mathcal{U}} = \mathbb{R}^2 \cap \{x \in X : Ax \leq b, \forall (A, b) \in \mathcal{U}\} = \{(0, 0)\},$$

and so, $(C \cap D_{\mathcal{U}} - x_0)^\circ = \mathbb{R}^2$. Note that one has $S^+ = \mathbb{R}_+^2$, and

$$\begin{aligned} & \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle Ax_0 - b, \lambda \rangle = 0\} \\ &= \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} : a \in [-1, 1], \lambda := (\lambda_1, \lambda_2) \in S^+ \right\} \\ &= \mathbb{R}_+^2 \cup (-\mathbb{R}_+^2). \end{aligned}$$

Therefore, since $(D_{\mathcal{U}} - x_0)^\circ$ is closed and convex, by an argument similar to the proof of (3.8), it follows that

$$\begin{aligned} (C \cap D_{\mathcal{U}} - x_0)^\circ = \mathbb{R}^2 &= \{(0, 0)\} + \text{cl co}(\mathbb{R}_+^2 \cup (-\mathbb{R}_+^2)) \\ &= (C - x_0)^\circ + \text{cl co}\{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+ \text{ and} \\ &\quad \langle Ax_0 - b, \lambda \rangle = 0\} \\ &\subseteq (C - x_0)^\circ + (D_{\mathcal{U}} - x_0)^\circ \\ &\subseteq (C \cap D_{\mathcal{U}} - x_0)^\circ. \end{aligned} \tag{3.9}$$

Hence,

$$(C \cap D_{\mathcal{U}} - x_0)^\circ = (C - x_0)^\circ + (D_{\mathcal{U}} - x_0)^\circ.$$

That is, the pair $\{C, D_{\mathcal{U}}\}$ has the strong CHIP at x_0 . But, the robust strong CHIP fails at the point x_0 , because in view of (3.9), one has

$$\begin{aligned} (C - x_0)^\circ + \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle Ax_0 - b, \lambda \rangle = 0\} &= \mathbb{R}_+^2 \cup (-\mathbb{R}_+^2) \\ &\neq \mathbb{R}^2 \\ &= (C \cap D_{\mathcal{U}} - x_0)^\circ. \end{aligned}$$

However, in the case of uncertainty-free data, where $\mathcal{U} = \{(\bar{A}, \bar{b})\}$, the robust strong CHIP reduces to the strong CHIP. To see this, note that

$$\begin{aligned} & \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle Ax_0 - b, \lambda \rangle = 0\} \\ &= \{(\bar{A})^*(\lambda) : \lambda \in S^+ \text{ and } \langle \bar{A}x_0 - \bar{b}, \lambda \rangle = 0\} \\ &= (D_{\mathcal{U}} - x_0)^\circ, \end{aligned}$$

where $D_{\mathcal{U}} = \{x \in X : \bar{A}x \leq \bar{b}\}$. So, in view of Definition 3.1,

$$(C \cap D_{\mathcal{U}} - x_0)^\circ = (C - x_0)^\circ + (D_{\mathcal{U}} - x_0)^\circ.$$

We now define the ‘‘characteristic cone M ’’ by:

$$M := \text{epi}\delta_C^* + \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle). \tag{3.10}$$

Remark 3.1. It should be noted that δ_C^* is the conjugate function of the indicator function δ_C , and we have $\delta_C^* = \sigma_C$, where σ_C is called the support function of C and is defined by:

$$\sigma_C : X \longrightarrow (-\infty, +\infty], \sigma_C(u) := \sup_{y \in C} \langle u, y \rangle, (u \in X).$$

As we establish below, we show that the pair $\{C, D_{\mathcal{U}}\}$ has the robust strong CHIP at a point $x \in X$, whenever the characteristic cone M is weakly closed and convex in $X \times \mathbb{R}$.

Remark 3.2. If A and B are two sets in a real vector space Z such that A is convex, then,

$$A + \text{co}(B) = \text{co}(A + B).$$

Theorem 3.1. Suppose that the characteristic cone M defined by (3.10) is weakly closed and convex. Then, for each $x \in C \cap D_{\mathcal{U}}$, the pair $\{C, D_{\mathcal{U}}\}$ has the robust strong CHIP at the point x .

Proof. Let $K_{\mathcal{U}} = C \cap D_{\mathcal{U}}$. Since $\delta_{K_{\mathcal{U}}} = \delta_C + \delta_{D_{\mathcal{U}}}$ and $\text{epi}\delta_C^*$ is a convex set, by using Lemma 2.1 and Remark 3.1, we observe that

$$\begin{aligned} \text{epi}\delta_{K_{\mathcal{U}}}^* &= w\text{-cl}(\text{epi}\delta_C^* + \text{epi}\delta_{D_{\mathcal{U}}}^*) \\ &= w\text{-cl}(\text{epi}\delta_C^* + w\text{-cl} \text{co}(\bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) + \{0\} \times \mathbb{R}_+)) \\ &= w\text{-cl}(\text{epi}\delta_C^* + \text{co}(\bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) + \{0\} \times \mathbb{R}_+)) \\ &= w\text{-cl} \text{co}(\text{epi}\delta_C^* + \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) + \{0\} \times \mathbb{R}_+) \\ &= w\text{-cl} \text{co}(\text{epi}\delta_C^* + \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle)). \end{aligned}$$

Hence, since M is weakly closed and convex, one has

$$\begin{aligned} M &= w\text{-cl} \text{co}M \\ &= w\text{-cl} \text{co}(\text{epi}\delta_C^* + \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle)) \\ &= \text{epi}\delta_{K_{\mathcal{U}}}^*. \end{aligned}$$

Now, let $x \in K_{\mathcal{U}}$ be arbitrary. Then, for each $a \in (K_{\mathcal{U}} - x)^\circ$, we have

$$\langle a, k - x \rangle \leq 0, \forall k \in K_{\mathcal{U}}.$$

This implies that $(a, \langle a, x \rangle) \in \text{epi}\delta_{K_{\mathcal{U}}}^* = M$. Hence, it follows that

$$(a, \langle a, x \rangle) \in \text{epi}\delta_C^* + \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle).$$

So, there exist $(b, \beta) \in \text{epi}\delta_C^*$, $(A, b') \in \mathcal{U}$ and $\lambda \in S^+$ such that

$$(a, \langle a, x \rangle) = (b, \beta) + (A^*(\lambda), \langle b', \lambda \rangle).$$

This gives us that, for all $z \in C$:

$$\begin{aligned} \langle b, z \rangle \leq \beta &= \langle a, x \rangle - \langle b', \lambda \rangle \\ &= \langle b + A^*(\lambda), x \rangle - \langle b', \lambda \rangle \\ &= \langle b, x \rangle + \langle \lambda, Ax - b' \rangle \leq \langle b, x \rangle. \end{aligned} \tag{3.11}$$

(It should be noted that for satisfying the last inequality in (3.11) one has: since $x \in K_{\mathcal{U}} = C \cap D_{\mathcal{U}} \subseteq D_{\mathcal{U}}$ and $(A, b') \in \mathcal{U}$, it follows from (3.4) that $Ax \leq b'$. Now, since $\lambda \in S^+$, it follows that $\langle \lambda, Ax - b' \rangle \leq 0$.) Thus, we get from (3.11) that $b \in (C - x)^\circ$. Moreover, letting $z := x$ in (3.11), we conclude that $\langle \lambda, Ax - b' \rangle = 0$. Then, we see that

$$a = b + A^*(\lambda) \in (C - x)^\circ + \{A^*(\lambda) : (A, b') \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle \lambda, Ax - b' \rangle = 0\}.$$

Therefore,

$$(K_{\mathcal{U}} - x)^\circ \subseteq (C - x)^\circ + \{A^*(\lambda) : (A, b') \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle \lambda, Ax - b' \rangle = 0\}.$$

Note that the reverse inclusion always holds, because in view of (3.8), we have

$$\begin{aligned} &(C - x)^\circ + \{A^*(\lambda) : (A, b') \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle \lambda, Ax - b' \rangle = 0\} \\ &\subseteq (C - x)^\circ + (D_{\mathcal{U}} - x)^\circ \\ &\subseteq (K_{\mathcal{U}} - x)^\circ. \end{aligned} \tag{3.12}$$

So, the conclusion follows. □

In the sequel, we show that the robust strong CHIP is a necessary and sufficient condition for characterizing the robust best approximation in terms of the so-called perturbation property.

Recall first that the basic characterization property of the best approximation of a point $x \in X$ from a closed convex set W in the Hilbert space X is that

$$w_0 = P_W(x) \text{ if and only if } x - w_0 \in (W - w_0)^\circ, \tag{3.13}$$

where $w_0 \in W$ (for more details, see [8, Theorem 4.1]). Recall also that for closed and convex sets C and D with $x \in C \cap D \neq \emptyset$, we always have

$$(C - x)^\circ + (D - x)^\circ \subseteq (C \cap D - x)^\circ. \tag{3.14}$$

Theorem 3.2. *Let $x_0 \in K_{\mathcal{U}} = C \cap D_{\mathcal{U}}$ be given. Then, the following assertions are equivalent:*

- (i) *The pair $\{C, D_{\mathcal{U}}\}$ has the robust strong CHIP at the point x_0 .*

(ii) For any $x \in X$,

$$x_0 = P_{K_{\mathcal{U}}}(x) \iff x_0 = P_C(x - A^*(\lambda)) \text{ for some } (A, b) \in \mathcal{U} \\ \text{and some } \lambda \in S^+ \text{ with } \langle \lambda, Ax_0 - b \rangle = 0.$$

Proof. [(i) \implies (ii)]. Suppose that the robust strong CHIP property holds at x_0 . For any $x \in X$ and $x_0 = P_{K_{\mathcal{U}}}(x)$, by the basic characterization property the equation (3.13) and robust strong CHIP, we have

$$x - x_0 \in (K_{\mathcal{U}} - x_0)^\circ = (C - x_0)^\circ + \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+, \langle \lambda, Ax_0 - b \rangle = 0\}.$$

Thus, there exist $(A, b) \in \mathcal{U}, \lambda \in S^+$ with $\langle \lambda, Ax_0 - b \rangle = 0$ such that $x - A^*(\lambda) - x_0 \in (C - x_0)^\circ$. Applying the basic characterization property the equation (3.13), again, we see that $x_0 = P_C(x - A^*(\lambda))$. Therefore, the following implication holds.

$$x_0 = P_{K_{\mathcal{U}}}(x) \implies x_0 = P_C(x - A^*(\lambda)) \text{ for some } (A, b) \in \mathcal{U} \\ \text{and some } \lambda \in S^+ \text{ with } \langle \lambda, Ax_0 - b \rangle = 0.$$

To see the reverse implication, let

$$M(x_0) := \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+, \langle \lambda, Ax_0 - b \rangle = 0\}.$$

Assume that $x_0 = P_C(x - l)$ for some $l \in M(x_0)$. Then, by the basic characterization property (3.13), $x - l - x_0 \in (C - x_0)^\circ$. We now show that

$$M(x_0) \subseteq (D_{\mathcal{U}} - x_0)^\circ. \tag{3.15}$$

To verify this, let $a \in M(x_0)$. Then, there exist $(\bar{A}, \bar{b}) \in \mathcal{U}, \lambda \in S^+$ with $\langle \lambda, \bar{A}x_0 - \bar{b} \rangle = 0$ such that $a = (\bar{A})^*(\lambda)$. So, for each $x \in D_{\mathcal{U}}$, we have

$$\langle a, x - x_0 \rangle = \langle (\bar{A})^*(\lambda), x - x_0 \rangle = \langle \lambda, \bar{A}x - \bar{A}x_0 \rangle = \langle \lambda, \bar{A}x - \bar{b} \rangle \leq 0.$$

Thus, (3.15) holds. Therefore, it follows that

$$x - x_0 \in (C - x_0)^\circ + l \subseteq (C - x_0)^\circ + M(x_0) \\ \subseteq (C - x_0)^\circ + (D_{\mathcal{U}} - x_0)^\circ \\ \subseteq (K_{\mathcal{U}} - x_0)^\circ.$$

Hence, $x - x_0 \in (K_{\mathcal{U}} - x_0)^\circ$. Again, by the basic characterization property (3.13), we get that $x_0 = P_{K_{\mathcal{U}}}(x)$. So, (ii) follows.

[(ii) \implies (i)]. In view of (3.12), we first observe that

$$(C - x_0)^\circ + \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle \lambda, Ax_0 - b \rangle = 0\} \subseteq (K_{\mathcal{U}} - x_0)^\circ.$$

Now, to see the reverse inclusion, let $a \in (K_{\mathcal{U}} - x_0)^\circ$. Then, $P_{K_{\mathcal{U}}}(x) = x_0$, where $x := x_0 + a$. Thus (ii) gives us that

$$x_0 = P_C(x - A^*(\lambda)),$$

for some $(A, b) \in \mathcal{U}$ and some $\lambda \in S^+$ with $\langle \lambda, Ax_0 - b \rangle = 0$. Then, the basic characterization property (3.13) implies that

$$a - A^*(\lambda) = x - A^*(\lambda) - x_0 \in (C - x_0)^\circ.$$

Thus,

$$a = (a - A^*(\lambda)) + A^*(\lambda) \in (C - x_0)^\circ + \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+, \langle \lambda, Ax_0 - b \rangle = 0\}.$$

Therefore, the conclusion follows. □

In the uncertainty-free case where \mathcal{U} is a singleton $\{(\bar{A}, \bar{b})\}$, Theorem 3.2 reduces to the following characterization of the strong CHIP.

Corollary 3.1. *Let $x_0 \in K := C \cap D$, where $D := \{x \in X : \bar{A}x \leq \bar{b}\}$. Then, the following assertions are equivalent:*

- (i) *The pair $\{C, D\}$ has the strong CHIP at the point x_0 .*
- (ii) *For any $x \in X$,*

$$x_0 = P_K(x) \iff x_0 = P_C(x - (\bar{A})^*(\lambda)) \text{ for some } \lambda \in S^+ \text{ with } \langle \lambda, \bar{A}x_0 - \bar{b} \rangle = 0.$$

Proof. Let $\mathcal{U} := \{(\bar{A}, \bar{b})\}$ in Theorem 3.2. Then, the conclusion follows from Theorem 3.2. □

We illustrate that the characterizations of the robust best approximations hold under a concave-like condition whenever the characteristic cone M which defined by (3.10) is weakly closed.

Theorem 3.3. *Let $x_0 \in K_{\mathcal{U}} = C \cap D_{\mathcal{U}}$ be arbitrary. Assume that the characteristic cone M defined by (3.10) is weakly closed and the concave-like condition:*

$$\begin{aligned} & (\forall ((A_1, b_1), \lambda_1), ((A_2, b_2), \lambda_2) \in \mathcal{U} \times S^+, \mu \in [0, 1]) (\exists ((A_3, b_3), \lambda_3) \in \mathcal{U} \times S^+) \\ & \text{such that } \langle A_3x - b_3, \lambda_3 \rangle \geq \mu \langle A_1x - b_1, \lambda_1 \rangle + (1 - \mu) \langle A_2x - b_2, \lambda_2 \rangle, \\ & \forall x \in X, \end{aligned} \tag{3.16}$$

holds. Then, for any $x \in X$,

$$x_0 = P_{K_{\mathcal{U}}}(x) \iff x_0 = P_C(x - A^*(\lambda)) \text{ for some } (A, b) \in \mathcal{U} \text{ and some } \lambda \in S^+ \text{ with } \langle \lambda, Ax_0 - b \rangle = 0.$$

Proof. The conclusion will follow from Theorem 3.1 and Theorem 3.2 if we show that the characteristic cone M is convex. In fact, we show that the hypothesis: $((A, b), \lambda) \mapsto \langle Ax - b, \lambda \rangle$ is concave-like on $\mathcal{U} \times S^+$ for each $x \in X$, ensures that M is convex. To this end, note that one has

$$\begin{aligned} M &= \text{epi}\delta_C^* + \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) \\ &= \text{epi}\delta_C^* + \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) + (\{0\} \times \mathbb{R}_+). \end{aligned}$$

Since $\text{epi}\delta_C^*$ is convex, it is enough to show that the set

$$\bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) + (\{0\} \times \mathbb{R}_+)$$

is convex. Now, let

$$(a_1, \alpha_1), (a_2, \alpha_2) \in \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) + (\{0\} \times \mathbb{R}_+)$$

and $\mu \in [0, 1]$ be arbitrary. Then, there exist $((A_1, b_1), \lambda_1) \in \mathcal{U} \times S^+$ and $((A_2, b_2), \lambda_2) \in \mathcal{U} \times S^+$ such that

$$a_i = A_i^*(\lambda_i) \text{ and } \alpha_i \geq \langle b_i, \lambda_i \rangle, \quad i = 1, 2. \tag{3.17}$$

So, for this $\mu \in [0, 1]$, in view of the hypothesis, we can find $((A_3, b_3), \lambda_3) \in \mathcal{U} \times S^+$ such that

$$\begin{aligned} &\mu(\langle a_1, x \rangle - \langle b_1, \lambda_1 \rangle) + (1 - \mu)(\langle a_2, x \rangle - \langle b_2, \lambda_2 \rangle) \\ &= \mu(\langle \lambda_1, A_1 x - b_1 \rangle) + (1 - \mu)(\langle \lambda_2, A_2 x - b_2 \rangle) \\ &\leq \langle A_3 x - b_3, \lambda_3 \rangle \\ &= \langle A_3^*(\lambda_3), x \rangle - \langle b_3, \lambda_3 \rangle, \quad \forall x \in X. \end{aligned}$$

This gives us that

$$\langle \mu a_1 + (1 - \mu)a_2 - A_3^*(\lambda_3), x \rangle \leq \mu \langle b_1, \lambda_1 \rangle + (1 - \mu) \langle b_2, \lambda_2 \rangle - \langle b_3, \lambda_3 \rangle, \tag{3.18}$$

for all $x \in X$.

Put $x = 0$ in (3.18), then,

$$\mu \langle b_1, \lambda_1 \rangle + (1 - \mu) \langle b_2, \lambda_2 \rangle \geq \langle b_3, \lambda_3 \rangle. \tag{3.19}$$

This together with the fact that $\alpha_i \geq \langle b_i, \lambda_i \rangle$ ($i = 1, 2$) implies that

$$\mu \alpha_1 + (1 - \mu) \alpha_2 \geq \langle b_3, \lambda_3 \rangle. \tag{3.20}$$

Let $\gamma > 0$ be an arbitrary positive real number. By putting $x := \gamma[\mu a_1 + (1 - \mu)a_2 - A_3^*(\lambda_3)]$ in (3.18), and letting $\gamma \rightarrow +\infty$, we conclude that

$$\mu a_1 + (1 - \mu)a_2 = A_3^*(\lambda_3). \tag{3.21}$$

Therefore, (3.20) and (3.21) implies that

$$\mu(a_1, \alpha_1) + (1 - \mu)(a_2, \alpha_2) \in \bigcup_{(A,b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle) + (\{0\} \times \mathbb{R}_+).$$

Thus, M is convex. □

Remark 3.3. *It is worth noting that if $\text{int}(S) \neq \emptyset$, $s \in \text{int}(S)$ and $\lambda \in S^+ \setminus \{0\}$, then, $\langle \lambda, s \rangle > 0$.*

In the following, we show that if $Y := \mathbb{R}^n$, then the characterizations of the robust best approximations hold under a strict feasibility-type condition and a concave-like condition.

Theorem 3.4. *Suppose that $Y = \mathbb{R}^n$ and \mathcal{U} is closed, bounded and convex. Let $x_0 \in K_{\mathcal{U}} = C \cap D_{\mathcal{U}}$ be given and let, for some $\bar{x} \in C$, $b - A\bar{x} \in \text{int}(S)$ for each $(A, b) \in \mathcal{U}$, where $S := \mathbb{R}_+^n$. Assume that the concave-like condition which defined by (3.16), holds. Then, for any $x \in X$,*

$$x_0 = P_{K_{\mathcal{U}}}(x) \iff x_0 = P_C(x - A^*(\lambda)) \text{ for some } (A, b) \in \mathcal{U} \text{ and some } \lambda \in S^+ \text{ with } \langle \lambda, Ax_0 - b \rangle = 0.$$

Note that, in the case that $Y = \mathbb{R}^n$, we have $S^+ = \mathbb{R}_+^n$.

Proof. The conclusion will follow from Theorem 3.3 if we show that the characteristic cone M is closed. Indeed, we show that the existence of $\bar{x} \in C$ with $b - A\bar{x} \in \text{int}(S)$ for all $(A, b) \in \mathcal{U}$, guarantees that the characteristic cone M is closed (in view of Theorem 3.3, one has M is convex, then, M is weakly closed if and only if it is norm closed). To see this, let $(x_k^*, r_k) \in M$ with $\|(x_k^*, r_k) - (x^*, r)\| \rightarrow 0$ for some $(x^*, r) \in X \times \mathbb{R}$ (note that by the definition of the product norm, one has $\|x_k^* - x^*\| \rightarrow 0$ and $|r_k - r| \rightarrow 0$). Then, there exist $(y_k^*, \alpha_k) \in \text{epi}\delta_C^*$, $(A_k, b_k) \in \mathcal{U}$ and $\lambda_k \in S^+$ such that

$$(x_k^*, r_k) = (y_k^*, \alpha_k) + (A_k^*(\lambda_k), \langle b_k, \lambda_k \rangle), \forall k = 1, 2, \dots \tag{3.22}$$

Since, in view of the hypothesis, \mathcal{U} is bounded, so the sequences $\{A_k\}_{k \geq 1}$ and $\{b_k\}_{k \geq 1}$ are also bounded, and hence, the sequence $\{A_k^*\}_{k \geq 1}$ is bounded. Therefore, by using [1, Lemma 2.37] and (3.22), by passing subsequences, we may assume that there exist $A \in L(X, \mathbb{R}^n)$ and $b \in \mathbb{R}^n$ such that $A_k \rightharpoonup A$ and $b_k \rightharpoonup b$ (we use the notation \rightharpoonup for weak convergence, for more details, see [1, pages 33-37]). Thus, $(A_k, b_k) \rightharpoonup (A, b)$, and so, since by the hypothesis \mathcal{U} is weakly closed (note that \mathcal{U} is convex), we have $(A, b) \in \mathcal{U}$. On the other hand, we have $A_k \rightharpoonup A$, i.e., $A_k x \rightharpoonup Ax$ (if and only if $\|A_k x - Ax\| \rightarrow 0$) in \mathbb{R}^n for each $x \in X$. This implies that $A_k^*(y) \rightharpoonup A^*(y)$ in X for each $y \in \mathbb{R}^n$. Now, since for each $y \in \mathbb{R}^n$ the sequences $\{A_k^*(y)\}_{k \geq 1}$ and $\{b_k\}_{k \geq 1}$ are bounded and converge weakly, it follows that

$$\|A_k^*(y) - A^*(y)\| \rightarrow 0 \text{ for each } y \in \mathbb{R}^n, \text{ and } \|b_k - b\| \rightarrow 0. \tag{3.23}$$

We claim that the sequence $\{\lambda_k\}_{k \geq 1} \subset S^+$ is bounded. Otherwise, we have $\|\lambda_k\| \rightarrow +\infty$. Put $\gamma_k := \frac{\lambda_k}{\|\lambda_k\|}$ ($k = 1, 2, \dots$). Since $\{\gamma_k\}_{k \geq 1} \subset S^+$ is a bounded sequence, then, by passing a subsequence,

there exists $\lambda \in S^+ \setminus \{0\}$ such that $\|\gamma_k - \lambda\| \rightarrow 0$. Since $\|x_k^*\| \rightarrow \|x^*\|$, $|r_k| \rightarrow |r|$ and $\|\lambda_k\| \rightarrow +\infty$, it is easy to see that

$$\frac{\|x_k^*\|}{\|\lambda_k\|} \rightarrow 0 \text{ and } \frac{r_k}{\|\lambda_k\|} \rightarrow 0. \tag{3.24}$$

Since $(y_k^*, \alpha_k) \in \text{epi}\delta_C^*$ for all $k \geq 1$, and $\text{epi}\delta_C^*$ is a cone, one has

$$\left(\frac{y_k^*}{\|\lambda_k\|}, \frac{\alpha_k}{\|\lambda_k\|} \right) \in \text{epi}\delta_C^*, \forall k \geq 1. \tag{3.25}$$

Also, we have A^* is linear and continuous, it follows from $\|\gamma_k - \lambda\| \rightarrow 0$ that $\|A^*(\gamma_k) - A^*(\lambda)\| \rightarrow 0$. This together with (3.23) implies that

$$\|A_k^*(\gamma_k) - A^*(\lambda)\| \rightarrow 0. \tag{3.26}$$

Moreover, since $\|b_k - b\| \rightarrow 0$ and $\|\gamma_k - \lambda\| \rightarrow 0$, in view of [1, Lemma 2.41(iii)], we conclude that

$$|\langle b_k, \gamma_k \rangle - \langle b, \lambda \rangle| \rightarrow 0. \tag{3.27}$$

Now, by using (3.22), (3.24), (3.26) and (3.27) it is not difficult to show that

$$\left\| \left(\frac{y_k^*}{\|\lambda_k\|}, \frac{\alpha_k}{\|\lambda_k\|} \right) - [-(A^*(\lambda), \langle b, \lambda \rangle)] \right\| \rightarrow 0. \tag{3.28}$$

Since $\text{epi}\delta_C^*$ is norm closed, it follows from (3.25) and (3.28) that

$$-(A^*(\lambda), \langle b, \lambda \rangle) \in \text{epi}\delta_C^*. \tag{3.29}$$

So,

$$\langle \lambda, -A\bar{x} \rangle = \langle -A^*(\lambda), \bar{x} \rangle \leq -\langle b, \lambda \rangle. \tag{3.30}$$

Hence, we see that $\langle \lambda, b - A\bar{x} \rangle \leq 0$. This contradicts Remark 3.3, because $\lambda \in S^+ \setminus \{0\}$ and $b - A\bar{x} \in \text{int}(S)$. Therefore, $\{\lambda_k\}_{k \geq 1} \subset S^+$ is bounded. So, by passing a subsequence, there exists $\lambda \in S^+$ such that $\|\lambda_k - \lambda\| \rightarrow 0$. By an argument similar to the above (the assertions (3.23)-(3.28)) and using (3.22), one has

$$\begin{aligned} \|(y_k^*, \alpha_k) - [(x^*, r) - (A^*(\lambda), \langle b, \lambda \rangle)]\| &= \|[x_k^*, r_k] - (A_k^*(\lambda_k), \langle b_k, \lambda_k \rangle)\| - \\ &\quad \|[x^*, r] - (A^*(\lambda), \langle b, \lambda \rangle)\| \rightarrow 0. \end{aligned}$$

Since $\text{epi}\delta_C^*$ is norm closed, in view of (3.25)), we have

$$(x^*, r) - (A^*(\lambda), \langle b, \lambda \rangle) \in \text{epi}\delta_C^*,$$

and hence,

$$(x^*, r) \in \text{epi}\delta_C^* + (A^*(\lambda), \langle b, \lambda \rangle) \subseteq M.$$

Thus, M is closed, which completes the proof. □

Remark 3.4. *The proof of Theorem 3.4 shows that the robust Slater condition that there exists $\bar{x} \in C$ with $b - A\bar{x} \in \text{int}(S)$ for each $(A, b) \in \mathcal{U}$, guarantees that the characteristic cone M is closed. Moreover, the concave-like condition (3.16), ensures that the characteristic cone M is a convex set.*

4. Robust Min-Max Duality

In this section, X and Y are real Hilbert spaces. We establish a strong duality theorem between the robust constrained best approximation Problem (1.1) and its associated dual problem. Consequently, in the case of ellipsoidal uncertainty, we show how the robust best approximation can be found in terms of the solution of a simple dual second order cone programming problem.

Recall that $x_0 \in C \cap D_{\mathcal{U}}$ is a robust best approximation for the point $x \in X$ from the set $C \cap D_{\mathcal{U}}$ whenever x_0 solves the robust best approximation problem:

$$\min_{z \in X} \{\|x - z\| : z \in C \cap D_{\mathcal{U}}\},$$

where the set $D_{\mathcal{U}}$ is defined by:

$$D_{\mathcal{U}} := \{z \in X : Az \leq b, \forall (A, b) \in \mathcal{U}\}.$$

Equivalently, x_0 is a solution of the following problem:

$$(P) \quad \min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C \cap D_{\mathcal{U}} \right\} = \\ \min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C, Az \leq b, \forall (A, b) \in \mathcal{U} \right\}.$$

The dual problem for the Problem (P) is given by:

$$(D) \quad \max_{(A, b) \in \mathcal{U}, \lambda \in S^+} \min_{z \in C} \left\{ \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle \right\}.$$

The related dualities for finite dimensional convex optimization problems and their inf-max duality results can be found in [2]. It is worth noting that in the Problem (D), the function $\frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle$ ($z \in X$, $\lambda \in Y$) is the Lagrange function L corresponding to the Problem (P), i.e.

$$L(z, \lambda) := \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle, \quad z \in X, \lambda \in Y.$$

We show that the robust strong CHIP characterizes strong duality between Problem (P) and Problem (D).

Theorem 4.1. *For the Problem (P), we assume that $K_{\mathcal{U}} = C \cap D_{\mathcal{U}}$. Then, the following assertions are equivalent:*

- (i) *The pair $\{C, D_{\mathcal{U}}\}$ has the robust strong CHIP at each point $z \in K_{\mathcal{U}}$.*
- (ii) *For any $x \in X$, $\min(P) = \max(D)$, i.e.,*

$$\min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C \cap D_{\mathcal{U}} \right\} = \max_{(A, b) \in \mathcal{U}, \lambda \in S^+} \min_{z \in C} \left\{ \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle \right\}.$$

Proof. [(i) \implies (ii)]. Suppose that the pair $\{C, D_{\mathcal{U}}\}$ has the robust strong CHIP at each point $z \in K_{\mathcal{U}}$. Note that the weak duality:

$$\min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C \cap D_{\mathcal{U}} \right\} \geq \max_{(A,b) \in \mathcal{U}, \lambda \in S^+} \min_{z \in C} \left\{ \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle \right\}.$$

always holds. We only need to verify the reverse inequality:

$$\min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C \cap D_{\mathcal{U}} \right\} \leq \max_{(A,b) \in \mathcal{U}, \lambda \in S^+} \min_{z \in C} \left\{ \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle \right\}.$$

To this end, let x_0 be a solution of the Problem (P). Then, clearly x_0 is a robust best approximation of x , and so, by Theorem 3.2, one has

$$\begin{aligned} x_0 &= P_C(x - A^*(\lambda)) \text{ for some } (A, b) \in \mathcal{U} \text{ and some } \lambda \in S^+ \\ &\text{with } \langle \lambda, Ax_0 - b \rangle = 0. \end{aligned} \tag{4.31}$$

This, after some simplifications implies that for each $z \in C$,

$$\frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle \geq \frac{1}{2} \|x - x_0\|^2 + \langle \lambda, Ax_0 - b \rangle = \frac{1}{2} \|x - x_0\|^2. \tag{4.32}$$

Indeed, for proving (4.32), first note that in view of (4.31), we have $x_0 = P_C(x - A^*(\lambda))$, and so, by using (3.13), one has $x - A^*(\lambda) - x_0 \in (C - x_0)^\circ$. This implies that

$$\langle x - A^*(\lambda) - x_0, z - x_0 \rangle \leq 0, \quad \forall z \in C. \tag{4.33}$$

By a simple calculation, it follows from (4.33) that

$$\langle x - x_0, z - x_0 \rangle + \langle \lambda, Ax_0 - b \rangle \leq \langle \lambda, Az - b \rangle, \quad \forall z \in C. \tag{4.34}$$

Note that since $x_0 \in K_{\mathcal{U}} (\subseteq C)$ is a robust best approximation of x , thus by (3.13), we conclude that

$$\langle x - x_0, z - x_0 \rangle \leq 0, \quad \forall z \in C. \tag{4.35}$$

On the other hand, we have

$$\frac{1}{2} \|x - z\|^2 = \frac{1}{2} \|x - x_0\|^2 + \frac{1}{2} \|x_0 - z\|^2 + \langle x - x_0, x_0 - z \rangle.$$

Therefore, this together with (4.34) and (4.35) and the fact that $\langle \lambda, Ax_0 - b \rangle = 0$ (see (4.31)) implies that

$$\begin{aligned} \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle &= \frac{1}{2} \|x - x_0\|^2 + \frac{1}{2} \|x_0 - z\|^2 + \langle x - x_0, x_0 - z \rangle \\ &\quad + \langle \lambda, Az - b \rangle \\ &\geq \frac{1}{2} \|x - x_0\|^2 + \frac{1}{2} \|x_0 - z\|^2 + \langle x - x_0, x_0 - z \rangle \\ &\quad + \langle x - x_0, z - x_0 \rangle + \langle \lambda, Ax_0 - b \rangle \\ &\geq \frac{1}{2} \|x - x_0\|^2 + \langle \lambda, Ax_0 - b \rangle \\ &= \frac{1}{2} \|x - x_0\|^2. \end{aligned} \tag{4.36}$$

Then, (4.32) holds.

Since by the hypothesis, x_0 is an optimal solution of the Problem (P), thus we get from (4.36) that

$$\min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C \cap D_{\mathcal{U}} \right\} = \frac{1}{2} \|x - x_0\|^2 \leq \max_{(A,b) \in \mathcal{U}, \lambda \in S^+} \min_{z \in C} \left\{ \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle \right\}.$$

[(ii) \implies (i)]. Fix an arbitrary $x_0 \in K_{\mathcal{U}}$. Let $a \in (K_{\mathcal{U}} - x_0)^\circ$ be arbitrary. Then, $P_{K_{\mathcal{U}}}(x) = x_0$, where $x := a + x_0$. Then, (ii) gives us that there exist $(A, b) \in \mathcal{U}$ and $\lambda \in S^+$ such that

$$\frac{1}{2} \|a\|^2 = \min_{z \in C} \left\{ \frac{1}{2} \|a + x_0 - z\|^2 + \langle \lambda, Az - b \rangle \right\}.$$

This implies that $\langle \lambda, Ax_0 - b \rangle \geq 0$, and so, $\langle \lambda, Ax_0 - b \rangle = 0$ because $x_0 \in K_{\mathcal{U}}$. This shows that for each $z \in C$,

$$\frac{1}{2} \|a + x_0 - z\|^2 + \langle \lambda, Az - b \rangle \geq \frac{1}{2} \|a\|^2 + \langle \lambda, Ax_0 - b \rangle.$$

Therefore, the convex function $f(z) := \frac{1}{2} \|a + x_0 - z\|^2 + \langle \lambda, Az - b \rangle + \delta_C(z)$ attains its global minimum at x_0 . So,

$$0 \in \partial f(x_0) = -a + A^*(\lambda) + (C - x_0)^\circ.$$

Thus, $a \in (C - x_0)^\circ + A^*(\lambda)$ for some $(A, b) \in \mathcal{U}$ and some $\lambda \in S^+$ with $\langle \lambda, Ax_0 - b \rangle = 0$. Hence,

$$(K_{\mathcal{U}} - x_0)^\circ \subseteq (C - x_0)^\circ + \{A^*(\lambda) : (A, b) \in \mathcal{U}, \lambda \in S^+ \text{ and } \langle \lambda, Ax_0 - b \rangle = 0\}.$$

Note that the reverse inclusion always holds (see (3.12)). So, the robust strong CHIP holds at each point $z \in K_{\mathcal{U}}$. □

Ellipsoidal Uncertainty

Consider now the special case of the Problem (P) :

$$\min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C, Az \leq b, \forall (A, b) \in \mathcal{U} \right\},$$

where $\mathcal{U} = \mathcal{U}^e$ is the ellipsoidal uncertainty [3] defined by:

$$\begin{aligned} \mathcal{U}^e &:= \left\{ (A, b) \in L(X, \mathbb{R}^n) \times \mathbb{R}^n : Ax := (\langle a_1, x \rangle, \dots, \langle a_n, x \rangle), \forall x \in X, \right. \\ &b := (\beta_1, \dots, \beta_n), (a_i, \beta_i) = (\bar{a}_i^{(0)}, \bar{\beta}_i^{(0)}) + \sum_{l=1}^q v_i^l (\bar{a}_i^{(l)}, \bar{\beta}_i^{(l)}), \\ &\left. \|(v_i^1, \dots, v_i^q)\|_{\mathbb{R}^q} \leq 1, i = 1, \dots, n \right\}, \end{aligned} \tag{4.37}$$

where $(\bar{a}_i^{(l)}, \bar{\beta}_i^{(l)}) \in L(X, \mathbb{R}) \times \mathbb{R} = X^* \times \mathbb{R} = X \times \mathbb{R}$ (note that since X is a Hilbert space, so $X^* = X$), $i = 1, \dots, n, l = 0, \dots, q$ and $q \in \mathbb{N}$. It should be noted that the continuous linear mapping

$A : X \rightarrow \mathbb{R}^n$ is naturally defined by (4.37) because $Ax \in \mathbb{R}^n$ for each $x \in X$. In this case, the dual Problem (D) can be expressed as a second order cone programming problem which can easily be solved.

To see this, define the convex function $f : X \rightarrow [0, +\infty]$ by:

$$f(y) = \frac{1}{2}\|x - y\|^2 + \delta_C(y), \quad y \in X,$$

where δ_C is the indicator function of the closed convex set C . We observe that

$$\begin{aligned} & \max_{(A,b) \in \mathcal{U}^e, \lambda_i \geq 0} \min_{z \in C} \left\{ \frac{1}{2}\|x - z\|^2 + \langle \lambda, Az - b \rangle \right\} \\ &= \max_{\substack{(a_i, \beta_i) \in \mathcal{U}_i, \\ \lambda_i \geq 0}} \inf_{z \in X} \left\{ f(z) + \sum_{i=1}^n \lambda_i (\langle a_i, z \rangle - \beta_i) \right\} \\ &= \max_{\substack{(a_i, \beta_i) \in \mathcal{U}_i, \\ \lambda_i \geq 0}} \left\{ -f^* \left(-\sum_{i=1}^n \lambda_i a_i \right) - \sum_{i=1}^n \lambda_i \beta_i \right\} \\ &= \max_{\substack{\|(v_i^1, \dots, v_i^q)\| \leq 1, \\ \lambda_i \geq 0}} \left\{ -f^* \left(-\sum_{i=1}^n \lambda_i (\bar{a}_i^{(0)} + \sum_{l=1}^q v_i^l \bar{a}_i^{(l)}) \right) - \sum_{i=1}^n \lambda_i (\bar{\beta}_i^{(0)} + \sum_{l=1}^q v_i^l \bar{\beta}_i^{(l)}) \right\}, \end{aligned}$$

where f^* denotes the conjugate function of f . Letting $z_i := (z_i^{(1)}, \dots, z_i^{(q)})$, $z_i^{(l)} := \lambda_i v_i^l$, $l = 1, \dots, q$ and $z_i^{(0)} := \lambda_i \geq 0$, $i = 1, \dots, n$, we see that

$$\|(v_i^1, \dots, v_i^q)\|_{\mathbb{R}^q} \leq 1, \lambda_i \geq 0 \iff \|(z_i^{(1)}, \dots, z_i^{(q)})\|_{\mathbb{R}^q} \leq z_i^{(0)}.$$

So, the dual problem can be expressed as the following second order cone problem in the finite dimensional space $\mathbb{R}^{n(q+1)}$:

$$\begin{aligned} & \max_{(z_i^{(0)}, \dots, z_i^{(q)}) \in \mathbb{R}^{q+1}} \left\{ -f^* \left(-\sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{a}_i^{(l)} \right) - \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{\beta}_i^{(l)} \right. \\ & \quad \left. : \|(z_i^{(1)}, \dots, z_i^{(q)})\|_{\mathbb{R}^q} \leq z_i^{(0)}, i = 1, \dots, n \right\}. \end{aligned} \tag{4.38}$$

It is worth noting that when $C = X$ or $C = L_+^2[0, 1] = \{x(t) \in L^2[0, 1] : x(t) \geq 0, \text{ a.e.}\}$, the conjugate function f^* can be computed explicitly [8], and so the dual problem can be efficiently solved. For example, when $C = X$ and $f^*(y) = \frac{1}{2}\|y\|^2 + \langle y, x \rangle$, $y \in X$. So, in this case, the dual problem can be explicitly written as the following concave quadratic maximization problem with second order cone constraints:

$$\begin{aligned} & \max_{(z_i^{(0)}, \dots, z_i^{(q)}) \in \mathbb{R}^{q+1}} \left\{ -\left\| \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{a}_i^{(l)} \right\|^2 - \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \langle \bar{a}_i^{(l)}, x \rangle - \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{\beta}_i^{(l)} \right. \\ & \quad \left. : \|(z_i^{(1)}, \dots, z_i^{(q)})\|_{\mathbb{R}^q} \leq z_i^{(0)}, i = 1, \dots, n \right\}. \end{aligned}$$

The following corollary illustrates how we can obtain the robust best approximation in terms of a solution of the dual Problem (D) in the case of ellipsoidal uncertainty.

Corollary 4.1. Let $\mathcal{U} := \mathcal{U}^e$. Suppose that the pair $\{C, D_{\mathcal{U}}\}$ has the robust strong CHIP at each point $z \in K_{\mathcal{U}} = C \cap D_{\mathcal{U}}$. Then, the robust best approximation $x^* \in K_{\mathcal{U}}$ of $x \in X$ is given by:

$$x^* = P_C(x - \sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^{(l)} \bar{a}_i^{(l)}),$$

where $\bar{z} := (\bar{z}_1^{(0)}, \dots, \bar{z}_1^{(q)}, \dots, \bar{z}_n^{(0)}, \dots, \bar{z}_n^{(q)})$ is a solution of the optimistic dual problem

$$\max_{(z_i^{(0)}, \dots, z_i^{(q)}) \in \mathbb{R}^{q+1}} \left\{ -f^* \left(- \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{a}_i^{(l)} \right) - \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{\beta}_i^{(l)} \right. \\ \left. : \|(z_i^{(1)}, \dots, z_i^{(q)})\|_{\mathbb{R}^q} \leq z_i^{(0)}, i = 1, \dots, n \right\}, \tag{4.39}$$

where $f(y) := \frac{1}{2} \|x - y\|^2 + \delta_C(y)$, ($y \in X$).

Proof. First of all, Theorem 4.1 ensures that the strong duality holds (i.e., $p^* = q^*$, where p^* is the optimal value of the primal Problem (P) and q^* is the optimal value of the dual Problem (D)), and so, we have:

$$p^* = \min_{z \in X} \left\{ \frac{1}{2} \|x - z\|^2 : z \in C \cap D_{\mathcal{U}} \right\} \\ = \max_{(A, b) \in \mathcal{U}, \lambda \in \mathcal{S}^+} \min_{z \in C} \left\{ \frac{1}{2} \|x - z\|^2 + \langle \lambda, Az - b \rangle \right\} \\ = \max_{(z_i^{(0)}, \dots, z_i^{(q)}) \in \mathbb{R}^{q+1}} \left\{ -f^* \left(- \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{a}_i^{(l)} \right) - \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{\beta}_i^{(l)} \right. \\ \left. : \|(z_i^{(1)}, \dots, z_i^{(q)})\|_{\mathbb{R}^q} \leq z_i^{(0)}, i = 1, \dots, n \right\} = q^*.$$

The conclusion will follow if we show that $x^* = P_C(x - \sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^{(l)} \bar{a}_i^{(l)})$ is feasible for the Problem (P) and $\frac{1}{2} \|x - x^*\|^2 = \max(D)$.

To verify feasibility, we first observe that $x^* \in K_{\mathcal{U}}$ (and so, $x^* \in C$). Now, for each

$$z := (z_1^{(0)}, \dots, z_1^{(q)}, \dots, z_m^{(0)}, \dots, z_m^{(q)}),$$

we define

$$h(z) := f^* \left(- \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{a}_i^{(l)} \right) + \sum_{i=1}^n \sum_{l=0}^q z_i^{(l)} \bar{\beta}_i^{(l)}.$$

Since \bar{z} is a solution of the optimistic dual problem, thus, \bar{z} is a global minimizer of the function h over $\prod_{i=1}^n \text{SOC}_q$, where SOC_q is the second order cone in \mathbb{R}^q , i.e.,

$$\text{SOC}_q := \{(x_0, x_1, \dots, x_q) \in \mathbb{R}^{q+1} : x_0 \geq \|(x_1, \dots, x_q)\|_{\mathbb{R}^q}\}.$$

So, we have

$$\nabla h(\bar{z}) \in - \prod_{i=1}^n \text{SOC}_q \text{ and } \bar{z}^T \nabla h(\bar{z}) = 0.$$

Note that (see [16]) f^* is differentiable and $\partial f^*(y) = \nabla f^*(y) = P_C(y + x)$. So, for each $i = 1, \dots, n$,

$$-\langle \bar{a}_i^{(0)}, x^* \rangle - \bar{\beta}_i^{(0)} \geq \| -\langle \bar{a}_i^{(1)}, x^* \rangle - \bar{\beta}_i^{(1)}, \dots, \langle \bar{a}_i^{(q)}, x^* \rangle - \bar{\beta}_i^{(q)} \|_{\mathbb{R}^q}.$$

But, for each $(a_i, \beta_i) \in \mathcal{U}^e$ (see the definition of \mathcal{U}^e), there exists $(w_i^{(1)}, \dots, w_i^{(q)}) \in \mathbb{R}^q$ with $\|(w_i^{(1)}, \dots, w_i^{(q)})\|_{\mathbb{R}^q} \leq 1$ such that $(a_i, \beta_i) = (\bar{a}_i^{(0)}, \bar{\beta}_i^{(0)}) + \sum_{l=1}^q w_i^{(l)}(\bar{a}_i^{(l)}, \bar{\beta}_i^{(l)})$. Therefore, for each $i = 1, \dots, n$,

$$\begin{aligned} \langle a_i, x^* \rangle - \beta_i &= \langle \bar{a}_i^{(0)}, x^* \rangle - \bar{\beta}_i^{(0)} + \sum_{l=1}^q w_i^{(l)}(\langle \bar{a}_i^{(l)}, x^* \rangle - \bar{\beta}_i^{(l)}) \\ &\leq \langle \bar{a}_i^{(0)}, x^* \rangle - \bar{\beta}_i^{(0)} + \| \langle \bar{a}_i^{(1)}, x^* \rangle - \bar{\beta}_i^{(1)}, \dots, \langle \bar{a}_i^{(q)}, x^* \rangle - \bar{\beta}_i^{(q)} \|_{\mathbb{R}^q} \\ &\leq 0. \end{aligned}$$

Thus, x^* is feasible for the Problem (P).

Now, we verify $\frac{1}{2}\|x - x^*\|^2 = \max(D)$. The complementarity condition: $\bar{z}^T \nabla h(\bar{z}) = 0$, gives us that $\sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^l (\langle \bar{a}_i^{(l)}, x^* \rangle - \bar{\beta}_i^{(l)}) = 0$. Hence,

$$\begin{aligned} \max(D) &= -f^* \left(- \sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^l \bar{a}_i^{(l)} \right) - \sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^l \bar{\beta}_i^{(l)} \\ &= -f^* \left(- \sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^l \bar{a}_i^{(l)} \right) + \left\langle - \sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^l \bar{a}_i^{(l)}, x^* \right\rangle \\ &= f(x^*) \\ &= \frac{1}{2} \|x - x^*\|^2, \end{aligned}$$

where the second equality follows by the fact that $x^* \in \partial f^* \left(- \sum_{i=1}^n \sum_{l=0}^q \bar{z}_i^l \bar{a}_i^{(l)} \right)$ and the Fenchel's equality $f^*(v) + f(x) = \langle v, x \rangle \iff x \in \partial f^*(v)$. □

Now, by the following example, we illustrate the results obtained in Section 3 and Section 4.

Example 4.1. Let $X := \ell^2$ and $Y := \ell^2$, where

$$\ell^2 = \ell^2(\mathbb{N}) := \{x := (x_1, x_2, \dots) : x_i \in \mathbb{R}, \sum_{i=1}^{\infty} |x_i|^2 < \infty\},$$

endowed with the ℓ^2 inner product $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$ for each $x, y \in \ell^2$. Moreover, the induced norm is defined by:

$$\|x\| := \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}, \quad \forall x := (x_1, x_2, \dots) \in \ell^2.$$

Let

$$S := \{\lambda := (\lambda_1, \lambda_2, \dots) \in Y : \lambda_i \geq 0, \forall i \geq 1\}.$$

Note that in this case, one has $S^+ = S$. Now, for each $v \in (0, 1]$, define the function $A_v : X \rightarrow Y$ by $A_v x := (vx_1, vx_2, \dots) = vx$ for all $x := (x_1, x_2, \dots) \in X$. It is clear that A_v is a well-defined continuous linear mapping in $L(X, Y)$ for each $v \in (0, 1]$. Let $b_v := (v, \frac{1}{2}v, \frac{1}{3}v, \dots)$ for each $v \in (0, 1]$. Clearly, $b_v \in Y$ for each $v \in (0, 1]$. Let $\mathcal{U} := \{(A_v, b_v) : v \in (0, 1]\}$. Therefore, $\mathcal{U} \subset L(X, Y) \times Y$.

Thus, one has

$$\begin{aligned} D_{\mathcal{U}} &= \{z := (z_1, z_2, \dots) \in X : A_v z \leq b_v, \forall v \in (0, 1]\} \\ &= \{z \in X : vz_i \leq \frac{1}{i}v, \forall i = 1, 2, \dots, \forall v \in (0, 1]\} \\ &= \{z \in X : z_i \leq \frac{1}{i}, \forall i = 1, 2, \dots\}. \end{aligned} \tag{4.40}$$

Now, let $v \in (0, 1]$ be arbitrary. Define $A_v^* : Y \rightarrow X$ by $\langle A_v^*(\lambda), z \rangle := \langle \lambda, A_v z \rangle$ for all $\lambda \in Y$ and all $z \in X$. Then, we have

$$\langle A_v^*(\lambda), z \rangle := \langle \lambda, A_v z \rangle = \langle \lambda, vz \rangle = \langle v\lambda, z \rangle, \forall z \in X.$$

This implies that $A_v^*(\lambda) = v\lambda = (v\lambda_1, v\lambda_2, \dots)$ for all $\lambda := (\lambda_1, \lambda_2, \dots) \in Y$ and all $v \in (0, 1]$.

In the sequel, put $x_0 := (1, \frac{1}{2}, \frac{1}{3}, \dots) \in X$. It is not difficult to check that $\mathcal{F}(x_0) = \mathcal{U} \times S^+$. Therefore,

$$\begin{aligned} \bigcup_{((A_v, b_v), \lambda) \in \mathcal{F}(x_0), v \in (0, 1]} \{A_v^*(\lambda)\} &= \bigcup_{((A_v, b_v), \lambda) \in \mathcal{U} \times S^+, v \in (0, 1]} \{A_v^*(\lambda)\} \\ &= \bigcup_{\lambda \in S^+, v \in (0, 1]} \{v\lambda\} \\ &= S^+. \end{aligned} \tag{4.41}$$

Let $C := \mathcal{I}^2$. Then,

$$(C - x_0)^\circ = C^\circ = \{0\}. \tag{4.42}$$

Also, in view of (4.40), we have

$$\begin{aligned} (C \cap D_{\mathcal{U}} - x_0)^\circ &= (D_{\mathcal{U}} - x_0)^\circ \\ &= \left(\{z - x_0 : z \in D_{\mathcal{U}}\}\right)^\circ \\ &= \left(\{z - x_0 : z \leq x_0\}\right)^\circ \\ &= \left(\{z - x_0 : z - x_0 \leq 0\}\right)^\circ \\ &= S^+. \end{aligned}$$

So, it follows that

$$(K_{\mathcal{U}} - x_0)^\circ = (C \cap D_{\mathcal{U}} - x_0)^\circ = S^+. \tag{4.43}$$

(Note that since $x_0 \in D_{\mathcal{U}}$ and $x_0 \in X = \ell^2 = C$, we conclude that $x_0 \in K_{\mathcal{U}}$.) Therefore, (4.43) together with (4.41) and (4.42) implies that

$$(K_{\mathcal{U}} - x_0)^\circ = (C - x_0)^\circ + \bigcup_{((A_v, b_v), \lambda) \in \mathcal{F}(x_0), v \in (0, 1]} \{A_v^*(\lambda)\}.$$

Thus, in view of Definition 3.1, “the robust strong conical hull intersection property“ holds at the point x_0 . Consequently, by using Theorem 3.2 (the implication [(i) \implies (ii)]), for any $x \in X$, we have:

$$x_0 = P_{K_{\mathcal{U}}}(x) \iff x_0 = P_C(x - A_v^*(\lambda)) \text{ for some } v \in (0, 1] \text{ with } (A_v, b_v) \in \mathcal{U} \\ \text{and some } \lambda \in S^+ \text{ such that } \langle \lambda, A_v x_0 - b_v \rangle = 0.$$

Indeed, we have $\langle \lambda, A_v x_0 - b_v \rangle = 0$ for all $\lambda \in S^+$ and all $v \in (0, 1]$. Also, if

$$x_v := (1 + v\lambda_1, \frac{1}{2} + v\lambda_2, \frac{1}{3} + v\lambda_3, \dots) \in X \text{ with } \lambda = (\lambda_1, \lambda_2, \dots) \in S^+ \text{ and } v \in (0, 1],$$

then, $x_0 = P_C(x_v - A_v^*(\lambda))$ holds for all $\lambda \in S^+$ and all $v \in (0, 1]$, and that $x_0 = P_{K_{\mathcal{U}}}(x_v)$ for each $v \in (0, 1]$.

In the following, we show that the “characteristic cone M “ which defined by (3.10):

$$M = \text{epi}\delta_C^* + \bigcup_{(A, b) \in \mathcal{U}, \lambda \in S^+} (A^*(\lambda), \langle b, \lambda \rangle)$$

is closed (weakly closed) and convex. First, note that since $C = \ell^2$, one can easily see that

$$\delta_C^*(u) := \begin{cases} 0, & \text{if } u = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $u \in \ell^2$. So, $\text{epi}\delta_C^* = \{0\}$. Therefore,

$$\begin{aligned} M &= \text{epi}\delta_C^* + \bigcup_{(A_v, b_v) \in \mathcal{U}, \lambda \in S^+, v \in (0, 1]} (A_v^*(\lambda), \langle b_v, \lambda \rangle) \\ &= \{0\} + \bigcup_{\lambda = (\lambda_1, \lambda_2, \dots) \in S^+, v \in (0, 1]} (v\lambda, v\lambda_1 + \frac{1}{2}v\lambda_2 + \dots) \\ &= \{0\} + (S^+ \times [0, +\infty)) \\ &= S^+ \times [0, +\infty). \end{aligned}$$

Thus, it is clear that M is closed and convex. Then, in view of Theorem 3.1, one has “the robust strong conical hull intersection property“ holds at the point x_0 , which (in the above) was directly shown that it holds by using Definition 3.1. Consequently, since M is closed and convex, so the results obtained in Section 3 and Section 4 hold and illustrated by this example.

Remark 4.1. *It is worth nothing that if we assume that*

$$\mathcal{U} := \{(a, b)\} \subset L(X, \mathbb{R}) \times \mathbb{R} = X^* \times \mathbb{R} = X \times \mathbb{R},$$

and so, $D_{\mathcal{U}} := \{z \in X : \langle a, z \rangle \leq b\}$ (call, the singleton case), then, all proofs of the results under consideration hold for the Problem (1.1) (given in page 3) with the constraint set $D_{\mathcal{U}}$ without any change. Now, if Y is an infinite dimensional separable Hilbert space, then, by [1, Theorem 4.19(2)], there exists $a_i \in L(X, \mathbb{R})(= X^* = X)$, $i = 1, 2, \dots$, such that

$$Ax = (\langle a_1, x \rangle, \langle a_2, x \rangle, \dots) \text{ for each } x \in X. \quad (4.44)$$

Let $b := (b_1, b_2, \dots) \in Y$. Therefore, $(A, b) \in L(X, Y) \times Y$. Let

$$\mathcal{U} := \{(A, b) \in L(X, Y) \times Y : A \text{ is defined by (4.44), } b \in Y\},$$

and

$$D_{\mathcal{U}} = \{z \in X : Az \leq b, \forall (A, b) \in \mathcal{U}\}.$$

Thus, the Problem (1.1) :

$$\min_{z \in X} \{\|x - z\| : z \in C, Az \leq b, \forall (A, b) \in \mathcal{U}, (x \in C \cap D_{\mathcal{U}})\}$$

is equivalent to the following Problem:

$$\min_{z \in X} \{\|x - z\| : z \in C, \langle a_i, z \rangle \leq b_i, \text{ for all } a_i, b_i \text{ for which } (A, b) \in \mathcal{U}, i = 1, 2, \dots\},$$

which is a “semi-infinite programming“. Indeed, in the particular case, the Problem (1.1) can be inverted to a “robust semi-infinite programming“.

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