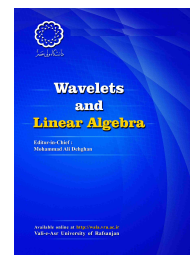


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Maps preserving mixed Jordan triple product of operators on prime algebras

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ABSTRACT

Let \mathcal{A} and \mathcal{B} be unital prime algebras and suppose \mathcal{A} contains a non-trivial idempotent P_1 . For arbitrary elements $A, B \in \mathcal{A}$, the product $A.B \circ A$ is called mixed Jordan triple product, where \cdot denotes the usual product and \circ denotes the Jordan product. We consider a bijective map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ which preserves the mixed Jordan triple product of elements. More precisely, ϕ satisfies $\phi(A.B \circ A) = \phi(A).\phi(B) \circ \phi(A)$, for every $A, B \in \mathcal{A}$.

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1. Introduction

Investigating mappings that preserve certain properties is a popular and interesting research field in mathematics. Most problems of this type are studied in operator theory. In this context, some issues are related to preserving the properties of product of elements on operator algebras, which we mention some of them.

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Let \mathcal{X} be a Banach space and $\mathcal{B}(\mathcal{X})$ denote the Banach algebra of all bounded linear operators on \mathcal{X} . Jordan product, triple Jordan product and skew Lie product of two operators $A, B \in \mathcal{B}(\mathcal{X})$ are defined as $A \circ B = \frac{1}{2}(AB + BA)$, ABA and $[A, B]_* = AB - BA^*$, respectively. Also two kinds of new products \bullet and \circ are defined by $A \bullet B = A^*B + B^*A$ and $A \circ B = A^*B - B^*A$.

The idea and motivation for this paper comes from the reference [7]. Authors in [7] considered the map ϕ on a standard operator algebra \mathcal{A} which satisfies the preserving property $\phi(\frac{1}{2}A^2B + \frac{1}{2}BA^2) = \frac{1}{2}\phi(A)^2\phi(B) + \frac{1}{2}\phi(B)\phi(A)^2$, which actually the multiplication of $\frac{1}{2}A^2B + \frac{1}{2}BA^2$ is the same as the mixed multiplication $A.A \circ B$, where ‘.’ denotes the usual product.

In this paper we consider the mixed product $A.B \circ A$. We call this product *mixed Jordan triple product*. In fact this research focuses on analyzing bijective maps on unital prime algebras that contain a nontrivial idempotent P_1 . An algebra \mathcal{A} is called prime if for arbitrary elements $A, B \in \mathcal{A}$ the condition $AAB = 0$ implies $A = 0$ or $B = 0$. Let \mathcal{A} and \mathcal{B} be unital prime algebras and \mathcal{A} contains a non-trivial idempotent P_1 . Our goal is to study the behavior of maps $\phi : \mathcal{A} \rightarrow \mathcal{B}$ which satisfy the condition:

$$\phi(A.B \circ A) = \phi(A).\phi(B) \circ \phi(A),$$

for all $A, B \in \mathcal{A}$.

Some mathematicians studied preservers of a specific product and also some, preservers of a combination of multiple products.

In [4], Molnar examined the multiplicative jordan triple isomorphism on the self-adjoint elements of von neumann algebras. Also in [3], Liu and Ji have proven that a bijective map Φ on factor von neumann algebra preserves Jordan product if and only if Φ is a *-isomorphism. Also in [5], Molnar showed that if a bijective map between standard operator algebras preserves jordan triple products, then it is either a linear or conjugate linear map.

In [6], Taghavi and Hosseinzadeh have characterized maps preserving a specific triple product of elements on ring. In [8] and [9], Yang and Zhang consider the maps preserving the mixed skew Lie triple products $[[A, B]_*, C]$ and $[[A, B], C]_*$ on factor von Neumann algebras. Authors in [2] studied maps preserving the mixed product $[A \bullet B, C]_*$ and in [10] studied the mixed product $A \bullet B \circ C$ on von Neumann algebras. In [1], authors proved that if a bijective map Θ between prime *-algebras satisfying a special mixed product, then Θ or $-\Theta$ is a *-ring isomorphism.

2. Proofs

Let \mathcal{A} be an algebra with unit I . Recall that an element P of \mathcal{A} is an idempotent whenever $P^2 = P$ and two idempotents P_1 and P_2 of \mathcal{A} are orthogonal idempotents whenever $P_1P_2 = P_2P_1 = 0$. Furthermore, two orthogonal idempotents P_1 and P_2 are called complete orthogonal idempotents if $P_1 + P_2 = I$.

Given an algebra \mathcal{A} containing a non-trivial idempotent P_1 , we set $P_2 := I - P_1$ and define $\mathcal{A}_{ij} := P_iAP_j$, which allows us to decompose any element $A \in \mathcal{A}$ into the sum $A = \sum_{1 \leq i, j \leq 2} A_{ij}$, where A_{ij} represents the elements of \mathcal{A}_{ij} .

The following lemma, which has a simple proof, is a key lemma in proving the main theorem

Lemma 2.1. *Let \mathcal{A} and \mathcal{B} be two algebras and let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a map which satisfies*

$$\phi(A.B \circ A) = \phi(A).\phi(B) \circ \phi(A),$$

for every $A, B \in \mathcal{A}$. Let $T, A_1, \dots, A_n \in \mathcal{A}$ such that

$$\phi(T) = \phi(A_1) + \dots + \phi(A_n).$$

Then

$$\phi(S.ToS) = \phi(S.A_1oS) + \dots + \phi(S.A_noS),$$

for every $S \in \mathcal{A}$.

Proof. Let $S \in \mathcal{A}$. Multiplying $\phi(T) = \phi(A_1) + \dots + \phi(A_n)$ by $\phi(S)$ from left and right and then multiply by $\phi(S)^2$ from the left yields

$$\begin{aligned} \phi(S)\phi(T)\phi(S) + \phi(S)^2\phi(T) = \\ \phi(S)\phi(A_1)\phi(S) + \phi(S)^2\phi(A_1) + \dots + \phi(S)\phi(A_n)\phi(S) + \phi(S)^2\phi(A_n) \end{aligned}$$

which implies

$$\phi(S).\phi(T)o\phi(S) = \phi(S).\phi(A_1)o\phi(S) + \dots + \phi(S).\phi(A_n)o\phi(S).$$

Now using the assumption follows that $\phi(S.ToS) = \phi(S.A_1oS) + \dots + \phi(S.A_noS)$. \square

One of the main results of this paper is the following theorem.

Theorem 2.2. *Let \mathcal{A} and \mathcal{B} be unital prime algebras and \mathcal{A} contains a non-trivial idempotent P_1 . Let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a bijective map which satisfies*

$$\phi(A.BoA) = \phi(A).\phi(B)o\phi(A),$$

for every $A, B \in \mathcal{A}$. Then ϕ is additive.

Proof. We divide the proof into several steps.

Step 1. $\phi(0) = 0$.

Setting $A = 0$ in preserving property of ϕ yields $\phi(0) = \frac{1}{2}\phi(0)\phi(B)\phi(0) + \frac{1}{2}\phi(0)^2\phi(B)$ and if B is an operator such that $\phi(B) = 0$, then $\phi(0) = 0$.

Step 2. $\phi(A_{11} + A_{12}) = \phi(A_{11}) + \phi(A_{12})$, for every $A_{11} \in \mathcal{A}_{11}$ and $A_{12} \in \mathcal{A}_{12}$.

Since ϕ is surjective, there exists a $T \in \mathcal{A}$ such that

$$\phi(T) = \phi(A_{11}) + \phi(A_{12}).$$

Using Lemma 2.1 for $S = P_1 - P_2$ gives

$$\begin{aligned} \phi((P_1 - P_2).To(P_1 - P_2)) = \phi((P_1 - P_2).A_{11}o(P_1 - P_2)) \\ + \phi((P_1 - P_2).A_{12}o(P_1 - P_2)). \end{aligned}$$

It is clear that

$$(P_1 - P_2).To(P_1 - P_2) = \frac{1}{2}(P_1 - P_2)T(P_1 - P_2) + \frac{1}{2}\underbrace{(P_1 - P_2)^2 T}_{P_1+P_2}$$

$$\begin{aligned}
&= \frac{1}{2}(T_{11} + T_{22}) + \frac{1}{2}(T_{11} + T_{22}) = T_{11} + T_{22}, \\
(P_1 - P_2).A_{11}o(P_1 - P_2) &= \frac{1}{2}(P_1 - P_2)A_{11}(P_1 - P_2) + \frac{1}{2}(P_1 - P_2)^2 A_{11} \\
&= A_{11}, \\
(P_1 - P_2).A_{12}o(P_1 - P_2) &= \frac{1}{2}(P_1 - P_2)A_{12}(P_1 - P_2) + \frac{1}{2}(P_1 - P_2)^2 A_{12} \\
&= 0.
\end{aligned}$$

Therefore, $\phi(T_{11} + T_{22}) = \phi(A_{11}) + \phi(0)$ and then by Step 1 and injectivity of ϕ , $T_{11} + T_{22} = A_{11}$. Therefore, $T_{11} = A_{11}$ and $T_{22} = 0$.

This time we apply Lemma 2.1 for $S = X_{21}$ and so we obtain

$$\begin{aligned}
&\phi\left(\frac{1}{2}X_{21}TX_{21} + \frac{1}{2}X_{21}^2T\right) = \\
&\phi\left(\frac{1}{2}X_{21}A_{11}X_{21} + \frac{1}{2}X_{21}^2A_{11}\right) + \phi\left(\frac{1}{2}X_{21}A_{12}X_{21} + \frac{1}{2}X_{21}^2A_{12}\right)
\end{aligned}$$

which implies that $\phi\left(\frac{1}{2}X_{21}TX_{21}\right) = \phi\left(\frac{1}{2}X_{21}A_{12}X_{21}\right)$. Injectivity of ϕ yields $\frac{1}{2}X_{21}TX_{21} = \frac{1}{2}X_{21}A_{12}X_{21}$ and since \mathcal{A} is a prime algebra, $T - A_{12} = 0$ which implies that $T_{12} = A_{12}$.

If we apply Lemma 2.1 for $S = X_{12}$, then we obtain $\phi\left(\frac{1}{2}X_{12}TX_{12}\right) = 0 = \phi(0)$ and then $T_{21} = 0$. Thus $T = T_{11} + T_{12} + T_{21} + T_{22} = A_{11} + A_{12}$. This together with $\phi(T) = \phi(A_{11}) + \phi(A_{12})$ completes the proof.

Step 3. $\phi(A_{21} + A_{22}) = \phi(A_{21}) + \phi(A_{22})$, for every $A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$.

The proof is similar to the previous step.

Step 4. $\phi(A_{11} + A_{12} + A_{21} + A_{22}) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22})$, for every $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$.

Since ϕ is surjective, there exists a $T \in \mathcal{A}$ such that

$$\phi(T) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22}).$$

Using Lemma 2.1 for $S = P_1$ gives

$$\phi\left(T_{11} + \frac{1}{2}T_{12}\right) = \phi(A_{11}) + \phi\left(\frac{1}{2}A_{12}\right).$$

Step 2 follows that

$$\phi\left(T_{11} + \frac{1}{2}T_{12}\right) = \phi\left(A_{11} + \frac{1}{2}A_{12}\right)$$

which by injectivity of ϕ , $T_{11} = A_{11}$ and $T_{12} = A_{12}$.

If we apply Lemma 2.1 for $S = P_2$ and use Step 3, then we obtain $T_{21} = A_{21}$ and $T_{22} = A_{22}$. Thus $T = T_{11} + T_{12} + T_{21} + T_{22} = A_{11} + A_{12} + A_{21} + A_{22}$. This together with $\phi(T) = \phi(A_{11}) + \phi(A_{12}) + \phi(A_{21}) + \phi(A_{22})$ completes the proof.

Step 5. $\phi(A_{ij} + B_{ij}) = \phi(A_{ij}) + \phi(B_{ij})$, for every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$, where $1 \leq i \neq j \leq 2$.

First assume that $i = 1$ and $j = 2$. It is easy to check that

$$\begin{aligned} & \frac{1}{2}(A_{12} + P_1)(2B_{12} + 2P_2)(A_{12} + P_1) + \frac{1}{2}(A_{12} + P_1)^2(2B_{12} + 2P_2) \\ & = A_{12} + B_{12}. \end{aligned}$$

Since $B_{12} \in \mathcal{A}_{12}$ and $P_2 \in \mathcal{A}_{22}$, by Step 4 we have

$$\phi(2B_{12} + 2P_2) = \phi(2B_{12}) + \phi(2P_2).$$

These equations together with assumption yield

$$\begin{aligned} & \phi(A_{12} + B_{12}) = \\ & \phi\left[\frac{1}{2}(A_{12} + P_1)(2B_{12} + 2P_2)(A_{12} + P_1) + \frac{1}{2}(A_{12} + P_1)^2(2B_{12} + 2P_2)\right] \\ & = \frac{1}{2}\phi(A_{12} + P_1)[\phi(2B_{12}) + \phi(2P_2)]\phi(A_{12} + P_1) \\ & \quad + \frac{1}{2}\phi(A_{12} + P_1)^2[\phi(2B_{12}) + \phi(2P_2)] \\ & = \left\{\frac{1}{2}\phi(A_{12} + P_1)\phi(2B_{12})\phi(A_{12} + P_1) + \frac{1}{2}\phi(A_{12} + P_1)^2\phi(2B_{12})\right\} \\ & \quad + \left\{\frac{1}{2}\phi(A_{12} + P_1)\phi(2P_2)\phi(A_{12} + P_1) + \frac{1}{2}\phi(A_{12} + P_1)^2\phi(2P_2)\right\} \\ & = \phi\left(\frac{1}{2}(A_{12} + P_1)2B_{12}(2A_{12} + P_1) + \frac{1}{2}(A_{12} + P_1)^2 2B_{12}\right) \\ & \quad + \phi\left(\frac{1}{2}(A_{12} + P_1)2P_2(2A_{12} + P_1) + \frac{1}{2}(A_{12} + P_1)^2 2P_2\right) \\ & = \phi(B_{12}) + \phi(A_{12}). \end{aligned}$$

The proof for the case $i = 2$ and $j = 1$ is similar.

Step 6. $\phi(A_{ii} + B_{ii}) = \phi(A_{ii}) + \phi(B_{ii})$, for every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, where $1 \leq i \leq 2$.

First assume that $i = 1$. Since ϕ is surjective, there exists a $T \in \mathcal{A}$ such that

$$\phi(T) = \phi(A_{11}) + \phi(B_{11}).$$

Using Lemma 2.1 for $S = X_{12}$ gives

$$\phi\left(\frac{1}{2}X_{12}T_{21}X_{12}\right) = 0 = \phi(0).$$

Injectivity of ϕ and primeness of \mathcal{A} yields that $T_{21} = 0$. Using again Lemma 2.1 for $S = X_{21}$ and $S = P_2$ follows that $T_{12} = 0$ and $T_{22} = 0$, respectively. Thus $T = T_{11}$ and then $\phi(T_{11}) = \phi(A_{11}) + \phi(B_{11})$. If we apply Lemma 2.1 for $S = I + X_{12}$ and use

$$\frac{1}{2}(I + X_{12})T_{11}(I + X_{12}) + \frac{1}{2}(I + X_{12})^2 T_{11} = T_{11} + \frac{1}{2}T_{11}X_{12},$$

then we obtain

$$\phi(T_{11} + \frac{1}{2}T_{11}X_{12}) = \phi(A_{11} + \frac{1}{2}A_{11}X_{12}) + \phi(B_{11} + \frac{1}{2}B_{11}X_{12})$$

and by Step 2

$$\phi(T_{11}) + \phi(\frac{1}{2}T_{11}X_{12}) = \phi(A_{11}) + \phi(\frac{1}{2}A_{11}X_{12}) + \phi(B_{11}) + \phi(\frac{1}{2}B_{11}X_{12}).$$

This relation together with $\phi(T_{11}) = \phi(A_{11}) + \phi(B_{11})$ and Step 5 follows

$$\phi(\frac{1}{2}T_{11}X_{12}) = \phi(\frac{1}{2}A_{11}X_{12} + \frac{1}{2}B_{11}X_{12}).$$

Injectivity of ϕ and primeness of \mathcal{A} yields that $T_{11} = A_{11} + B_{11}$. The proof for the case $i = 2$ is similar.

Step 7. ϕ is additive.

Let $A, B \in \mathcal{A}$. So there exist $A_{11}, B_{11} \in \mathcal{A}_{11}, A_{12}, B_{12} \in \mathcal{A}_{12}, A_{21}, B_{21} \in \mathcal{A}_{21}$ and $A_{22}, B_{22} \in \mathcal{A}_{22}$ such that $A = \sum_{1 \leq i, j \leq 2} A_{ij}$ and $B = \sum_{1 \leq i, j \leq 2} B_{ij}$. By Steps 4, 5 and 6 we have

$$\begin{aligned} \phi(A + B) &= \phi\left(\sum_{1 \leq i, j \leq 2} A_{ij} + \sum_{1 \leq i, j \leq 2} B_{ij}\right) \\ &= \phi\left(\sum_{1 \leq i, j \leq 2} [A_{ij} + B_{ij}]\right) \\ &= \phi(A_{11} + B_{11}) + \phi(A_{12} + B_{12}) + \phi(A_{21} + B_{21}) + \phi(A_{22} + B_{22}) \\ &= \phi(A_{11}) + \phi(B_{11}) + \phi(A_{12}) + \phi(B_{12}) \\ &\quad + \phi(A_{21}) + \phi(B_{21}) + \phi(A_{22}) + \phi(B_{22}) \\ &= \phi(A_{11} + A_{12} + A_{21} + A_{22}) + \phi(B_{11} + B_{12} + B_{21} + B_{22}) \\ &= \phi(A) + \phi(B). \end{aligned}$$

□

Using of the above theorem we prove that every mixed Jordan triple product preserver is a multiplicative map.

Theorem 2.3. *Let \mathcal{A} and \mathcal{B} be unital prime algebras and \mathcal{A} contains a non-trivial idempotent P_1 and $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a bijective map which satisfies*

$$\phi(A \cdot B \circ A) = \phi(A) \cdot \phi(B) \circ \phi(A)$$

for every $A, B \in \mathcal{A}$. Then ϕ or $-\phi$ is multiplicative.

Proof. We divide the proof into several steps.

Step 1. $\phi(I) = I$ or $\phi(I) = -I$.

By the preserving property of ϕ , for $A = I$ and arbitrary element B

$$\phi(B) = \phi(I) \cdot \phi(B) \circ \phi(I) = \frac{1}{2} \phi(I) \phi(B) \phi(I) + \frac{1}{2} \phi(I)^2 \phi(B).$$

Since ϕ is surjective, there exists an operator B_0 such that $\phi(B_0) = I$. Replacing B_0 instead B in last relation yields that $\phi(I)^2 = I$ and then $\phi(I)\phi(B)\phi(I) = \phi(B)$ and $\phi(I)\phi(B) = \phi(B)\phi(I)$. Thus $(\phi(I) - I)\phi(B)(\phi(I) + I) = 0$ for every B . Surjectivity of ϕ and primeness of \mathcal{B} imply that $\phi(I) - I = 0$ or $\phi(I) + I = 0$ and this completes the proof.

* Since $-\phi$ satisfies the assumptions on ϕ , so without loss of generality, next we assume that $\phi(I) = I$.

Step 2. $\phi(A^2) = \phi(A)^2$ and $\phi(A \circ B) = \phi(A) \circ \phi(B)$.

Using preserving property of ϕ for $B = I$ and arbitrary element A and Step 1 gives $\phi(A^2) = \phi(A)^2$. This relation together with additivity of ϕ by Theorem 2.1 implies that $\phi(A \circ B) = \phi(A) \circ \phi(B)$.

Step 3. $Q_i = \phi(P_i)$ and $Q_j = \phi(P_j)$ are orthogonal idempotents in \mathcal{B} , where $1 \leq i \neq j \leq 2$. Also $\mathcal{B} = \sum_{1 \leq i, j \leq 2} \mathcal{B}_{ij}$, where $\mathcal{B}_{ij} = Q_i \mathcal{B} Q_j$.

Step 1 and additivity of ϕ imply that $Q_i + Q_j = \phi(P_i + P_j) = \phi(I) = I$ and by step 2, since Q_i and Q_j are idempotents, so they are orthogonal in \mathcal{B} and this completes the proof.

Step 4. $\phi(A_{ij}) \in \mathcal{B}_{ij}$ for $1 \leq i, j \leq 2$.

Let $i \neq j$. It is clear that

$$\frac{1}{2} A_{ij} = P_i \cdot A_{ij} \circ P_i \tag{2.1}$$

and

$$\frac{1}{2} A_{ij} = A_{ij} \circ P_j. \tag{2.2}$$

2.1 together with preserving property of ϕ and idempotency of $\phi(P_i)$ implies

$$\phi(A_{ij}) = \phi(P_i) \phi(A_{ij}) \phi(P_i) + \phi(P_i) \phi(A_{ij})$$

which multiplying by $\phi(P_i)$ from left yields that $\phi(P_i) \phi(A_{ij}) \phi(P_i) = 0$ and then

$$\phi(A_{ij}) = \phi(P_i) \phi(A_{ij}). \tag{2.3}$$

Since by Step 2, ϕ preserves jordan product, from 2.2 we have

$$\phi(A_{ij}) = \phi(A_{ij}) \phi(P_j) + \phi(P_j) \phi(A_{ij})$$

which multiplying by $\phi(P_i)$ from left and using $\phi(P_i) \phi(P_j) = 0$ yields that

$$\phi(P_i) \phi(A_{ij}) = \phi(P_i) \phi(A_{ij}) \phi(P_j)$$

and then from 2.3

$$\phi(A_{ij}) = \phi(P_i) \phi(A_{ij}) \phi(P_j).$$

and then $\phi(A_{ij}) \in \mathcal{B}_{ij}$.

Relation $A_{ii} = P_i A_{ii} P_i$ and preserving property of ϕ follow that

$$\phi(A_{ii}) = \frac{1}{2}\phi(P_i)\phi(A_{ii})\phi(P_i) + \frac{1}{2}\phi(P_i)\phi(A_{ii}).$$

Multiplying by $\phi(P_i)$ from left yields that $\phi(P_i)\phi(A_{ii}) = \phi(P_i)\phi(A_{ii})\phi(P_i)$ and then $\phi(A_{ii}) = \phi(P_i)\phi(A_{ii})\phi(P_i) \in \mathcal{B}_{ii}$.

Step 5. $\phi(A_{ii})\phi(B_{ij}) = \phi(A_{ii}B_{ij})$, $\phi(A_{ii})\phi(B_{ii}) = \phi(A_{ii}B_{ii})$, $\phi(A_{ij})\phi(B_{jj}) = \phi(A_{ij}B_{jj})$ and $\phi(A_{ij})\phi(B_{ji}) = \phi(A_{ij}B_{ji})$, for every $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ and $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ where $1 \leq i \neq j \leq 2$.

Since ϕ preserves Jordan product, we have

$$\phi(A_{ii}B_{ij}) = \phi(A_{ii}B_{ij} + B_{ij}A_{ii}) = \phi(A_{ii})\phi(B_{ij}) + \phi(B_{ij})\phi(A_{ii}).$$

By Step 4, $\phi(B_{ij})\phi(A_{ii}) = 0$ and so $\phi(A_{ii}B_{ij}) = \phi(A_{ii})\phi(B_{ij})$.

Let $M_{ij} \in \mathcal{A}_{ij}$ be arbitrary. By previous part we have

$$\begin{aligned} \phi(A_{ii}B_{ii})\phi(M_{ij}) &= \phi(A_{ii}B_{ii}M_{ij}) = \phi(A_{ii})\phi(B_{ii}M_{ij}) \\ &= \phi(A_{ii})\phi(B_{ii})\phi(M_{ij}) \end{aligned}$$

which implies that

$$[\phi(A_{ii}B_{ii}) - \phi(A_{ii})\phi(B_{ii})]\phi(M_{ij}) = 0.$$

Since by Step 4, $\phi(M_{ij}) \in \mathcal{B}_{ij}$, from the primeness of \mathcal{B} we obtain $\phi(A_{ii}B_{ii}) = \phi(A_{ii})\phi(B_{ii})$.

Since ϕ preserves Jordan product, we have

$$\phi(A_{ij}B_{jj}) = \phi(A_{ij}B_{jj} + B_{jj}A_{ij}) = \phi(A_{ij})\phi(B_{jj}) + \phi(B_{jj})\phi(A_{ij}).$$

By Step 4, $\phi(B_{jj})\phi(A_{ij}) = 0$ and so $\phi(A_{ij}B_{jj}) = \phi(A_{ij})\phi(B_{jj})$.

Let $M_{ij} \in \mathcal{A}_{ij}$ be arbitrary. Since ϕ preserves Jordan product and is additive, we have

$$\begin{aligned} [\phi(A_{ij}B_{ji}) + \phi(B_{ji}A_{ij})]\phi(M_{ij}) &= [\phi(A_{ij}B_{ji} + B_{ji}A_{ij})]\phi(M_{ij}) \\ &= [\phi(A_{ij})\phi(B_{ji}) + \phi(B_{ji})\phi(A_{ij})]\phi(M_{ij}). \end{aligned}$$

Since by Step 4, $\phi(B_{ji}A_{ij})\phi(M_{ij}) = \phi(B_{ji})\phi(A_{ij})\phi(M_{ij}) = 0$, we obtain

$$[\phi(A_{ij}B_{ji}) - \phi(A_{ij})\phi(B_{ji})]\phi(M_{ij}) = 0$$

and this together with primeness of \mathcal{B} follows that $\phi(A_{ij}B_{ji}) = \phi(A_{ij})\phi(B_{ji})$.

Step 6. $\phi(AB) = \phi(A)\phi(B)$ for every $A, B \in \mathcal{A}$.

Let $A = \sum_{1 \leq i, j \leq 2} A_{ij}$ and $B = \sum_{1 \leq i, j \leq 2} B_{ij}$. Assertion follows from relation $AB = A_{11}B_{11} + A_{12}B_{21} + A_{11}B_{12} + A_{12}B_{22} + A_{21}B_{11} + A_{22}B_{21} + A_{22}B_{22} + A_{21}B_{12}$, Step 5 and additivity of ϕ . □

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