Wavelets and Linear Algebra 12(2) (2025) 1-14

Vali-e-Asr University
of Rafsanjan

Wavelets and Linear Algebra

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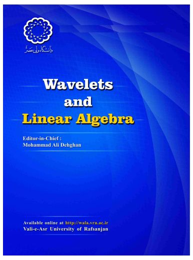

Redundancy and frame potential of finite frames

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Article Info

Article history:

Accepted 28 October 2024
Available online 31 December 2025

Keywords:
frame
frame potential
lower redundancy
upper redundancy
redundancy function.

2000 MSC:
42C15.

Abstract

This paper is concentrated on redundancy and frame potential of finite frames in n-dimensional Hilbert space $\mathcal{H}_n$. More precisely, all possible finite frame redundancies are characterized. Also, all possible frame potential of finite frames with prescribed norms is characterized. Finally, the results are presented for dimensions $n = 2$ and $n = 3$.

1. Introduction

The concept of frames in a Hilbert space $\mathcal{H}_n$ space was originally introduced by Duffin and Schaeffer in the context of the non-harmonic Fourier series [11]. From the last decade, various generalizations of the frames have been proposed such as frame of subspaces, g-frames, and so on[7, 14]. The concept of equal norm Parseval frames on finite-dimensional Hilbert spaces was

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http://doi.org/10.22072/wala.2024.2033719.1456

first introduced by Casazza and Leonhard in [8] and it been developed very fast over the last ten years, especially in the context of wavelets and Gabor systems.

Given a n-dimensional Hilbert space $\mathcal{H}_n$ with inner product $\langle ., . \rangle$, a sequence $\{f_i\}_{i=1}^m$ is called a frame for $\mathcal{H}_n$ if there exist constants $A > 0$, $B < \infty$ such that for all $f \in \mathcal{H}_n$,

$$A \|f\|^2 \leq \sum_{i=1}^{m} |\langle f, f_i \rangle|^2 \leq B \|f\|^2 , \tag{1.1}$$

where A, B are respectively the lower and upper frame bounds. The second inequality of the frame condition (1.1) is also known as the Bessel condition for $\{f_i\}_{i=1}^m$. The frame $\{f_i\}_{i=1}^m$ is called a A-tight frame, if $A = B$. A tight frame with $\|f_i\| = c$, for all $1 \leq i \leq m$, is called a c-equal norm tight frame and a tight frame with frame bound 1 is called a Parseval frame. A sequence $\{f_i\}_{i=1}^m$ is called a frame sequence in $\mathcal{H}_n$, if it is a frame for $\overline{span}\,\{f_i\}_{i=1}^m$.

The bounded linear operator S defined by

$$S : \mathcal{H}_n \to \mathcal{H}_n, \qquad S f = \sum_{i=1}^{m} \langle f, f_i \rangle f_i$$

is called the frame operator of $\{f_i\}_{i=1}^m$. For more information concerning frames refer to[1, 10, 13, 14, 15].

Frames are redundant sets of vectors in a Hilbert space, which yield one natural representation of each vector in the space, but may have infinitely many different representations for any given vector. It is this redundancy that makes frames useful in applications. In signal processing, this concept has become very useful in analyzing the completeness and stability of linear discrete signal representations.

The number of frame vectors per dimension is defined as the redundancy of a frame in the finite-dimensional setting which is not an unsatisfactory definition. A more precise quantitative notion of redundancy for finite frames (lower and upper redundancies) has been introduced in [3]. This quantitative notion of redundancy is generalized to infinite frames in [4].

The frame potential of a finite frame is introduced by Benedetto and Fickus in [2]. Actually, the frame potential of a finite frame is a criterion to measure the orthogonality. It is shown that, A-tight frames are the minimizers of the frame potential over $\mathcal{K}$, where $\mathcal{K}$ is the family of frames with lower frame bound A[5].

In this paper, we characterize all possible finite frame redundancies(Theorem 2.9). Also, we characterize all possible frame potential of finite frames with prescribed norms in $\mathcal{H}_n$(Theorem 2.11). Finally, the results are presented for special cases $\mathcal{H}_2$ and $\mathcal{H}_3$(Theorem 3.1 and Theorem 3.3).

2. Main results

Bodmann, Casazza, and Kutyniok introduced a quantitative notion of redundancy for finite frames, and Cahill, Casazza, and Heinecke generalized it to infinite frames.

Definition 2.1. [3] Let $\mathcal{F} = \{f_i\}_{i=1}^m$ be a frame for Hilbert space $\mathcal{H}_n$. The redundancy function of $\mathcal{F}$ is defined on the unit sphere $\mathbb{S} := \{x \in \mathcal{H}_n; \|x\| = 1\}$ in $\mathcal{H}_n$ by

$$\mathcal{R}_{\mathcal{F}} : \mathbb{S} \to \mathbb{R}^+, \quad \mathcal{R}_{\mathcal{F}}(x) := \sum_{i=1}^m \|P_{<f_i>}(x)\|^2,$$

where $P_{<f_i>}$ is the orthogonal projection onto $< f_i >:= span\{f_i\}$.
The upper and lower redundancy of $\mathcal{F}$ are defined by

$$\mathcal{R}_{\mathcal{F}}^+ := \sup_{x\in\mathbb{S}} \mathcal{R}_{\mathcal{F}}(x) \quad and \quad \mathcal{R}_{\mathcal{F}}^- := \inf_{x\in\mathbb{S}} \mathcal{R}_{\mathcal{F}}(x),$$

respectively.
Moreover, $\mathcal{F}$ has a uniform redundancy, if $\mathcal{R}_{\mathcal{F}}^- = \mathcal{R}_{\mathcal{F}}^+$.

The properties of lower and upper redundancy for frames can be found in [3, Theorem 2.1] and [4, Theorem 3.1].
Since, zero vectors have no effect on redundancy, throughout this paper, we assume that $f_i \neq 0$, for all $1 \leq i \leq m$. Thus

$$\mathcal{R}_{\mathcal{F}}(x) := \sum_{i=1}^m \frac{|\langle x, f_i\rangle|^2}{\|f_i\|^2}.$$

Benedetto and Fickus defined the frame potential of a frame as follows:

Definition 2.2. [2] Let $\mathcal{F} = \{f_i\}_{i=1}^m$ be a frame for Hilbert space $\mathcal{H}_n$. The frame potential of $\mathcal{F}$ is defined by

$$FP(\mathcal{F}) = \sum_{i,j=1}^m \left|\left\langle f_i, f_j\right\rangle\right|^2.$$

The following theorem presented a relation between the frame potential and redundancy of a frame.

Proposition 2.3. *If* $\mathcal{F} = \{f_i\}_{i=1}^m$ *is a frame for* $\mathcal{H}_n$*, then*

$$mC^4\mathcal{R}_{\mathcal{F}}^- \leq FP(\mathcal{F}) \leq mD^4\mathcal{R}_{\mathcal{F}}^+,$$

where $C = \min_{i=1}^m \|f_i\|$ *and* $D = \max_{i=1}^m \|f_i\|$.

Proof. For any $1 \leq j \leq m$, we have

$$\mathcal{R}_{\mathcal{F}}^- \leq \mathcal{R}_{\mathcal{F}}\left(\frac{f_j}{\|f_j\|}\right) = \sum_{i=1}^m \left|\left\langle \frac{f_j}{\|f_j\|}, \frac{f_i}{\|f_i\|}\right\rangle\right|^2 \leq \mathcal{R}_{\mathcal{F}}^+$$

and hence

$$m\mathcal{R}_{\mathcal{F}}^{-} \leq \sum_{i,j=1}^{m} \left| \left\langle \frac{f_j}{\|f_j\|}, \frac{f_i}{\|f_i\|} \right\rangle \right|^2 \leq m\mathcal{R}_{\mathcal{F}}^{+}.$$

Thus

$$\begin{aligned} m\mathcal{R}_{\mathcal{F}}^{-} &\leq \sum_{i,j=1}^{m} \left| \left\langle \frac{f_j}{\|f_j\|}, \frac{f_i}{\|f_i\|} \right\rangle \right|^2 \\ &= \sum_{i,j=1}^{m} \frac{1}{\|f_j\|^2} \frac{1}{\|f_i\|^2} \left| \left\langle f_j, f_i \right\rangle \right|^2 \\ &\leq \frac{1}{C^4} \sum_{i,j=1}^{m} \left| \left\langle f_j, f_i \right\rangle \right|^2 \\ &= \frac{1}{C^4} FP(\mathcal{F}) \end{aligned}$$

and

$$\begin{aligned} m\mathcal{R}_{\mathcal{F}}^{+} &\geq \sum_{i,j=1}^{m} \left| \left\langle \frac{f_j}{\|f_j\|}, \frac{f_i}{\|f_i\|} \right\rangle \right|^2 \\ &= \sum_{i,j=1}^{m} \frac{1}{\|f_j\|^2} \frac{1}{\|f_i\|^2} \left| \left\langle f_j, f_i \right\rangle \right|^2 \\ &\geq \frac{1}{D^4} \sum_{i,j=1}^{m} \left| \left\langle f_j, f_i \right\rangle \right|^2 \\ &= \frac{1}{D^4} FP(\mathcal{F}). \end{aligned}$$

□

The following remark is immediate from Proposition 2.3:

Remark 2.4. If $\mathcal{F} = \{f_i\}_{i=1}^{m}$ is a c-equal norm tight frame for $\mathcal{H}_n$, then

$$\mathcal{R}_{\mathcal{F}}^{-} = \mathcal{R}_{\mathcal{F}}^{+} = \frac{1}{mc^4} FP(\mathcal{F}).$$

The frame potential of the union of two frames is bigger than of summation of their frame potentials:

Proposition 2.5. *If* $\mathcal{F} = \{f_i\}_{i=1}^{m_1}$ *and* $\mathcal{G} = \{g_i\}_{i=1}^{m_2}$ *are frames for* $\mathcal{H}_n$, *then*

$$FP(\mathcal{F} \cup \mathcal{G}) > FP(\mathcal{F}) + FP(\mathcal{G}).$$

Proof. Let

$$h_i := \begin{cases} f_i \quad , & 1 \le i \le m_1, \\ \\ g_{i-m_1}, & m_1 + 1 \le i \le m_1 + m_2 \end{cases} .$$

Then

$$\begin{aligned} FP(\mathcal{F} \cup \mathcal{G}) &= \sum_{i,j=1}^{m_1+m_2} \left|\left\langle h_i, h_j \right\rangle\right|^2 \\ &> \sum_{i,j=1}^{m_1} \left|\left\langle h_i, h_j \right\rangle\right|^2 + \sum_{i,j=m_1+1}^{m_1+m_2} \left|\left\langle h_i, h_j \right\rangle\right|^2 \\ &= \sum_{i,j=1}^{m_1} \left|\left\langle f_i, f_j \right\rangle\right|^2 + \sum_{i,j=1}^{m_2} \left|\left\langle g_i, g_j \right\rangle\right|^2 \\ &= FP(\mathcal{F}) + FP(\mathcal{G}). \end{aligned}$$

(In the above inequality used of the completeness of frames $\mathcal{F}$ and $\mathcal{G}$.) □

The following two examples show that redundancy and frame potential of finite frames are not necessarily stable under small perturbations of the frame set.

Example 2.6. Let $0 < \epsilon < 1$ and let $\{e_i\}_{i=1}^n$ be an orthonormal basis for $\mathcal{H}_n$, $\mathcal{F} = \{f_i\}_{i=1}^n$ be the frame defined by

$$f_i := \begin{cases} \epsilon e_1 + \sqrt{1-\epsilon^2} e_2, & i = 2, \\ \\ e_i, & i \neq 2 \end{cases} .$$

and $\mathcal{G} = \{g_i\}_{i=1}^n$ be the frame defined by

$$g_i := \begin{cases} \sqrt{1-\epsilon^2} e_2, & i = 2, \\ \\ e_i, & i \neq 2 \end{cases} .$$

It is easy to show that $\max_{i=1}^n \|f_i - g_i\| \le \epsilon$ and $\mathcal{R}_{\mathcal{G}}^- = \mathcal{R}_{\mathcal{G}}^+ = 1$. But $\mathcal{R}_{\mathcal{F}}^- = 1 - \epsilon$ and $\mathcal{R}_{\mathcal{F}}^+ = 1 + \epsilon$, by [12, Example 2.3].

Example 2.7. Let $\epsilon > 0$ be given. Assume that $\mathcal{F} := \left\{\left(1 + \frac{\epsilon}{2}\right) e_1, \left(1 + \frac{\epsilon}{2}\right) e_2, ..., \left(1 + \frac{\epsilon}{2}\right) e_n\right\}$, where $\mathcal{E} := \{e_1, e_2, ..., e_n\}$ is the orthonormal basis for $\mathcal{H}_n$. Then $\max_{i=1}^n \|f_i - e_i\| \le \epsilon$, but $FP(\mathcal{F}) = n\left(1 + \frac{\epsilon}{2}\right)^4 > n = FP(\mathcal{E})$.

For a given positive self adjoint operator S on n-dimensional Hilbert space $\mathcal{H}_n$ and $m \ge n$, Casazza and Leon give necessary and sufficient conditions on $\alpha_1 \ge \alpha_2 \ge ... \ge \alpha_m > 0$ so that there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_n$ with frame operator S and $\|f_i\| = \alpha_i$, for all $1 \le i \le m$:

Theorem 2.8. *[9] Let S be a positive self adjoint operator on n-dimensional Hilbert space $\mathcal{H}_n$. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalues of S. Fixed $m \geq n$ and real numbers $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_m > 0$. The following are equivalent:*

(1) *There is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_n$ with frame operator S and $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$.*

(2) *For every $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \alpha_i^2 \leq \sum_{i=1}^{k} \lambda_i \qquad \textit{and} \qquad \sum_{i=1}^{m} \alpha_i^2 = \sum_{i=1}^{n} \lambda_i.$$

Using the above theorem, all possible finite frame redundancies is characterized in the following theorem.

Theorem 2.9. *Let $0 < \mu \leq \lambda$ and positive integers $m \geq n$ be given. Then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_n$ such that $\mathcal{R}_{\mathcal{F}}^- = \mu$ and $\mathcal{R}_{\mathcal{F}}^+ = \lambda$ if and only if $\lambda \geq 1$ and $(n-1)\mu + \lambda \leq m \leq \mu + (n-1)\lambda$.*

Proof. We first assume that $\mathcal{F} = \{f_i\}_{i=1}^m$ is a frame for $\mathcal{H}_n$ such that $\mathcal{R}_{\mathcal{F}}^- = \mu$ and $\mathcal{R}_{\mathcal{F}}^+ = \lambda$. Then for the frame $\mathcal{G} := \left\{\frac{f_i}{\|f_i\|}\right\}_{i=1}^m$ we have

$$\lambda = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n = \mu \qquad \textit{and} \qquad \alpha_i := \left\|\frac{f_i}{\|f_i\|}\right\| = 1, \quad \forall i \in \{1, 2, ..., m\},$$

where $\lambda_1, \lambda_2, ...\lambda_n$ are eigenvalues of the frame operator of the frame $\mathcal{G}$. Using Theorem 2.8 we have $\lambda \geq 1$, $m = \lambda = \mu$, for $n = 1$ and $\lambda \geq 1$, $m = \lambda + \mu$, for $n = 2$. Also for $n > 2$, we have

$$\begin{cases} 1 & \leq \lambda \\ j & \leq \lambda + \sum_{i=2}^{j} \lambda_i, \quad 2 \leq j \leq n-1, \\ n & \leq \lambda + \mu + \sum_{i=2}^{n-1} \lambda_i, \\ m & = \lambda + \mu + \sum_{i=2}^{n-1} \lambda_i. \end{cases} \tag{2.1}$$

Now using relationships (2.1) we have

$$(n-2)\mu \leq \sum_{i=2}^{n-1} \lambda_i = m - \lambda - \mu = \sum_{i=2}^{n-1} \lambda_i \leq (n-2)\lambda$$

and hence

$$(n-1)\mu + \lambda \leq m \leq \mu + (n-1)\lambda.$$

Conversely, let $\lambda \geq 1$ and $(n-1)\mu + \lambda \leq m \leq \mu + (n-1)\lambda$. If $\mu \leq 1$, then $m - \mu \geq n - 1$ and if $\mu \geq 1$, then $m - \mu \geq (n-2)\mu + \lambda \geq n - 2 + 1 = n - 1$. Thus we have $m - \mu \geq n - 1$. Set $\lambda_1 = \lambda = \mu$, for $n = 1$,

$$\lambda_j := \begin{cases} \lambda, & j = 1, \\ \mu, & j = 2 \end{cases}$$

for $n = 2$ and

$$\lambda_j := \begin{cases} \lambda, & j = 1, \\ \mu, & j = n, \\ \frac{m-\lambda-\mu}{n-2}, & 2 \leq j \leq n-1 \end{cases}$$

for $n > 2$. Also set

$$\alpha_i = 1, \qquad \forall\ 1 \leq i \leq m.$$

If $n > 2$, then using inequality $m - \mu \geq n - 1$, we have

$$\begin{aligned} \sum_{i=1}^{j} \lambda_i &= \lambda + \sum_{i=2}^{j} \lambda_i \\ &= \lambda + (j-1)\,\frac{m-\lambda-\mu}{n-2} \\ &= \frac{(n-1-j)\,\lambda + (j-1)\,(m-\mu)}{n-2} \\ &\geq \frac{n-1-j+(j-1)\,(n-1)}{n-2} \\ &= j \\ &= \sum_{i=1}^{j} \alpha_i^2, \end{aligned}$$

for any $1 \leq j \leq n-1$. Also

$$\begin{aligned} \sum_{i=1}^{n} \lambda_i &= \lambda + \mu + \sum_{i=2}^{j} \lambda_{n-1} \\ &= \lambda + \mu + (n-2)\,\frac{m-\lambda-\mu}{n-2} \\ &= m \\ &= \sum_{i=1}^{m} \alpha_i^2. \end{aligned}$$

For $n = 1$ and $n = 2$, the above relations hold, obviously.

Now using Theorem 2.8, there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_n$ with $\|f_i\| = 1$ for $1 \leq i \leq m$ such that the largest eigenvalue of the frame operator of $\mathcal{F}$ is λ and the smallest eigenvalue of the frame operator of $\mathcal{F}$ is μ. It follows that $\mathcal{R}_{\mathcal{F}}^- = \mu$ and $\mathcal{R}_{\mathcal{F}}^+ = \lambda$. □

All possible finite frame redundancies with special frame potential is characterized in the following theorem.

Theorem 2.10. *Let $0 < \mu \leq \lambda$, $\gamma > 0$ and positive integers $m \geq n$ be given. Then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_n$ such that $\mathcal{R}_{\mathcal{F}}^- = \mu$, $\mathcal{R}_{\mathcal{F}}^+ = \lambda$ and $FP(\mathcal{F}) = \gamma$ if and only if $\lambda \geq 1$ and $(n-1)\mu + \lambda \leq m \leq \mu + (n-1)\lambda$.*

Proof. If $\lambda \geq 1$ and $(n-1)\mu + \lambda \leq m \leq \mu + (n-1)\lambda$, then there is a frame $\mathcal{F}_0 = \{f_i\}_{i=1}^m$ for $\mathcal{H}_n$ such that $\mathcal{R}_{\mathcal{F}_0}^- = \mu$, $\mathcal{R}_{\mathcal{F}_0}^+ = \lambda$, by Theorem 2.9. Now let $\mathcal{F} := \left\{\sqrt[4]{\frac{\gamma}{FP(\mathcal{F}_0)}} f_i\right\}_{i=1}^m$. For any $x \in \mathbb{S}$ we have

$$\begin{aligned}\mathcal{R}_{\mathcal{F}}(x) :&= \sum_{i=1}^m \frac{\left|\left\langle x, \sqrt[4]{\frac{\gamma}{FP(\mathcal{F}_0)}} f_i\right\rangle\right|^2}{\left\|\sqrt[4]{\frac{\gamma}{FP(\mathcal{F}_0)}} f_i\right\|^2} \\ &= \sum_{i=1}^m \frac{|\langle x, f_i\rangle|^2}{\|f_i\|^2} \\ &= \mathcal{R}_{\mathcal{F}_0}(x).\end{aligned}$$

Thus $\mathcal{R}_{\mathcal{F}}^- = \mathcal{R}_{\mathcal{F}_0}^- = \mu$ and $\mathcal{R}_{\mathcal{F}}^+ = \mathcal{R}_{\mathcal{F}_0}^+ = \lambda$. Also

$$\begin{aligned}FP(\mathcal{F}_0) &= \sum_{i,j=1}^m \left|\left\langle \sqrt[4]{\frac{\gamma}{FP(\mathcal{F}_0)}} f_i, \sqrt[4]{\frac{\gamma}{FP(\mathcal{F}_0)}} f_j\right\rangle\right|^2 \\ &= \frac{\gamma}{FP(\mathcal{F}_0)} \sum_{i,j=1}^m \left|\left\langle f_i, f_j\right\rangle\right|^2 \\ &= \frac{\gamma}{FP(\mathcal{F}_0)} FP(\mathcal{F}_0) \\ &= \gamma.\end{aligned}$$

The converse of the theorem holds by Theorem 2.9. □

All possible frame potential of finite frames with prescribed norms is characterized in the next theorem.

Theorem 2.11. *Let positive integers $m \geq n$, real numbers $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m > 0$ and $\gamma > 0$ be given. Then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_n$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and*

$FP(\mathcal{F}) = \gamma$ if and only if there exist positive numbers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ such that

$$\begin{cases} \sum_{i=1}^{k} \alpha_i^2 \leq \sum_{i=1}^{k} \lambda_i, \qquad 1 \leq k \leq n, \\ \sum_{i=1}^{m} \alpha_i^2 = \sum_{i=1}^{n} \lambda_i, \\ \sum_{i=1}^{n} \lambda_i^2 = \gamma. \end{cases} \tag{2.2}$$

Proof. If there exist positive numbers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ such that (2.2) is satisfied, then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^{m}$ for $\mathcal{H}_n$ with $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$, such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ are the eigenvalues of the frame operator S with respect to an orthonarmal basis $\{e_i\}_{i=1}^{n}$ for $\mathcal{H}_n$, by Theorem 2.8.

Using spectral mapping Theorem and equality $trace\left(S^2\right) = FP(\mathcal{F})$[6], we have

$$\begin{aligned} FP(\mathcal{F}) &= trace\left(S^2\right) \\ &= \sum_{i=1}^{n} \lambda_i^2 \\ &= \gamma. \end{aligned}$$

Conversely, suppose that $\mathcal{F} = \{f_i\}_{i=1}^{m}$ is a frame for $\mathcal{H}_n$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ be the eigenvalues of the frame operator S with respect to an orthonarmal basis $\{e_i\}_{i=1}^{n}$ for $\mathcal{H}_n$. Thus $\sum_{i=1}^{n} \lambda_i^2 = trace\left(S^2\right) = FP(\mathcal{F}) = \gamma$, and

$$\begin{cases} \sum_{i=1}^{k} \alpha_i^2 \leq \sum_{i=1}^{k} \lambda_i, \qquad 1 \leq k \leq n, \\ \sum_{i=1}^{m} \alpha_i^2 = \sum_{i=1}^{n} \lambda_i \end{cases}$$

by Theorem 2.8. □

Corollary 2.12. *If $\mathcal{F} = \{f_i\}_{i=1}^{m}$ is a frame for $\mathcal{H}_n$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$, then $\frac{1}{n}\alpha^2 \leq \gamma < \alpha^2$, where $\alpha = \sum_{i=1}^{m} \alpha_i^2$.*

Proof. There exist positive numbers $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ such that

$$\begin{cases} \sum_{i=1}^{m} \alpha_i^2 = \sum_{i=1}^{n} \lambda_i, \\ \sum_{i=1}^{n} \lambda_i^2 = \gamma. \end{cases}$$

by Theorem 2.11. Thus

$$\alpha^2 = (\lambda_1 + \lambda_2 + ... + \lambda_n)^2 > \lambda_1^2 + \lambda_2^2 + ... + \lambda_n^2 = \gamma$$

and

$$\begin{aligned}\alpha^2 &= (\lambda_1+\lambda_2+...+\lambda_n)^2\\ &= \lambda_1^2+\lambda_2^2+...+\lambda_n^2+2\sum_{i<j}\lambda_i\lambda_j\\ &\leq \lambda_1^2+\lambda_2^2+...+\lambda_n^2+\sum_{i<j}\left(\lambda_i^2+\lambda_j^2\right)\\ &= n\left(\lambda_1^2+\lambda_2^2+...+\lambda_n^2\right)\\ &= n\gamma.\end{aligned}$$

□

Corollary 2.13. *If $\gamma = \sum_{i=1}^{n} \alpha_i^4$, then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^{n}$ for $\mathcal{H}_n$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq n$ and $FP(\mathcal{F}) = \gamma$.*

Proof. Set $\lambda_i := \alpha_i^2$, for all $1 \leq i \leq n$ and apply Theorem 2.11. □

Corollary 2.14. *If $\gamma = \frac{1}{n}\left(\sum_{i=1}^{m} \alpha_i^2\right)^2$, such that $\sum_{i=1}^{k} \alpha_i^2 \leq \frac{k}{n}\sum_{i=1}^{m} \alpha_i^2$, for all $1 \leq k \leq n$, then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^{m}$ for $\mathcal{H}_n$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$.*

Proof. Set $\lambda_i := \frac{1}{n}\sum_{i=1}^{m} \alpha_i^2$, for all $1 \leq i \leq n$ and apply Theorem 2.11. □

3. Possible frame potential of frames with prescribed norms in $\mathcal{H}_2$ and $\mathcal{H}_3$

In this section, we focus on special cases $n = 2$ and $n = 3$. All possible frame potential of frames with prescribed norms in $\mathcal{H}_2$ is characterized in the following theorem.

Theorem 3.1. *Let integer $m \geq 2$, real numbers $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_m > 0$ and $\gamma > 0$ be given. Then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^{m}$ for $\mathcal{H}_2$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$ if and only if*

$$\begin{cases} \frac{1}{2}\alpha^2 \leq \gamma < \alpha^2, \\ \sqrt{2\gamma-\alpha^2} \geq 2\alpha_1^2 - \alpha \end{cases},$$

where $\alpha = \sum_{i=1}^{m} \alpha_i^2$.

Proof. If $\mathcal{F} = \{f_i\}_{i=1}^{m}$ is a frame for $\mathcal{H}_2$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$, then there exist positive numbers $\lambda_1 \geq \lambda_2$ such that

$$\begin{cases} \alpha_1^2 \leq \lambda_1, \\ \lambda_1 + \lambda_2 = \alpha, \\ \lambda_1^2 + \lambda_2^2 = \gamma \end{cases}$$

by Theorem 2.11.

We have $\frac{1}{2}\alpha^2 \leq \gamma < \alpha^2$, by Corollary 2.12. Also, using inequality $\alpha_1^2 \leq \lambda_1$ and equality $\lambda_1 + \lambda_2 = \alpha$, we have

$$\lambda_1 - \lambda_2 \geq 2\alpha_1^2 - \alpha$$

and hence

$$\begin{aligned}\sqrt{2\gamma - \alpha^2} &= \sqrt{2\lambda_1^2 + 2\lambda_2^2 - (\lambda_1 + \lambda_2)^2}\\ &= \sqrt{(\lambda_1 - \lambda_2)^2}\\ &= \lambda_1 - \lambda_2\\ &\geq 2\alpha_1^2 - \alpha.\end{aligned}$$

Conversely, if

$$\begin{cases} \frac{1}{2}\alpha^2 \leq \gamma < \alpha^2, \\ \sqrt{2\gamma - \alpha^2} \geq 2\alpha_1^2 - \alpha \end{cases},$$

then for

$$\begin{cases} \lambda_1 := \frac{\alpha + \sqrt{2\gamma - \alpha^2}}{2}, \\ \lambda_2 := \frac{\alpha - \sqrt{2\gamma - \alpha^2}}{2} \end{cases}$$

we have

$$\begin{cases} \lambda_1 \geq \lambda_2 > 0 \\ \alpha_1^2 \leq \lambda_1, \\ \lambda_1 + \lambda_2 = \alpha, \\ \lambda_1^2 + \lambda_2^2 = \gamma \end{cases}.$$

Then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_2$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$, by Theorem 2.11. □

Corollary 3.2. *If $\mathcal{F} = \{f_i\}_{i=1}^m$ is a frame for $\mathcal{H}_n (n \geq 2)$ with $FP(\mathcal{F}) = \gamma$, then*

$$\begin{cases} \frac{1}{2}\alpha^2 \leq \gamma < \alpha^2, \\ \sqrt{2\gamma - \alpha^2} \geq 2\alpha_1^2 - \alpha \end{cases},$$

where $\alpha = \sum_{i=1}^m \|Pf_i\|^2$, $\alpha_1 = \|Pf_1\|$ and P is the orthogonal projection of $\mathcal{H}_n$ onto $\mathcal{H}_2$.

Proof. For any $f \in \mathcal{H}_2$, we have

$$\begin{aligned} Sf &= \sum_{i=1}^{m} \langle f, Pf_i \rangle Pf_i \\ &= P\left(\sum_{i=1}^{m} \langle P^* f, f_i \rangle f_i \right) \\ &= PS_0P^*f, \end{aligned}$$

where S and S_0 are the frame operators of $\{Pf_i\}_{i=1}^m$ and $\mathcal{F}$, respectively. We have $S = PS_0P^*$ and hence

$$\begin{aligned} FP\left(\{Pf_i\}_{i=1}^m\right) &= trace\left(S^2\right) \\ &= trace\left(PS_0P^*PS_0P^*\right) \\ &= trace\left(PS_0^2P^*\right) \\ &= trace\left(S_0^2P^*P\right) \\ &= trace\left(S_0^2\right) \\ &= FP(\mathcal{F}) \\ &= \gamma. \end{aligned}$$

Now the result is obtained from Theorem 3.1. □

A similar proof as the proof of the Theorem 3.1, shows that, for $m \geq 1$, there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_1$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$ if and only if $\gamma = \alpha^2$.

All possible frame potential of frames with prescribed norms in $\mathcal{H}_3$ is characterized in the next theorem.

Theorem 3.3. *Let integer $m \geq 3$, real numbers $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_m > 0$ and $\gamma > 0$ be given. Then there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_3$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$ if and only if*

$$\begin{cases} \frac{1}{3}\alpha^2 \leq \gamma < \frac{1}{2}\alpha^2, \\ \gamma \geq \frac{1}{2}\left(3\alpha_1^4 + \alpha^2 - 2\alpha_1^2\alpha\right), \\ \cap_{j=1}^{5} I_j \neq \emptyset \end{cases} \tag{3.1}$$

or

$$\begin{cases} \frac{1}{2}\alpha^2 \leq \gamma, \\ \gamma \geq \frac{1}{2}\left(3\alpha_1^4 + \alpha^2 - 2\alpha_1^2\alpha\right), \\ \cap_{j=1}^{6} I_j \neq \emptyset \end{cases} \tag{3.2}$$

, where $\alpha = \sum_{i=1}^{m} \alpha_i^2$ *and*

$$\begin{cases} I_1 = \left[\frac{\alpha-\sqrt{6\gamma-2\alpha^2}}{3}, \frac{\alpha+\sqrt{6\gamma-2\alpha^2}}{3}\right], \\ I_2 = \left[\alpha - 2\alpha_1^2, \infty\right), \\ I_3 = \left(-\infty, \frac{\alpha-\alpha_1^2-\sqrt{2\gamma-3\alpha_1^4-\alpha^2+2\alpha_1^2\alpha}}{2}\right] \cup \left[\frac{\alpha-\alpha_1^2+\sqrt{2\gamma-3\alpha_1^4-\alpha^2+2\alpha_1^2\alpha}}{2}, \infty\right), \\ I_4 = \left(0, \frac{1}{3}\alpha\right], \\ I_5 = \left(-\infty, \frac{2\alpha-\sqrt{6\gamma-2\alpha^2}}{6}\right] \cup \left[\frac{2\alpha+\sqrt{6\gamma-2\alpha^2}}{6}, \infty\right), \\ I_6 = \left(-\infty, \frac{\alpha-\sqrt{2\gamma-\alpha^2}}{2}\right) \cup \left(\frac{\alpha+\sqrt{2\gamma-\alpha^2}}{2}, \infty\right) \end{cases}.$$

Proof. Using Theorem 2.11, there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_3$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$, if and only if there exist positive numbers $\lambda_1 \geq \lambda_2 \geq \lambda_3$ such that

$$\begin{cases} \alpha_1^2 \leq \lambda_1, \\ \lambda_3 \leq \alpha - \alpha_1^2 - \alpha_2^2, \\ \lambda_1 + \lambda_2 = \alpha - \lambda_3, \\ \lambda_1^2 + \lambda_2^2 = \gamma - \lambda_3^2 \end{cases}.$$

Now using the proof of the Theorem 3.1, there is a frame $\mathcal{F} = \{f_i\}_{i=1}^m$ for $\mathcal{H}_3$ such that $\|f_i\| = \alpha_i$, for all $1 \leq i \leq m$ and $FP(\mathcal{F}) = \gamma$, if and only if the system of inequalities

$$\begin{cases} \frac{1}{2}(\alpha - x)^2 \leq \gamma - x^2, \\ \gamma - x^2 < (\alpha - x)^2, \\ \sqrt{2(\gamma - x^2) - (\alpha - x)^2} \geq 2\alpha_1^2 - \alpha + x, \\ \frac{\alpha - x - \sqrt{2(\gamma - x^2) - (\alpha - x)^2}}{2} \geq x > 0 \end{cases}$$

has a solution in variable x.

It is easy to show that the above system of inequalities has a solution in variable x, if and only if (3.1) or (3.2) hold. □

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