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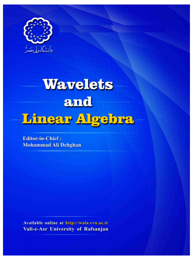

Wavelets
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Linear Algebra

Using of rational Haar wavelet to solve of nonlinear integro-differential equations

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ABSTRACT

This article uses the rational Haar wavelet and the successive method to solve the nonlinear Fredholm integral differential equation. Additionally, we have proved the convergence and order of convergence in this method by using the fixed point Banach theorem. In this way, numerical integration is not used. We also talk about two examples. We solved, drawing the absolute error, and plot of the exact and numerical solution. Finally, the results show that the proposed method is powerful for solving this equation.

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1. Introduction

Many fields of the problem can be modeled and used in many fields of science by Integro-differential equations (IDE), and nonlinear integro-differential equations (NIDE), involving chemical kinetics, bioinformatics, engineering, robotics and automation systems, physical science, Darbox and telegrahph equation, quantum systems, design and analysis of circuits, fluid dynamics, materials with memory. Also, the nonlinear integro-differential equation was formulated for a robotic arm's motion, boundary value problems, Biological Systems eigenvalue problems, models involving population dynamics, inverse problems, and mathematical models. Many different techniques have been developed for solving an approximation of this equation, such as Wavelet-Galerkin, Adomian decomposition, successive approximation method, the method of moments, spline functions, Homotopy perturbation, method of quadrature, Meshless methods, finite element or boundary element methods, Dzhumabaev's parametrization method, and trigonometric polynomials[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15]. Almost all the main characters of functions are used in this method, such as polynomial series, differing problems of dynamic problems or systems, and orthogonal functions, the element of solving a system of algebraic equations. [11, 12, 15, 17]. In this work, we study the numerical solution of the nonlinear Fredholm integral differential equation (NFIDE), specifically of the second kind, as follows

$$\Lambda'(t) = \omega(t) + \alpha \int_0^1 R(t, s, \Lambda(s))ds, \qquad \alpha \in \mathbb{R}, \tag{1.1}$$

Where $\Lambda(t)$, is the unknown function, $\omega(t)$ is a known continuous function, $R(t, s, \Lambda(s))$ is the kernel function, that

$$R(t, s, \Lambda(s)) \in C([0, 1]^2 \times \mathbb{R}, \mathbb{R}).$$

Also, R is continuous and Lipschitzian function as

$$|R(s, t, \Lambda_1(t)) - R(s, t, \Lambda_2(t))| \leq q|\Lambda_1(t) - \Lambda_2(t)|, \tag{1.2}$$

that $\Lambda_1, \Lambda_2 \in C([0, 1], \mathbb{R})$, and q is a Lipschitz constant, and α is a parameter determining the nature of the integral.

At first, we define the rational Haar (RH) wavelet as

Definition 1.1. The (RH) wavelet has a function defined by

$$h_r(x) = RH(2^r x - s), \quad s = 0, 1, \ldots, 2^{l-1}, \quad r = 2^l + s,$$

that

$$RH(x) = \begin{cases} 1, & 0 \leq x < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq x < 1, \\ 0, & otherwise. \end{cases} \tag{1.3}$$

Definition 1.2 ([16])**.** Expanded of function $f(t) \in C([0, 1])$ by RH function as

$$f(t) = \sum_{i=0}^{m-1} \mu_i h_i(t), \quad m = 2^{n+1}, \quad n = 0, 1, \ldots, \tag{1.4}$$

which as coefficients μ_i are given by

$$\mu_i = 2^j \int_0^1 \eta(t)h_i(t)dx = 2^j\langle\eta, h_i\rangle, \quad j = 0, 1, \ldots, n,$$

and the sequence $\{h_i\}_{i=0}^{\infty}$ is a complete orthonormal system in $L^2([0, 1])$.

Similar to the problem posed with P. L. Ulyanov in [16], the series $\sum_i 2^j\langle\eta, h_i\rangle h_i$, for $f(t) \in C[0, 1]$, converges uniformly to f. If (1.4) truncated up to its first m terms that $m = 2^{n+1}$, $n = 0, 1, \ldots$ and $j = 0, 1, \ldots, n$ then we have:

$$f(t) = \sum_{i=0}^{m-1} \mu_i h_i(t) = \mathbf{A}^T\mathbf{h}(t), \tag{1.5}$$

where

$$\mathbf{A} = [\mu_0, \mu_1, \ldots, \mu_{m-1}]^T, \qquad \mathbf{h}(t) = [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T.$$

By integrating the equations (1.1) from 0 to t we have

$$\Lambda(t) = \Lambda(0) + \int_0^t \omega(x)dx + \alpha \int_0^t \int_0^1 R(x, s, \Lambda(s))dsdx, \tag{1.6}$$

By using RH wavelet bases and Eq. (1.6) we have

$$\varphi_{i-1}(x, s) = R(x, s, \Lambda_{i-1}(s)) = \sum_{l=1}^{\infty}\sum_{q=1}^{\infty} b_{lq}h_l(x)h_q(s). \tag{1.7}$$

We assume W_m has an orthogonal projection, then we have

$$W_m(\varphi)(x, s) = \sum_{l=1}^{m-1}\sum_{q=1}^{m-1} b_{lq}h_l(x)h_q(s),$$

or

$$Q_m(\varphi_{n-1})(t, s) = \mathbf{h}^T(t)\mathbf{B}\mathbf{h}(s), \tag{1.8}$$

where

$$\mathbf{B} = [b_{lq}]_{m\times m}. \tag{1.9}$$

That $l = 2^j + k$, $q = 2^i + k'$ for $i, j = 0, 1, \ldots$, $k = 0, 1, \ldots, 2^j - 1$, $k' = 0, 1, \ldots, 2^i - 1$, and $0 \le l, q \le m - 1$.
By applying the RH function the matrice $\mathbf{B}$ are given as:

$$\mathbf{B} = (H_m^{-1})^T\hat{\mathbf{B}}H_m^{-1}, \tag{1.10}$$

where $\hat{\mathbf{B}} = [\hat{b}_{ij}]_{m\times m}$, and

$$\hat{b}_{ij} = \varphi_{n-1}(\frac{2i-1}{2m}, \frac{2j-1}{2m}), \ \ i, j = 1, 2, \ldots, m. \tag{1.11}$$

Here, by applying the RH function and the vector $\mathbf{h}(t)$, the matrix H_m is constructed as follows:

$$H_m = \left[\mathbf{h}(\frac{1}{2m}), \mathbf{h}(\frac{3}{2m}), \ldots, \mathbf{h}(\frac{2m-1}{2m})\right]. \tag{1.12}$$

Thus

$$\Lambda_i(t) = \Lambda(0) + \int_0^t \omega(x)dx + \alpha \int_0^t \int_0^1 W_m(\varphi_{i-1}(x,s))dsdx, \quad i = 1, 2, ... \tag{1.13}$$

2. Error analysis

In this section, we discuss how to prove the convergence of our method, for this purpose we used the fixed point Banach theorem. The first let T has an integral operator that T is a continuous and Lipschitzian function as

$$T : (C([0,1]), \|\cdot\|_\infty) \longrightarrow (C([0,1]), \|\cdot\|_\infty)$$

Thus:

$$T\Lambda(t) = \Lambda(0) + \int_0^t \omega(x)dx + \alpha \int_0^t \int_0^1 R(x,s,\Lambda(s))dsdx. \tag{2.1}$$

Lemma 2.1. *Let $R(t,s,\Lambda(s))$ are defined in (1.2), and Λ is a fixed point of T. Then for all $\Lambda_0 \in C([0,1],\mathbb{R})$ we have:*

$$\|\Lambda - T^n(\Lambda_0)\|_\infty \leq \|T(\Lambda_0) - \Lambda_0\|_\infty \sum_{j=n}^{\infty} q^j. \tag{2.2}$$

Also, T in (2.1) has a unique fixed point.

Proof: By applying (2.1) gives:

$$\begin{aligned}\left|T\Lambda_1(t) - T\Lambda_2(t)\right| &= \left|\alpha \int_0^t \int_0^1 (R(x,s,\Lambda_1(s)) - R(x,s,\Lambda_2(s)))dsdx\right| \\ &\leq |\alpha| \int_0^t \int_0^1 \left|R(x,s,\Lambda_1(s)) - R(x,s,\Lambda_2(s))\right|dsdx \\ &\leq |\alpha| \int_0^t \int_0^1 q|\Lambda_1(s) - \Lambda_2(s)|dsdx \leq |\alpha|q\|\Lambda_1 - \Lambda_2\|_\infty. \end{aligned} \tag{2.3}$$

Thus we have

$$\left|T\Lambda_1(t) - T\Lambda_2(t)\right| \leq |\alpha|q\|\Lambda_1 - \Lambda_2\|_\infty, \tag{2.4}$$

By induction on $n \in \mathbb{N}$ we have:

$$\|T^n\Lambda_1 - T^n\Lambda_2\|_\infty \leq (|\alpha|q)^n\|\Lambda_1 - \Lambda_2\|_\infty.$$

Thus, Eq. (2.1) has a unique solution, if we assume $|\alpha|q < 1$ then T is a contraction mapping, and

$$\sum_{n=1}^{\infty} \|T^n\Lambda_1 - T^n\Lambda_2\|_\infty < \infty.$$

.

Theorem 2.2. *Assume that* $\varphi_{i-1} \in \mathbb{C}([0,1]^2)$ *and* $\{\Lambda_i\}_{i\geq 1}$ *is a subset of* $C([0,1])$*, we have:*

$$\|\Lambda - \Lambda_i\|_\infty \leq \|T(\Lambda_0) - \Lambda_0\|_\infty \sum_{j=i}^{\infty} q^j + \sum_{j=1}^{i} q^{i-j}\varepsilon_j, \tag{2.5}$$

that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i > 0$ *for* $i \geq 1$.

Proof: Suppose that

$$\begin{aligned} &\left\|T(\Lambda_i) - \Lambda_i\right\|_\infty \leq \\ &|\alpha|\left\|\int_0^t \int_0^1 \varphi_{i-1}(x,s) - W_m(\varphi_{i-1})(x,s)dxds\right\|_\infty \leq |\alpha|\|\varphi_{i-1} - W_m(\varphi_{i-1})\|_\infty. \end{aligned} \tag{2.6}$$

We assume

$$f(t,s) := \varphi_{i-1} - W_m(\varphi_{i-1}).$$

By applying of interpolating property for variables t_l, s_j that $l, j \leq m-1$, we have:

$$\begin{aligned} t_0 &= 0, \quad t_l = \frac{1}{2^{n_1+1}} + \frac{v_1}{2^{n_1}}, \qquad l = 2^{n_1} + v_1, \qquad n_1, n_2 \geq 1, \\ s_0 &= 0, \quad s_j = \frac{1}{2^{n_2+1}} + \frac{v_2}{2^{n_2}}, \qquad j = 2^{n_2} + v_2, \end{aligned}$$

by applied mean-value theorem for function $f(\cdot,\cdot)$ we have

$$\begin{aligned} &\|\varphi_{i-1} - W_m(\varphi_{i-1})\|_\infty = \\ &\left\|f(t_l, s_j) + \frac{\partial f}{\partial t}(\xi,\gamma)(\xi - t_l) + \frac{\partial f}{\partial s}(\xi,\gamma)(\gamma - s_j)\right\|_\infty \\ = &\left\|(I - W_m)\frac{\partial \varphi_{i-1}}{\partial t}(\xi,\gamma) + (I - W_m)\frac{\partial \varphi_{i-1}}{\partial s}(\xi,\gamma)\right\|_\infty \max\{\|\xi - t_l\|_\infty, \|\gamma - s_j\|_\infty\} \\ \leq &\frac{2}{2^i}\|(I - W_m)\|_\infty \left\|\frac{\partial \varphi_{i-1}}{\partial t}(\xi,\gamma) + \frac{\partial \varphi_{i-1}}{\partial s}(\xi,\gamma)\right\|_\infty. \end{aligned} \tag{2.7}$$

We assume

$$L_{i-1} = \max\{\|\frac{\partial \varphi_{i-1}}{\partial t}\|_\infty, \|\frac{\partial \varphi_{i-1}}{\partial s}\|_\infty\}, \quad i = 1, 2, \ldots, \tag{2.8}$$

thus in Eq. (2.7) by setting Eq. (2.8) we have

$$\|\varphi_{i-1} - W_m(\varphi_{i-1})\|_\infty \leq \frac{4L_{i-1}}{2^i}.$$

Therefore, in the inequality (2.6) we have:

$$\|T(\Lambda_{i-1}) - \Lambda_i\|_\infty \leq |\alpha|\frac{4L_{i-1}}{2^i}.$$

If

$$|\alpha|\frac{4L_{i-1}}{2^i} < \varepsilon_k, \quad k = 1, 2, ..., i,$$

thus

$$\|T(\Lambda_{i-1}) - \Lambda_i\|_\infty < \varepsilon_i. \tag{2.9}$$

Applying the triangle inequality we achieve:

$$\|\Lambda - \Lambda_i\|_\infty \leq \|\Lambda - T^i(\Lambda_0)\|_\infty + \sum_{j=1}^{i} q^j \|T(\Lambda_{j-1}) - \Lambda_j\|_\infty.$$

Finally, based on equation (2.2) in Lemma 2.1 and equation (2.9) we conclude that:

$$\|\Lambda - \Lambda_i\|_\infty \leq \|T(\Lambda_0) - \Lambda_0\|_\infty \sum_{j=i}^{\infty} q^j + \sum_{j=1}^{i} q^{i-j}\varepsilon_j. \tag{2.10}$$

Thus, the result follows.

3. Illustrative examples

In this section, we give two numerical examples. In our examples, we use the method discussed in section 1.

Example 3.1. Let the NFIDE equation as follows:

$$\Lambda'(x) - \frac{1}{38}x^5 sin(5x) = \frac{1}{40}\int_0^1 x^2 sin^2(3t) t^8 \Lambda^4(t)dt,$$

the precise solution is $\Lambda(x) = \frac{1}{25}x + \frac{1}{30}sin^2(3x)$.

We calculated the absolute error of example 3.1 in $i = 2$, and $i = 3$, and show it in Table 1. The comparison between numerical and exact solutions in Figure 1 is depicted. The absolute errors for $i = 3$ are traced in Figure 2.

Table 1: Absolute errors of Example 3.1.

x_i	$m = 2^3$	$m = 2^4$
0.1	$6.12E{-6}$	$5.80E{-6}$
0.2	$6.12E{-6}$	$5.80E{-6}$
0.3	$6.12E{-6}$	$5.81E{-6}$
0.4	$6.14E{-6}$	$5.82E{-6}$
0.5	$6.18E{-6}$	$5.86E{-6}$
0.6	$6.23E{-6}$	$5.92E{-6}$
0.7	$6.28E{-6}$	$5.97E{-6}$
0.8	$6.29E{-6}$	$5.97E{-6}$
0.9	$6.089E{-6}$	$5.76E{-6}$
CPU-Time (s)	0.094	0.765

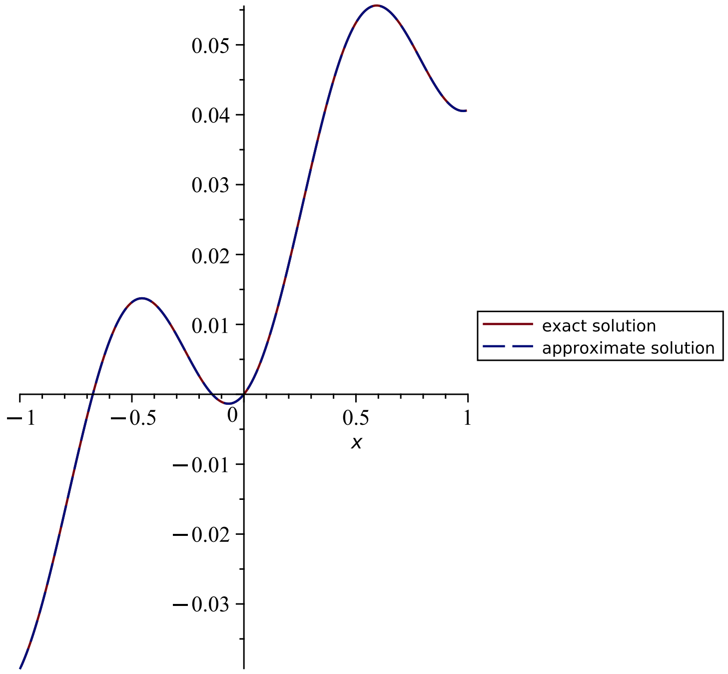

Figure 1: The depicted of exact and numerical solution for $i = 3$ in Example 3.1.

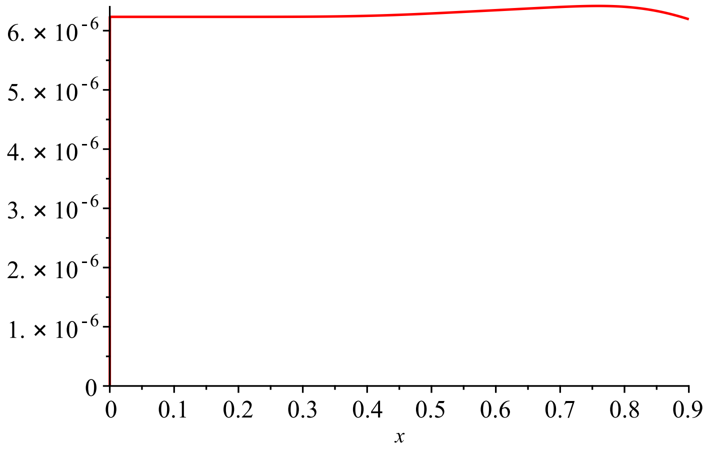

Figure 2: Plot of absolute errors of Example 3.1.

Example 3.2. Let the NFIDE equation as follows:

$$\Lambda'(x) - \frac{1}{35}x^3 ln(x) = \frac{1}{40}\int_0^1 x^3 ln^4(t)(1+t)^4\Lambda(t)dt,$$

the precise solution is $\Lambda(x) = \frac{1}{6}x^5 - \frac{1}{8}x^4$.

We calculated the absolute error of example 3.2 in $I = 2$, and $I = 3$, and show it in Table 2. The comparison between numerical and exact solutions in Figure 3 is depicted. The absolute errors for $I = 3$ are traced in Figure 4. The CPU time is represented in the lost row of the table 2.

Table 2: Absolute errors of Example 3.2.

x_i	$m = 2^3$	$m = 2^4$
0.1	$-4.81E{-7}$	$-4.30E{-7}$
0.2	$5.21E{-7}$	$5.71E{-7}$
0.3	$3.34E{-7}$	$3.85E{-7}$
0.4	$1.29E{-6}$	$1.34E{-6}$
0.5	$1.37E{-6}$	$1.42E{-6}$
0.6	$-9.60E{-7}$	$-9.10E{-7}$
0.7	$1.10E{-6}$	$1.15E{-6}$
0.8	$-2.35E{-6}$	$-2.30E{-6}$
0.9	$4.96E{-6}$	$4.74E{-6}$
CPU-Time (s)	0.0237	0.15

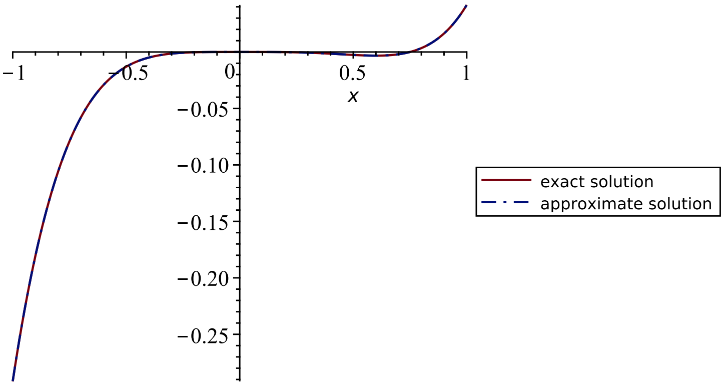

Figure 3: The plot of exact and numerical solution for $i = 3$ in Example 3.2.

4. Conclusion

NFIDE stands at the confluence of theoretical mathematics and practical application. In many cases, analysis of solutions (NFIDE) is challenging or even impossible. As scientific challenges evolve, so too will the methodologies for their resolution, paving the way for more advanced applications across diverse fields of study. In many instances, we calculate approximate solutions. This paper introduces a novel approach utilizing the RH wavelet for solving this kind of equation. Also, this method we approach avoids numerical integration and proves the convergence of this method. Additionally, we have included numerical results in the illustrative examples section. The CPU times required for problem-solving have been computed, and the study results are elaborated in some tables.

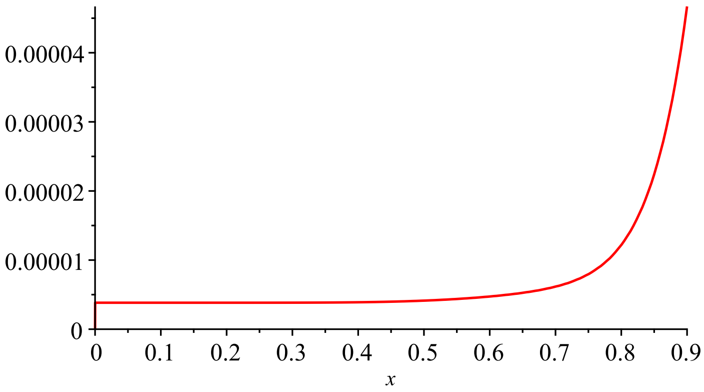

Figure 4: Absolute errors of Example 3.2.

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