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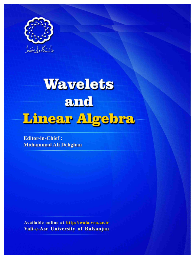

Characterizing Lagrange Multipliers with Set Valued Constraints by Using Contingent Epiderivatives

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Abstract

In this paper, we employ the generalized Guignard's constraint qualification to present the dual cone characterizations of the constraint set S with set valued constraints in $\mathbb{R}^n$. The obtained results provide sufficient conditions for which the "strong conical hull intersection property" (strong CHIP, in short) holds. Moreover, we establish necessary and sufficient conditions for characterizing "perturbation property" of the constrained best approximation to any point $x \in \mathbb{R}^n$ from a convex set $\tilde{S} := K \cap S$ by the strong CHIP of K and S at a reference point, where K is a non-empty closed convex set in $\mathbb{R}^n$. Finally, under the generalized Guignard's constraint qualification we derive the Lagrange multipliers characterizations of the constrained best approximation with set valued constraints. The clarification of our results is illustrated by the numerical experiments.

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1. Introduction

This paper concerns to the study of the Lagrange multipliers characterizations of constrained best approximation with C-convex set valued inequality constraints that is equivalent to special classes of set valued optimization problems. Set valued optimization is an effective and beneficial branch of applied mathematics that deals with optimization problems which have set valued objectives and/or set valued constraints. Furthermore, the set optimization gives not only a new spectrum of powerful techniques to many applications in applied sciences, but also many applications in pure and applied mathematics itself. For example, optimal control, duality principles in vector optimization, inverse problem for variational inequalities, image processing problems, fuzzy optimization problems, game theory, robust optimization, stochastic optimization, mathematical finance and welfare economics all lead to the set valued optimization problems. We refer the reader to [2, 6, 15, 18, 19, 20, 21, 22, 25, 26, 35] for more details on the history of set valued and nonlinear analysis. Recently, many researchers have been interested in investigating set valued optimization problems by using different kinds of generalized derivatives of set valued mappings (see [20, 21, 22, 36] and the references therein). On the other hand, convexity plays a crucial role in set valued optimization. In recent years, many scholars have introduced some convexity versions and its generalizations using different types of set relations to deal with set valued optimization problems. We refer the reader to [8, 15, 26, 33, 35] and the references therein for various types of set relations and convexity versions.

In many optimization problems, the Lagrange multipliers play a central role, moreover, constraint qualifications are essential ingredients of optimization and approximation theory. The main purpose of this paper is to present necessary and sufficient conditions for the constrained best approximation problem according to the Lagrange multipliers using the generalized Guignard's constraint qualification in the finite dimensional spaces that inequality constraints are C-convex set valued mappings and sets are compared with the upper set less order relation, where C is a non-empty pointed closed convex cone in $\mathbb{R}^n$. Indeed, we show under the generalized Guignard's constraint qualification that the "perturbation property" of the constrained best approximation from a non-empty closed set $K \cap S$ is characterized by the "strong conical hull intersection property" (strong CHIP, in short) at a reference feasible point $x \in \tilde{S}$, where K is a non-empty closed convex set in $\mathbb{R}^n$ and $\tilde{S} := K \cap S$. We do this by first establishing the dual cone characterizations of the constraint set S. Finally, we present the "Lagrange multipliers characterizations" of the constrained best approximation. Our results extend and solve the constrained best approximation problem with set valued constraints, in a general case, and not only recapture the corresponding known results of [3, 4, 10, 16, 23, 24, 27, 28, 29, 30, 32] and the references therein, but (in a particular case) also allow applications to problems that their constraint functions g_j, $j = 1, 2, \ldots, k$, are single valued quasiconvex as they guarantee the convexity of the constraint set S. Simple numerical examples illustrate the nature of our results. It is worth noting that the Guignard's constraint qualification is much less known and can hardly be found in any textbook. However, it was introduced in [14] and noted in [31] that it is the weakest constraint qualification, which guarantees that at the local minimum of an optimization problem there exist Lagrange multipliers such that the Karush-Kuhn-Tucker conditions are first-order optimality conditions. Because the dual cone characterization theorem is a key factor in providing the necessary and sufficient conditions for the constrained best approximation problem, therefore in this paper, we first introduce the generalized Guignard's constraint qualification which plays a fundamental role to present the dual cone characterizations of the constraint set S (using the contingent epiderivative concept for set valued mappings) with C-convex set valued inequality constraints. The main motivation behind considering the contingent epiderivative is the fact that the epigraph of a set valued mapping has significantly better structure than the graph of a set valued mapping. In the classical case, when the inequality constraints are real valued, convex/nonconvex and differentiable functions, many fundamental notions of so-called constraint qualifications for the constrained best approximation problem have been introduced (see [9, 11, 12, 13, 23, 24, 27, 28] and the references therein).

The layout of the paper is organized as follows. In Section 2, we provide some basic notions, definitions and preliminary results. Establishing the dual cone characterizations of the constraint set S and sufficient conditions for which the strong CHIP property holds are given in Section 3. Characterizing the "perturbation property" of the constrained best approximation are presented in Section 4. Finally, under the generalized Guignard's constraint qualification we give the "Lagrange multipliers characterizations" of the constrained best approximation with set valued constraints. Also, by simple numerical examples we illustrate the nature of our results. In Section 5, the conclusions and applications are presented.

2. Preliminaries

We recall from [26] the notion of the upper set less order relation $\precsim^u$ on a real normed space Y that is defined as follows: let $A, B \in \mathcal{P}(Y)$ ($\mathcal{P}(Y)$ is the set of all subsets of Y) be non-empty and arbitrary subsets of Y. One says that

$$A \precsim^u_C B \quad \Longleftrightarrow \forall\, x \in A\; \exists\, y \in B \;\ni\; x \leq_C y, \tag{2.1}$$

or equivalently, $A \subseteq B - C$, where the proper closed convex and pointed cone $C \subset Y$ induces a partial ordering $\leq_C$ on Y, which is defined as follows: for each $x, y \in Y$,

$$x \leq_C y \Longleftrightarrow y - x \in C.$$

We now state the basic concepts and some results of set optimization (see [15, 18, 25, 33] and the references therein for more details). For presenting the definition and properties of the contingent epiderivative, we give the following standard assumption.

Assumption 2.1. Let X and Y be real normed spaces, and let Y be partially ordered by a closed convex and pointed cone $C \subset Y$, which is assumed to be proper, i.e., $\{0\} \neq C \neq Y$, and let $F : X \rightrightarrows Y$ be a set valued mapping. Note that a cone C is called pointed if $C \cap (-C) = \{0\}$.

In the sequel, we present some concepts of the set valued mapping $F : X \rightrightarrows Y$ [3]. We define the domain of F by:

$$\mathrm{dom}\,(F) := \{x \in X : F(x) \neq \emptyset\}.$$

The graph of F is defined as follows:

$$Graph(F) := \{(x, y) \in X \times Y : x \in \mathrm{dom}\,(F),\ \text{and } y \in F(x)\}.$$

The epigraph of F is defined by:

$$\text{epi}\,(F) := \{(x, y) \in X \times Y : x \in \text{dom}\,(F), \text{ and } y \in F(x) + C\}.$$

In the case that we use the upper set less order relation $\precsim_C^u$, we can define

$$\text{epi}\,(F) := \{(x, y) \in X \times Y : x \in \text{dom}\,(F), \text{ and } F(x) \subseteq y - C\}.$$

The set valued mapping $F : X \rightrightarrows Y$ is called C-convex if for all $x_1, x_2 \in \text{dom}\,(F)$ and all $0 \leq \alpha \leq 1$,

$$\alpha F(x_1) + (1 - \alpha)F(x_2) \subseteq F(\alpha x_1 + (1 - \alpha)x_2) + C.$$

Therefore, we say that the set valued mapping $F : X \rightrightarrows Y$ is $\precsim_C^u$-convex if for all $x_1, x_2 \in \text{dom}\,(F)$ and all $0 \leq \alpha \leq 1$,

$$F(\alpha x_1 + (1 - \alpha)x_2) \subseteq \alpha F(x_1) + (1 - \alpha)F(x_2) - C.$$

Remark 2.1. It is not difficult to check that if the set valued mapping $F : X \rightrightarrows Y$ is $\precsim_C^u$-convex, then, epi$\,(F)$ is convex with respect to the upper set less order relation $\precsim_C^u$. Also, the set valued mapping F is C-convex if and only if its epigraph is a convex set in $X \times Y$.

We now recall the definition and properties of the contingent epiderivative of the set valued mapping $F : X \rightrightarrows Y$ that will be used throughout the paper.

Definition 2.1. [1, 18, 25] Let X and Y be real normed spaces and Assumption 2.1 hold. Let $(\bar{x}, \bar{y}) \in Graph(F)$ be given. A single valued mapping $D_E F(\bar{x}, \bar{y}) : X \longrightarrow Y$ is called the contingent epiderivative of F at the point $(\bar{x}, \bar{y})$, if the following identity holds.

$$epi(D_E F(\bar{x}, \bar{y})) = T(epi(F); (\bar{x}, \bar{y})),$$

where $T(S\,; x)$ is called the contingent cone of a subset S of a real normed space V at a point $x \in S$, and is defined by:

$$\begin{aligned} T(S\,; x) \quad &:= \{v \in V : \exists\, \lambda_m > 0, \exists\, v_m \in V \ s.t.\ \lambda_m \longrightarrow 0^+, \\ &v_m \longrightarrow v, x + \lambda_m v_m \in S, \forall\, m \geq 1\}. \end{aligned} \tag{2.2}$$

It is easy to show that the contingent cone $T(S\,; x)$ can be represented as follows:

$$T(S\,; x) = \{v \in V : \exists\, \lambda_m > 0, \exists\, x_m \in S \ \ s.t.\ \lambda_m \longrightarrow 0^+, x_m \longrightarrow x, \frac{x_m - x}{\lambda_m} \longrightarrow v\}.$$

Remark 2.2. It should be noted that if the point $(\bar{x}, \bar{y})$ belongs to the interior of $Graph(F)$, then the contingent cone of $epi(F)$ at $(\bar{x}, \bar{y})$ equals to the product space $X \times Y$ and, in this case, the contingent epiderivative $D_E F(\bar{x}, \bar{y})$ does not exist.

In what follows is an existence theorem for contingent epiderivative, in the special case, when $Y := \mathbb{R}$.

Theorem 2.1. [18, 25] Suppose that X is a real normed space and Assumption 2.1 is satisfied. Let $(\bar{x}, \bar{y}) \in Graph(F)$ be given. Assume that there are functionals $f_1, f_2 : X \longrightarrow \mathbb{R}$ such that

$$epi(f_1) \supseteq T(epi(F); (\bar{x}, \bar{y})) \supseteq epi(f_2).$$

Then the contingent epiderivative $D_E F(\bar{x}, \bar{y})$ is given by:

$$D_E F(\bar{x}, \bar{y})(x) = \min\{y \in \mathbb{R} : (x, y) \in T(epi(F); (\bar{x}, \bar{y}))\}, \ \forall\, x \in X,$$

where for the real valued function $f : X \longrightarrow \mathbb{R}$, the epigraph of f is defined by:

$$\text{epi}\,(f) := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}.$$

In the following, we give some properties of the contingent epiderivative.

Theorem 2.2. [18] Suppose that X and Y are real normed spaces and Assumption 2.1 is satisfied. Let $(\bar{x}, \bar{y}) \in Graph(F)$ be given. If the contingent epiderivative $D_E F(\bar{x}, \bar{y})$ exists, then it is unique.

Theorem 2.3. [18] Suppose that X and Y are real normed spaces and Assumption 2.1 is satisfied. Let the set valued mapping $F : X \rightrightarrows Y$ be $\precsim_C^u$-convex. If the contingent epiderivative $D_E F(\bar{x}, \bar{y})$ exists at the point $(\bar{x}, \bar{y}) \in Graph(F)$, then, $D_E F(\bar{x}, \bar{y})$ is positively homogeneous and subadditive.

We now state a generalization of the concept of the subdifferential of a convex function for a set valued mapping. We first give the following assumption.

Assumption 2.2. Let X and Y be real normed spaces and Assumption 2.1 be satisfied. Let the set valued mapping $F : X \rightrightarrows Y$ be C-convex, and let $(\bar{x}, \bar{y}) \in Graph(F)$ be such that the contingent epiderivative $D_E F(\bar{x}, \bar{y})$ of F at the point $(\bar{x}, \bar{y})$ exists.

Definition 2.2. [18] Suppose that Assumption 2.2 holds. A linear mapping $L : X \longrightarrow Y$ is called a subgradient of the set valued mapping $F : X \rightrightarrows Y$ at the point $(\bar{x}, \bar{y}) \in Graph(F)$, if

$$L(x) \leq_C D_E F(\bar{x}, \bar{y})(x), \ \forall\, x \in X.$$

The subdifferential of F at the point $(\bar{x}, \bar{y})$ is the set of all subgradients L of F at $(\bar{x}, \bar{y})$, and is defined by:

$$\partial F(\bar{x}, \bar{y}) := \{L : X \longrightarrow Y : L \text{ is a linear mapping and } L(x) \leq_C D_E F(\bar{x}, \bar{y})(x), \forall\, x \in X\}. \qquad (2.3)$$

Note that Definition 2.2 is a natural extension of the convex subdiffferential of a convex function $f : X \longrightarrow \mathbb{R}$ (for example, see [3, 17]). Here the directional derivative is replaced by the contingent epiderivative and the usual ordering $\leq$ is replaced by the partial ordering $\leq_C$ induced by the proper closed convex and pointed cone C.

Remark 2.3. Note that the subdifferential (which defined by (2.3)) is not defined if the contingent epiderivative does not exist. Moreover, the convexity of F is actually not needed in Definition 2.2.

We next recall some basic properties of the subdifferential which defined by (2.3).

Theorem 2.4. [18] Suppose that Assumption 2.2 holds and that Y has the least upper bound property (i.e., every non-empty subset of Y with an upper bound has a least upper bound). Then, $\partial F(\bar{x}, \bar{y}) \neq \emptyset$ at every point $(\bar{x}, \bar{y}) \in Graph(F)$.

Theorem 2.5. [18] Suppose that Assumption 2.2 holds. Then the subdifferential (defined by (2.3)) is a convex set. Moreover, if all subgradients are bounded, then the subdifferential is closed in the space of all bounded linear mappings.

Remark 2.4. It should be noted that whenever $X := \mathbb{R}^n$ and $Y := \mathbb{R}^m$, then every linear mapping $L : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is bounded and, in this case, the subdifferential is closed.

The following theorem presents a condition under which the subdifferential (defined by (2.3)) is a singleton.

Theorem 2.6. [18] Suppose that Assumption 2.2 is satisfied. If the contingent epiderivative $D_E F(\bar{x}, \bar{y})$ of F at the point $(\bar{x}, \bar{y}) \in Graph(F)$ is linear, then, $\partial F(\bar{x}, \bar{y}) = \{D_E F(\bar{x}, \bar{y})\}$.

We now state the relationship between the above subdifferential and the classical subdifferential.

Theorem 2.7. [18] Suppose that Assumption 2.2 holds. Then every subgradient $L \in \partial F(\bar{x}, \bar{y})$ of the set valued mapping $F : X \rightrightarrows Y$ at the point $(\bar{x}, \bar{y}) \in Graph(F)$ satisfies the following inequality:

$$L(x - \bar{x}) \leq_C y - \bar{y}, \ \forall\, x \in dom(F), \forall\, y \in F(x).$$

From now on, throughout the paper, we assume that $X := \mathbb{R}^n$ and $Y := \mathbb{R}$, where $\mathbb{R}^n$ is the Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. We also replace the partial ordering $\leq_C$ induced by the proper closed convex and pointed cone C by the upper set less order relation $\precsim^u_C$, where $C := \mathbb{R}_+ \subset \mathbb{R}$, and $\mathbb{R}_+ := \{x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \ \forall\, i = 1, 2, \ldots, n\}$. Let $W \subseteq \mathbb{R}^n$ be a non-empty set. We denote by clW, $convW$, $conicW$ and $coneW$, the closure, convex hull, conical hull of W and the cone generated by W, respectively.

We define the constraint set S by:

$$S := \bigcap_{j=1}^{k} S_j, \tag{2.4}$$

where

$$S_j := \{x \in \mathbb{R}^n : x \in dom(G_j) \text{ and } G_j(x) \precsim^u_{\mathbb{R}_+} \{0\}\},$$

for all $j = 1, 2, \ldots, k$ ($k \in \mathbb{N}$), and the constraint set valued mappings $G_j : \mathbb{R}^n \rightrightarrows \mathbb{R}$ are $\precsim^u_{\mathbb{R}_+}$-convex and their contingent epiderivative exist at a given point of $Graph(G_j)$, $j = 1, 2, \ldots, k$ (we assume that the contingent epiderivative exists for each constraint set valued mapping G_j at every point of its graph). Obviously, for each $j = 1, 2, \ldots, k$, the set S_j is convex, and hence, the feasible set S is also convex. For each $x \in S$, let $I(x)$ be denoted the set of active indices at the point x, and is defined by:

$$I(x) := \{j \in \{1, 2, \ldots, k\} : \ 0 \in G_j(x)\}. \tag{2.5}$$

For each $x \in S$ and each $j \in I(x)$ (note that in this case, $0 \in G_j(x)$), we denote the conical hull of the contingent epiderivative $D_E G_j(x,0)$ ($j \in I(x)$) by $C_E(x)$, and is defined as follows:

$$C_E(x) := conic\{D_E G_j(x,0) : j \in I(x)\}. \tag{2.6}$$

It should be noted that the convex cone $C_E(x)$ has the following form:

$$C_E(x) = \left\{ \sum_{j \in I(x)} \lambda_j D_E G_j(x,0) : \lambda_j \geq 0, \ \forall \ j \in I(x) \right\}. \tag{2.7}$$

For a subset E of $\mathbb{R}^n$, we define [7, 11] the negative polar cone of E by:

$$E^\circ := \{y \in \mathbb{R}^n : \langle y, z \rangle \leq 0, \ \forall \ z \in E\}, \tag{2.8}$$

and the convex cone $E^{\circ\circ} := (E^\circ)^\circ$ is called the bipolar of E. We also define the normal cone to a convex set $E \subseteq \mathbb{R}^n$ at a point $x \in E$ by:

$$N_E(x) := \{y \in \mathbb{R}^n : \langle y, z - x \rangle \leq 0, \ \forall \ z \in E\}. \tag{2.9}$$

It is easy to see that $N_E(x) = (E - x)^\circ$.

The following theorem is well known and can be found in [7, 11].

Theorem 2.8. Let E be a non-empty subset of $\mathbb{R}^n$. Then, $E^{\circ\circ} = cl(conv(cone(E)))$.

We now present the properties of the negative polar cone in the following lemma.

Lemma 2.1. [11, 12] Let E, E_1 and E_2 be non-empty subsets of $\mathbb{R}^n$. Then,

(i) E° is a non-empty closed convex cone.

(ii) $E^\circ = (cl(E))^\circ$.

(iii) $E^\circ = (cone(E))^\circ = (cl(cone(E)))^\circ$.

(iv) $E^{\circ\circ\circ} = E^\circ$.

(v) If E is a closed convex cone, then, $E^{\circ\circ} = E$.

(vi) If $E_1 \subseteq E_2$, then, $E_1^\circ \supseteq E_2^\circ$.

Theorem 2.9. [1, 18] Let D be a non-empty subset of $\mathbb{R}^n$, and let $x \in D$. Then the following assertions hold.

(i) $T(D; x)$ is a closed cone.

(ii) $T(D; x) \subseteq cl(cone(D - x))$.

(iii) $T(D; x) = cl(cone(D - x))$, if D is convex.

In the sequel, let S be given as in (2.4), and let K be a non-empty closed convex set in $\mathbb{R}^n$ such that $K \cap S \neq \emptyset$. Let $\tilde{S} := K \cap S$.

Lemma 2.2. Suppose that $\tilde{x} \in \tilde{S}$ is given. Then, $T(\tilde{S}; \tilde{x}) \subseteq (C_E(\tilde{x}))^\circ$.

Proof: Let $u \in T(\tilde{S};\tilde{x})$ be arbitrary. Then, in view of (2.2), there exist sequences $\{\alpha_m\}_{m\in\mathbb{N}} \subset (0,+\infty)$ and $\{u_m\}_{m\in\mathbb{N}} \subset \mathbb{R}^n$ such that $\alpha_m \longrightarrow 0^+$, $u_m \longrightarrow u$ and $\tilde{x} + \alpha_m u_m \in \tilde{S}$ for all $m \in \mathbb{N}$. Put, $x_m := \tilde{x} + \alpha_m u_m$ for each $m \in \mathbb{N}$. Thus, $x_m \in \tilde{S}$ for all $m \in \mathbb{N}$, and

$$u_m = \frac{x_m - \tilde{x}}{\alpha_m} \ (m \in \mathbb{N}), \text{ and } x_m \longrightarrow \tilde{x}. \tag{2.10}$$

Now, let $z \in C_E(\tilde{x})$ be arbitrary. Then, there exist $\lambda_j \geq 0$, $j \in I(\tilde{x})$ such that

$$z = \sum_{j\in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0). \tag{2.11}$$

So, by using $u_m \longrightarrow u$ as $m \longrightarrow +\infty$, and in view of (2.10), (2.11), Theorem 2.3, Theorem 2.6 and Theorem 2.7, we obtain that

$$\begin{aligned}
\langle u, z\rangle &= \langle u, \sum_{j\in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0)\rangle \\
&= \langle \lim_{m\longrightarrow+\infty} u_m, \sum_{j\in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0)\rangle \\
&= \lim_{m\longrightarrow+\infty} \sum_{j\in I(\tilde{x})} \lambda_j \langle u_m, D_E G_j(\tilde{x}, 0)\rangle \\
&= \lim_{m\longrightarrow+\infty} \sum_{j\in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0)(u_m) \\
&= \lim_{m\longrightarrow+\infty} \sum_{j\in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0)(\frac{x_m - \tilde{x}}{\alpha_m}) \\
&= \lim_{m\longrightarrow+\infty} \frac{1}{\alpha_m} \sum_{j\in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0)(x_m - \tilde{x}) \\
&\leq \lim_{m\longrightarrow+\infty} \frac{1}{\alpha_m} \sum_{j\in I(\tilde{x})} \lambda_j (g_m^j - 0),
\end{aligned} \tag{2.12}$$

for all $g_m^j \in G_j(x_m)$ $(j \in I(\tilde{x}))$ and all $m \in \mathbb{N}$. Since $x_m \in \tilde{S}$ for all $m \in \mathbb{N}$, it follows that $G_j(x_m) \precsim^u_{\mathbb{R}_+} \{0\}$ for each $j \in \{1, 2, \ldots, k\}$ and each $m \in \mathbb{N}$. Therefore, by the definition of the upper set less order relation $\precsim^u_{\mathbb{R}_+}$, we conclude that $g_m^j \leq 0$ for all $j \in \{1, 2, \ldots, k\}$ and all $m \in \mathbb{N}$. Thus, (2.12) implies that

$$\begin{aligned}
\langle u, z\rangle &\leq \lim_{m\longrightarrow+\infty} \frac{1}{\alpha_m} \sum_{j\in I(\tilde{x})} \lambda_j g_m^j \\
&\leq 0.
\end{aligned}$$

Therefore, $\langle u, z\rangle \leq 0$ for all $z \in C_E(\tilde{x})$, i.e., $u \in (C_E(\tilde{x}))^\circ$. Hence, $T(\tilde{S};\tilde{x}) \subseteq (C_E(\tilde{x}))^\circ$. ■

In the following, we present the generalized Guingard's constraint qualification for the constraint set S with $\precsim^u_{\mathbb{R}_+}$-convex inequality set valued constraints.

Definition 2.3. Let $\tilde{x} \in \tilde{S}$ be fixed and arbitrary. We say that the generalized Guignard's constraint qualification ($GGCQ$, in short) holds at $\tilde{x}$, if

$$(C_E(\tilde{x}))^\circ \subseteq cl(conv(T(\tilde{S};\tilde{x}))).$$

Remark 2.5. It should be noted that the generalized Guignard's constraint qualification which defined by Definition 2.3 coincides with its counterparts whenever the constraint functions $G_j's$, $j = 1, 2, \ldots, k$, are single valued, convex and differentiable. To this end, we assume that the constraint functions $G_j's$, $j = 1, 2, \ldots, k$, are single valued, convex and Fréchet differentiable. So, in view of Lemma 2.1 and Theorem 2.8, one has

$$\begin{aligned}(C_E(\tilde{x}))^\circ &= (conv(cone(A(\tilde{x}))))^\circ \\ &= (cl(conv(cone(A(\tilde{x})))))^\circ \\ &= ((A(\tilde{x}))^{\circ\circ})^\circ \\ &= (A(\tilde{x}))^\circ,\end{aligned}$$

where for each $x \in \tilde{S}$, we define the set of gradients $\nabla G_j(x)$ ($j \in I(x)$), by

$$A(x) := \{\nabla G_j(x) : j \in I(x)\}, \tag{2.13}$$

and $I(x)$ denotes the set of active indices at x, i.e.,

$$I(x) := \{j \in \{1, 2, \ldots, k\} : G_j(x) = 0\} \text{ (see [16])}. \tag{2.14}$$

We now recall the concept of the strong conical hull intersection property (strong CHIP, in short) which is often used in optimization and approximation theory. The strong CHIP property was first introduced in [12] (see also, [11]).

Definition 2.4. A finite collection $\{C_1, C_2, \ldots, C_m\}$ of non-empty closed convex sets in $\mathbb{R}^n$ which has a non-empty intersection is said to have the strong conical hull intersection property (strong CHIP) at a point $x \in \bigcap_{j=1}^m C_j$, if

$$\Big(\bigcap_{j=1}^m C_j - x\Big)^\circ = \sum_{j=1}^m \Big(C_j - x\Big)^\circ.$$

The definition of the best approximation is given in the following (see [11, 34]).

Definition 2.5. Let D be a non-empty subset of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $p \in D$. We say that p is a best approximation (projection) of x in D, if $\|x - p\| = d_D(x)$, where $d_D(x) := \inf_{y \in D} \|x - y\|$. In this case, the set of all best approximations of x in D is given by:

$$P_D(x) := \{p \in D : \|x - p\| = d_D(x)\}.$$

The following characterization of the best approximation in $\mathbb{R}^n$ is well known [11].

Theorem 2.10. Let D be a non-empty closed convex subset of $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and $p \in D$. Then,

$$p = P_D(x) \iff \langle y - p, x - p \rangle \leq 0, \ \forall\, y \in D. \tag{2.15}$$

Moreover, we conclude from (2.8) and (2.15) that

$$p = P_D(x) \iff x - p \in (D - p)^\circ. \tag{2.16}$$

3. Dual Cone Characterizations of the Constraint Set S

This section is devoted to the study of dual cone characterizations of the constraint set S, where S is given by (2.4). Moreover, we give sufficient conditions for which the strong CHIP property holds.

Now, for each $x \in S$, put

$$M(x) := \Big\{\sum_{j\in I}\lambda_j D_E G_j(x, g_j) : g_j \in G_j(x),\ \lambda_j \geq 0 \text{ with } \lambda_j g_j = 0,\ \forall\ j \in I\Big\}, \tag{3.1}$$

where $I := \{1, 2, \ldots, k\}$.

In view of [5, Proposition B.16(b)], it can be shown that the set $M(x)$ is a closed cone in $\mathbb{R}^n$.

From now on, we assume that S is given as in (2.4), K is a non-empty closed convex subset of $\mathbb{R}^n$ such that $K \cap S \neq \emptyset$ and $\tilde{S} := K \cap S$.

Lemma 3.1. Let $\tilde{x} \in \tilde{S}$ be fixed and arbitrary. Then, $M(\tilde{x}) = C_E(\tilde{x})$, where $C_E(\tilde{x})$ and $M(\tilde{x})$ defined by (2.7) and (3.1), respectively.

Proof: Let $z \in C_E(\tilde{x})$ be arbitrary. Then, in view of (2.7), there exist $\lambda_j \geq 0$, $j \in I(\tilde{x})$ such that

$$z = \sum_{j\in I(\tilde{x})}\lambda_j D_E G_j(\tilde{x}, 0). \tag{3.2}$$

But, one has $0 \in G_j(\tilde{x})$ for each $j \in I(\tilde{x})$, so take $\tilde{g}_j := 0 \in G_j(\tilde{x})$ for each $j \in I(\tilde{x})$. Also, put $\lambda_j := 0$ for each $j \in I \setminus I(\tilde{x})$, and choose $\tilde{g}_j \in G_j(\tilde{x})$ arbitrary for each $j \in I \setminus I(\tilde{x})$. Then, $\lambda_j\tilde{g}_j = 0$ for all $j \in I$. Therefore, by using (3.2), we have

$$z = \sum_{j\in I}\lambda_j D_E G_j(\tilde{x}, \tilde{g}_j), \tag{3.3}$$

for some $\tilde{g}_j \in G_j(\tilde{x})$ and some $\lambda_j \geq 0$ with $\lambda_j\tilde{g}_j = 0$ for all $j \in I$. Hence, (3.3) together with (3.1) implies that $z \in M(\tilde{x})$, and so, $C_E(\tilde{x}) \subseteq M(\tilde{x})$.

Conversely, let $z \in M(\tilde{x})$ be arbitrary. Thus, by (3.1), there exist $\tilde{g}_j \in G_j(\tilde{x})$ and $\lambda_j \geq 0$ with $\lambda_j\tilde{g}_j = 0$ for all $j \in I$ such that

$$z = \sum_{j\in I}\lambda_j D_E G_j(\tilde{x}, \tilde{g}_j). \tag{3.4}$$

We claim that $\lambda_j = 0$ for all $j \in I \setminus I(\tilde{x})$. Assume if possible that there exists $j_0 \in I \setminus I(\tilde{x})$ such that $\lambda_{j_0} > 0$. Since $\lambda_{j_0}\tilde{g}_{j_0} = 0$, it follows that $0 = \tilde{g}_{j_0} \in G_{j_0}(\tilde{x})$, and hence, $j_0 \in I(\tilde{x})$, which is a contradiction. So, $\lambda_j = 0$ for all $j \in I \setminus I(\tilde{x})$. Therefore, we conclude from (3.4) that

$$z = \sum_{j\in I(\tilde{x})}\lambda_j D_E G_j(\tilde{x}, \tilde{g}_j). \tag{3.5}$$

Now, if $\tilde{g}_j = 0$ for all $j \in I(\tilde{x})$, then, in view of (2.7) and (3.5), one has $z \in C_E(\tilde{x})$. Assume that $\tilde{g}_{j_0} \neq 0$ for some $j_0 \in I(\tilde{x})$. Thus, by using the fact that $\lambda_j\tilde{g}_j = 0$ for all $j \in I$, it follows that $\lambda_{j_0} = 0$.

Hence, (3.5) together with (2.7) implies that $z \in C_E(\tilde{x})$, and so, $M(\tilde{x}) \subseteq C_E(\tilde{x})$, and the proof is complete. ■

We now present the dual cone characterizations of the constraint set S, which has a crucial role for characterizing the constrained best approximation by the set $\tilde{S}$.

Theorem 3.1. Let $\tilde{x} \in \tilde{S}$ be fixed and arbitrary, and let $M(\tilde{x})$ be given as in (3.1). If the generalized Guignard's constraint qualification holds at the point $\tilde{x}$, then, $M(\tilde{x}) = (S - \tilde{x})^{\circ} = (\tilde{S} - \tilde{x})^{\circ}$.

Proof: Since $\tilde{S} \subseteq S$, it is easy to check that $(S - \tilde{x})^{\circ} \subseteq (\tilde{S} - \tilde{x})^{\circ}$. Now, we show that $M(\tilde{x}) \subseteq (S - \tilde{x})^{\circ}$. To this end, let $z \in M(\tilde{x})$ be arbitrary. Thus, by (2.7) and Lemma 3.1, there exist $\lambda_j \geq 0$, $j \in I(\tilde{x})$ such that

$$z = \sum_{j \in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0). \tag{3.6}$$

Let $x \in S$ be arbitrary. It follows from (3.6), Theorem 2.6 and Theorem 2.7 that

$$\begin{aligned} \langle z, x - \tilde{x} \rangle &= \langle \sum_{j \in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0), x - \tilde{x} \rangle \\ &= \sum_{j \in I(\tilde{x})} \lambda_j D_E G_j(\tilde{x}, 0)(x - \tilde{x}) \\ &\leq \sum_{j \in I(\tilde{x})} \lambda_j (g_j - 0), \end{aligned} \tag{3.7}$$

for all $g_j \in G_j(x)$ and all $j \in I(\tilde{x})$. Because of $g_j \in G_j(x)$ ($j \in I(\tilde{x})$), so by the definition of the upper set less order relation $\precsim^{u}_{\mathbb{R}_+}$, we conclude that $g_j \leq 0$ for all $j \in I(\tilde{x})$. Thus, in view of (3.7), we have

$$\begin{aligned} \langle z, x - \tilde{x} \rangle &\leq \sum_{j \in I(\tilde{x})} \lambda_j g_j \\ &\leq 0, \ \forall \ x \in S. \end{aligned}$$

Therefore, $z \in (S - \tilde{x})^{\circ}$, and so, $M(\tilde{x}) \subseteq (S - \tilde{x})^{\circ}$.

Finally, we prove that $(\tilde{S} - \tilde{x})^{\circ} \subseteq M(\tilde{x})$. By the validity of the generalized Guignard's constraint qualification at the point $\tilde{x}$, one has

$$(C_E(\tilde{x}))^{\circ} \subseteq cl(conv(T(\tilde{S}; \tilde{x}))). \tag{3.8}$$

On the other hand, by Theorem 2.9(i), $T(\tilde{S}; \tilde{x})$ is a cone, and so, it follows from Theorem 2.8 and (3.8) that

$$(C_E(\tilde{x}))^{\circ} \subseteq (T(\tilde{S}; \tilde{x}))^{\circ\circ}. \tag{3.9}$$

Thus, by using (3.9), Lemma 2.1, Lemma 3.1 and Theorem 2.8, we obtain that

$$\begin{aligned} & (T(\tilde{S}; \tilde{x}))^{\circ\circ\circ} \subseteq (C_E(\tilde{x}))^{\circ\circ}, \\ \Longrightarrow \ & (T(\tilde{S}; \tilde{x}))^{\circ} \subseteq cl(conv(cone(C_E(\tilde{x})))), \\ \Longrightarrow \ & (T(\tilde{S}; \tilde{x}))^{\circ} \subseteq cl(C_E(\tilde{x})), \\ \Longrightarrow \ & (T(\tilde{S}; \tilde{x}))^{\circ} \subseteq cl(M(\tilde{x})). \end{aligned} \tag{3.10}$$

Since the set $M(\tilde{x})$ is closed in $\mathbb{R}^n$, it follows from (3.10) that

$$(T(\tilde{S};\tilde{x}))^{\circ} \subseteq M(\tilde{x}). \tag{3.11}$$

Also, in view of Lemma 2.1(*iii*) and Theorem 2.9(*ii*), we have

$$(\tilde{S}-\tilde{x})^{\circ} = (cl(cone(\tilde{S}-\tilde{x})))^{\circ} \subseteq (T(\tilde{S};\tilde{x}))^{\circ}. \tag{3.12}$$

Hence, we conclude from (3.11) and (3.12) that $(\tilde{S}-\tilde{x})^{\circ} \subseteq M(\tilde{x})$, which completes the proof. ■

Remark 3.1. It should be noted that in Theorem 3.1, the inclusion $M(\tilde{x}) \subseteq (S-\tilde{x})^{\circ}$ holds without the validity of the generalized Guignard's constraint qualification at the point $\tilde{x}$.

In the following example, we show that Theorem 3.1 may not be satisfied, if the generalized Guignard's constraint qualification is omitted.

Example 3.1. Let $G_1, G_2 : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ be defined by:

$$G_1(x) := [-\|x\|, 0],$$

and

$$G_2(x) := \begin{cases} [-\|x\|, -x_1], & x_1 \geq 0, \\ \emptyset, & x_1 < 0, \end{cases}$$

for all $x := (x_1, x_2) \in \mathbb{R}^2$.

Clearly, $dom(G_1) = \mathbb{R}^2$, and $dom(G_2) = \mathbb{R}_+ \times \mathbb{R}$. Since the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by $g(x) := -\|x\|$ $(x \in \mathbb{R}^2)$ is concave, it is not difficult to check that G_1 and G_2 are $\precsim^u_{\mathbb{R}_+}$-convex set valued mappings. Moreover, one has

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^2 : x \in dom(G_1), \text{ and } G_1(x) \subseteq -\mathbb{R}_+\} \\ &= \{x \in \mathbb{R}^2 : [-\|x\|, 0] \subseteq -\mathbb{R}_+\} = \mathbb{R}^2, \end{aligned}$$

and

$$\begin{aligned} S_2 &= \{x \in \mathbb{R}^2 : x \in dom(G_2), \text{ and } G_2(x) \subseteq -\mathbb{R}_+\} \\ &= \{x \in \mathbb{R}^2 : x \in dom(G_2), \text{ and } [-\|x\|, -x_1] \subseteq -\mathbb{R}_+\} \\ &= \mathbb{R}_+ \times \mathbb{R}. \end{aligned}$$

Therefore, $S := S_1 \cap S_2 = \mathbb{R}_+ \times \mathbb{R}$. Let $K := \mathbb{R}^2_+$. Then, $\tilde{S} := K \cap S = \mathbb{R}^2_+$. Put $\tilde{x} := (0,0) \in \tilde{S}$. One can easily see that $\tilde{g}_1 = 0 \in G_1(\tilde{x})$ and $\tilde{g}_2 = 0 \in G_2(\tilde{x})$. Thus, $I(\tilde{x}) = \{1,2\}$, and $(0,0)$ is an element of $Graph(G_1)$ and $Graph(G_2)$. We have

$$\begin{aligned} epi(G_1) &= \{(x,\alpha) \in \mathbb{R}^2 \times \mathbb{R} : x \in dom(G_1), \text{ and } G_1(x) \subseteq \alpha - \mathbb{R}_+\} \\ &= \{(x,\alpha) \in \mathbb{R}^2 \times \mathbb{R} : x \in dom(G_1), \text{ and } [-\|x\|, 0] \subseteq (-\infty, \alpha]\} \\ &= \{(x,\alpha) \in \mathbb{R}^2 \times \mathbb{R} : x \in \mathbb{R}^2, \text{ and } \alpha \geq 0\}, \end{aligned}$$

and

$$\begin{aligned} epi(G_2) &= \{(x,\alpha)\in\mathbb{R}^2\times\mathbb{R} : x\in dom(G_2), \text{ and } G_2(x)\subseteq \alpha-\mathbb{R}_+\} \\ &= \{(x,\alpha)\in\mathbb{R}^2\times\mathbb{R} : x\in dom(G_2), \text{ and } [-\|x\|,-x_1]\subseteq(-\infty,\alpha]\} \\ &= \{(x,\alpha)\in\mathbb{R}^2\times\mathbb{R} : x\in \mathbb{R}_+\times\mathbb{R}, \text{ and } -x_1\le\alpha\}. \end{aligned}$$

Since $epi(G_1)$ and $epi(G_2)$ are closed convex cones, by using Theorem 2.9 (*iii*), we obtain that

$$T(epi(G_1);(0,0)) = \{(x,\alpha)\in\mathbb{R}^2\times\mathbb{R} : x\in\mathbb{R}^2, \text{ and } \alpha\ge 0\},$$

and

$$T(epi(G_2);(0,0)) = \{(x,\alpha)\in\mathbb{R}^2\times\mathbb{R} : x\in\mathbb{R}_+\times\mathbb{R}, \text{ and } -x_1\le\alpha\}.$$

Then, in view of Theorem 2.1, it is not difficult to show that

$$D_EG_1(0,0)(x) = \min\{y\in\mathbb{R} : (x,y)\in T(epi(G_1);(0,0))\} = 0, \ \forall\, x=(x_1,x_2)\in\mathbb{R}^2,$$

and

$$D_EG_2(0,0)(x) = \min\{y\in\mathbb{R} : (x,y)\in T(epi(G_2);(0,0))\} = -x_1, \ \forall\, x=(x_1,x_2)\in\mathbb{R}^2.$$

Note that the effective domain of the contingent epiderivative $D_EG_2(0,0)$ is equal to $\mathbb{R}_+\times\mathbb{R}$. We can consider the contingent epiderivatives $D_EG_1(0,0)$ and $D_EG_2(0,0)$ as vectors $\mathbf{a}_1 := (0,0)$ and $\mathbf{a}_2 := (-1,0)$, respectively. Therefore, in view of (2.7),

$$C_E(\tilde{x}) = \{\lambda_1 D_EG_1(0,0)+\lambda_2 D_EG_2(0,0) : \lambda_1\ge 0, \text{ and } \lambda_2\ge 0\} = \mathbb{R}_-\times\{0\}.$$

Then, $(C_E(\tilde{x}))^\circ = \mathbb{R}_+\times\mathbb{R}$, and by using Lemma 3.1, one has $M(\tilde{x}) = \mathbb{R}_-\times\{0\}$, which is a closed set in $\mathbb{R}^2$. On the other hand, $T(\tilde{S};\tilde{x}) = \mathbb{R}^2_+$, and so, $(C_E(\tilde{x}))^\circ \nsubseteq \mathbb{R}^2_+ = cl(conv(T(\tilde{S};\tilde{x})))$, i.e., the generalized Guignard's constraint qualification does not hold at the point $\tilde{x}$. Furthermore, it is easy to see that $(\tilde{S}-\tilde{x})^\circ = \mathbb{R}^2_- \nsubseteq \mathbb{R}_-\times\{0\} = M(\tilde{x})$. Hence, Theorem 3.1 does not hold.

In the following, we give sufficient conditions for which the strong CHIP property holds.

Theorem 3.2. Let $\tilde{x}\in\tilde{S}$ be fixed and arbitrary. Suppose that the generalized Guignard's constraint qualification holds at the point $\tilde{x}$. Then, $\{K,S\}$ has the strong CHIP property at the point $\tilde{x}$.

Proof: It is easy to check that one has always

$$(K-\tilde{x})^\circ + (S-\tilde{x})^\circ \subseteq (K\cap S-\tilde{x})^\circ = (\tilde{S}-\tilde{x})^\circ,$$

where $\tilde{S} := K\cap S$.

For the converse inclusion, in view of Theorem 3.1, we have $(\tilde{S}-\tilde{x})^\circ = (S-\tilde{x})^\circ$. Therefore, since $\{0\}\subseteq(K-\tilde{x})^\circ$, it follows that

$$(\tilde{S}-\tilde{x})^\circ \subseteq (K-\tilde{x})^\circ + (S-\tilde{x})^\circ,$$

which completes the proof. ■

By the following example, we show that the converse statement to Theorem 3.2 does not necessarily hold.

Example 3.2. Consider Example 3.1, and let S, K, $\tilde{S}$ and $\tilde{x}$ be given as in Example 3.1. Then, in view of Example 3.1, $\tilde{S} := K \cap S = \mathbb{R}^2_+$, and so, $(K \cap S - \tilde{x})^\circ = \mathbb{R}^2_-$. Also, it is not difficult to show that $(K - \tilde{x})^\circ = \mathbb{R}^2_-$ and $(S - \tilde{x})^\circ = \mathbb{R}_- \times \{0\}$. Thus,

$$(K \cap S - \tilde{x})^\circ = (K - \tilde{x})^\circ + (S - \tilde{x})^\circ.$$

Therefore, $\{K, S\}$ has the strong CHIP property at the point $\tilde{x}$, while the generalized Guignard's constraint qualification does not hold at the point $\tilde{x}$ (see Example 3.1).

4. Characterizations of the Constrained Best Approximation with Set Valued Constraints

In this section, we apply the obtained results in Section 3, and present characterizations of the constrained best approximation with $\precsim^u_{\mathbb{R}_+}$-convex set valued inequality constraints under the weakest constraint qualification (generalized Guignard's constraint qualification). Let S be as in (2.4), given by:

$$S = \{x \in \mathbb{R}^n : G_j(x) \precsim^u_{\mathbb{R}_+} \{0\},\ \forall\ j = 1, 2, \ldots, k\},$$

where the constraint mappings $G_j : \mathbb{R}^n \rightrightarrows \mathbb{R}$, $j = 1, 2, \ldots, k$, are $\precsim^u_{\mathbb{R}_+}$-convex and its contingent epiderivative at a reference point of $Graph(G_j)$ exists. Obviously, the feasible set S is convex. Let K be a non-empty closed convex subset of $\mathbb{R}^n$ such that $K \cap S \neq \emptyset$, and let $\tilde{S} := K \cap S$. Note that S is a convex set, and so, $\tilde{S}$ is also a convex set.

Theorem 4.1. (**Generalized Perturbation Property**). Let $\tilde{x} \in \tilde{S}$ be fixed and arbitrary. Suppose that $\tilde{S}$ is closed. If the generalized Guignard's constraint qualification holds at the point $\tilde{x}$, then the following assertions are equivalent.

(i) The pair $\{K, S\}$ has the strong CHIP property at the point $\tilde{x}$.

(ii) For any $x \in \mathbb{R}^n$, $\tilde{x} = P_{\tilde{S}}(x)$ if and only if there exist $\tilde{g} := (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and $\tilde{\lambda} := (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ with $\tilde{\lambda}_j \geq 0$ and $\tilde{\lambda}_j \tilde{g}_j = 0$ for all $j = 1, 2, \ldots, k$ such that

$$\tilde{x} = P_K(x - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j)).$$

Proof: $[(i) \implies (ii)]$. Suppose that (i) holds. Then, by Definition 2.4,

$$(\tilde{S} - \tilde{x})^\circ = (K - \tilde{x})^\circ + (S - \tilde{x})^\circ. \tag{4.1}$$

Also, in view of Theorem 3.1, we have $M(\tilde{x}) = (S - \tilde{x})^\circ$. So, it follows from (4.1) that

$$(\tilde{S} - \tilde{x})^\circ = (K - \tilde{x})^\circ + M(\tilde{x}). \tag{4.2}$$

Now, let $x \in \mathbb{R}^n$ be arbitrary. Assume that $\tilde{x} = P_{\tilde{S}}(x)$. Thus, by Theorem 2.10 and the fact that $\tilde{S}$ is closed and convex, one has $x - \tilde{x} \in (\tilde{S} - \tilde{x})^\circ$. Therefore, in view of (4.2), $x - \tilde{x} \in (K - \tilde{x})^\circ + M(\tilde{x})$. So, by (3.1) there exist $\tilde{g} := (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and $\tilde{\lambda} := (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ with $\tilde{\lambda}_j \geq 0$ and $\tilde{\lambda}_j \tilde{g}_j = 0$ for all $j = 1, 2, \ldots, k$ such that

$$x - \tilde{x} - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j) \in (K - \tilde{x})^{\circ}. \tag{4.3}$$

Thus, by closedness and convexity of K and the validity of Theorem 2.10, it follows that $\tilde{x} = P_K(x - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j))$. Hence, the following implication holds.

$$\tilde{x} = P_{\tilde{S}}(x) \implies \tilde{x} = P_K(x - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j)),$$

for some $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and some $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ with $\tilde{\lambda}_j \geq 0$ and $\tilde{\lambda}_j \tilde{g}_j = 0$ for all $j = 1, 2, \ldots, k$.

Conversely, assume that there exist $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ with $\tilde{\lambda}_j \geq 0$ and $\tilde{\lambda}_j \tilde{g}_j = 0$ for all $j = 1, 2, \ldots, k$ such that

$$\tilde{x} = P_K(x - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j)).$$

This together with (4.2) and (3.1) implies that

$$\begin{aligned} x - \tilde{x} \quad & \in (K - \tilde{x})^{\circ} + \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j) \\ & \subseteq (K - \tilde{x})^{\circ} + M(\tilde{x}) \\ & = (\tilde{S} - \tilde{x})^{\circ}. \end{aligned}$$

Therefore, $x - \tilde{x} \in (\tilde{S} - \tilde{x})^{\circ}$, and hence, by Theorem 2.10, $\tilde{x} = P_{\tilde{S}}(x)$.

[$(ii) \implies (i)$]. Assume that (ii) holds. We first note that one has always

$$(K - \tilde{x})^{\circ} + (S - \tilde{x})^{\circ} \subseteq (K \cap S - \tilde{x})^{\circ} = (\tilde{S} - \tilde{x})^{\circ}. \tag{4.4}$$

For proving the inverse inclusion, let $y \in (\tilde{S} - \tilde{x})^{\circ}$ be arbitrary, and let $x := \tilde{x} + y$. Then, $x - \tilde{x} \in (\tilde{S} - \tilde{x})^{\circ}$, and so, by closedness and convexity of $\tilde{S}$ and Theorem 2.10, $\tilde{x} = P_{\tilde{S}}(x)$. Thus, in view of the hypothesis (ii), there exist $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ with $\tilde{\lambda}_j \geq 0$ and $\tilde{\lambda}_j \tilde{g}_j = 0$ for all $j = 1, 2, \ldots, k$ such that

$$\tilde{x} = P_K(x - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j)).$$

Again, by Theorem 2.10,

$$x - \tilde{x} - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j) \in (K - \tilde{x})^\circ. \tag{4.5}$$

Therefore, it follows from (4.5), (3.1) and Theorem 3.1 that

$$x - \tilde{x} \in (K - \tilde{x})^\circ + M(\tilde{x}) = (K - \tilde{x})^\circ + (S - \tilde{x})^\circ. \tag{4.6}$$

So, $y \in (K - \tilde{x})^\circ + (S - \tilde{x})^\circ$, and hence,

$$(\tilde{S} - \tilde{x})^\circ \subseteq (K - \tilde{x})^\circ + (S - \tilde{x})^\circ,$$

which completes the proof. ■

In the sequel, let $x^* \in \mathbb{R}^n$ be fixed, and define the function $f : \mathbb{R}^n \longrightarrow [0, +\infty)$ by

$$f(x) := \|x - x^*\|, \ \forall\, x \in \mathbb{R}^n.$$

For $\tilde{x} \in \mathbb{R}^n$, we recall (see [3]) that $\partial\|\cdot - x^*\|(\tilde{x}) =: \partial f(\tilde{x})$ is given by

$$\partial\|\cdot - x^*\|(\tilde{x}) = \begin{cases} \{\frac{\tilde{x}-x^*}{\|\tilde{x}-x^*\|}\}, & \tilde{x} \neq x^*, \\ B(0,1), & \tilde{x} = x^*, \end{cases} \tag{4.7}$$

where $B(0,1) := \{y \in \mathbb{R}^n : \|y\| \leq 1\}$ is the closed unit ball of $\mathbb{R}^n$. We also denote by $\partial h(x)$ the convex subdifferential of a convex function $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ at some point $x \in \mathbb{R}^n$.

In the following, we give the Lagrange multipliers characterizations of the constrained best approximation with $\precsim^{u}_{\mathbb{R}_+}$-convex set valued inequality constraints under the generalized Guignard's constraint qualification.

Theorem 4.2. Let $\tilde{x} \in \tilde{S}$ and $x \in \mathbb{R}^n$ be fixed and arbitrary. Suppose that $\tilde{S}$ is closed. If the generalized Guignard's constraint qualification holds at the point $\tilde{x}$, then the following assertions are equivalent.

(i) $\tilde{x} = P_{\tilde{S}}(x)$.

(ii) There exist $\tilde{g} := (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and $\tilde{\lambda} := (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ with

$$\tilde{\lambda}_j \geq 0, \ \forall\, j = 1, 2, \ldots, k, \tag{4.8}$$

$$\tilde{\lambda}_j \tilde{g}_j = 0, \ \forall\, j = 1, 2, \ldots, k, \tag{4.9}$$

$$0 \in \partial\|\cdot - x\|(\tilde{x}) + (K - \tilde{x})^\circ + \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j). \tag{4.10}$$

Proof: $[(i) \implies (ii)]$. Suppose that (i) holds, i.e., $\tilde{x} = P_{\tilde{S}}(x)$. If $x = \tilde{x}$, then in view of (4.7), one has $0 \in \partial\|\cdot - x\|(\tilde{x})$ and since $0 \in (K - \tilde{x})^\circ$, it is enough to take $\tilde{\lambda}_j = 0$ and choose $\tilde{g}_j \in G_j(\tilde{x})$ arbitrary for all $j = 1, 2, \ldots, m$. Therefore, (ii) holds. Thus, we may assume without loss of generality that

$x \neq \tilde{x}$. Since $\tilde{x} = P_{\tilde{S}}(x)$, we conclude from Theorem 2.10 that $x - \tilde{x} \in (\tilde{S} - \tilde{x})^{\circ}$. On the other hand, in view of Theorem 3.1, one has

$$(\tilde{S} - \tilde{x})^{\circ} = M(\tilde{x}),$$

and hence, $x - \tilde{x} \in M(\tilde{x})$. Because of $M(\tilde{x})$ is a cone and $x \neq \tilde{x}$, one has

$$\frac{x - \tilde{x}}{\|x - \tilde{x}\|} \in M(\tilde{x}).$$

Therefore, by using (3.1), there exist $\tilde{g} := (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and $\tilde{\lambda} := (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ with $\tilde{\lambda}_j \geq 0$ and $\tilde{\lambda}_j \tilde{g}_j = 0$ for all $j = 1, 2, \ldots, k$ such that

$$\frac{x - \tilde{x}}{\|x - \tilde{x}\|} = \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j). \tag{4.11}$$

Put, $u := \frac{\tilde{x} - x}{\|\tilde{x} - x\|}$. So, $u \in \mathbb{R}^n$, and this together with (4.7) implies that

$$u \in \partial \| \cdot - x\|(\tilde{x}). \tag{4.12}$$

Thus, it follows from (4.11) and (4.12) that

$$0 \in \partial \| \cdot - x\|(\tilde{x}) + \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j), \tag{4.13}$$

for some $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and some $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ such that $\tilde{g}$ and $\tilde{\lambda}$ satisfy (4.8) and (4.9). Hence, by (4.13) and the fact that $0 \in (K - \tilde{x})^{\circ}$, we conclude that (ii) holds.

[$(ii) \implies (i)$]. Assume that (ii) holds. Then there exist $\tilde{g} := (\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_k) \in G_1(\tilde{x}) \times G_2(\tilde{x}) \times \ldots \times G_k(\tilde{x})$ and $\tilde{\lambda} := (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k) \in \mathbb{R}^k$ such that $\tilde{g}$ and $\tilde{\lambda}$ satisfy (4.8), (4.9) and (4.10). We show that $\|\tilde{x} - x\| = \inf_{w \in \tilde{S}} \|w - x\|$. To this end, let $w \in \tilde{S}$ be arbitrary. Thus, $w \in S$ and $w \in K$. On the other hand, by (4.10), there exists $y \in (K - \tilde{x})^{\circ}$ such that

$$-y - \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j) \in \partial \| \cdot - x\|(\tilde{x}). \tag{4.14}$$

Thus, by using the definition of the classic convex subdifferential and (4.14), we conclude that

$$\|\tilde{x} - x\| - \|w - x\| \leq \langle y + \sum_{j=1}^{k} \tilde{\lambda}_j D_E G_j(\tilde{x}, \tilde{g}_j), w - \tilde{x} \rangle. \tag{4.15}$$

Since $y \in (K - \tilde{x})^{\circ}$ and $w \in K$, it follows from (2.8) that

$$\langle y, w - \tilde{x} \rangle \leq 0. \tag{4.16}$$

So, it follows from (4.8), (4.9), (4.15), (4.16), Theorem 2.6 and Theorem 2.7 that

$$\begin{aligned}
\|\tilde{x}-x\|-\|w-x\| &\leq \langle y+\sum_{j=1}^{k}\tilde{\lambda}_j D_E G_j(\tilde{x},\tilde{g}_j), w-\tilde{x}\rangle \\
&= \langle y, w-\tilde{x}\rangle + \langle \sum_{j=1}^{k}\tilde{\lambda}_j D_E G_j(\tilde{x},\tilde{g}_j), w-\tilde{x}\rangle \\
&\leq \langle \sum_{j=1}^{k}\tilde{\lambda}_j D_E G_j(\tilde{x},\tilde{g}_j), w-\tilde{x}\rangle \\
&= \sum_{j=1}^{k}\tilde{\lambda}_j D_E G_j(\tilde{x},\tilde{g}_j)(w-\tilde{x}) \\
&\leq \sum_{j=1}^{k}\tilde{\lambda}_j (g_j-\tilde{g}_j) \\
&= \sum_{j=1}^{k}\tilde{\lambda}_j g_j, \qquad (4.17)
\end{aligned}$$

for all $g_j \in G_j(w)$, $j = 1, 2, \ldots, k$. Because of $w \in S$ and $g_j \in G_j(w)$, so by the definition of the upper set less order relation $\precsim^u_{\mathbb{R}_+}$, we conclude that $g_j \leq 0$ for all $j = 1, 2, \ldots, k$. In view of (4.17),

$$\|\tilde{x}-x\|-\|w-x\| \leq \sum_{j=1}^{k}\tilde{\lambda}_j g_j \leq 0.$$

Therefore, $\|\tilde{x}-x\|-\|w-x\| \leq 0$ for all $w \in \tilde{S}$. This implies that $\|\tilde{x}-x\| = \inf_{w\in\tilde{S}}\|w-x\| = d_{\tilde{S}}(x)$. Hence, $\tilde{x} = P_{\tilde{S}}(x)$, i.e., (i) holds. ■

The following example illustrates Theorem 4.2.

Example 4.1. Let $G_j : \mathbb{R} \rightrightarrows \mathbb{R}$ $(j = 1, 2)$ be set valued mappings which are defined by:

$$G_1(t) := (-\infty, t],$$

and

$$G_2(t) := (-\infty, -2t-1],$$

for all $t \in \mathbb{R}$.

Clearly, $dom(G_1) = dom(G_2) = \mathbb{R}$, and it is not difficult to check that G_1 and G_2 are $\precsim^u_{\mathbb{R}_+}$-convex set valued mappings. Moreover,

$$S_1 := \{t \in \mathbb{R} : G_1(t) \subseteq -\mathbb{R}_+\} = \{t \in \mathbb{R} : (-\infty, t] \subseteq (-\infty, 0]\} = (-\infty, 0],$$

and

$$S_2 := \{t \in \mathbb{R} : G_2(t) \subseteq -\mathbb{R}_+\} = \{t \in \mathbb{R} : (-\infty, -2t-1] \subseteq (-\infty, 0]\} = [-\frac{1}{2}, +\infty).$$

So, $S := S_1 \cap S_2 = [-\frac{1}{2}, 0]$. Let $K := (-\infty, 0]$. Thus, $\tilde{S} := K \cap S = [-\frac{1}{2}, 0]$. Let $\tilde{t} := 0 \in \tilde{S}$. It is easy to see that $0 \in G_1(\tilde{t})$ and $0 \notin G_2(\tilde{t})$. Thus, $I(\tilde{t}) = \{1\}$ and $(0,0) \in Graph(G_1)$. We also have

$$\begin{aligned} \text{epi}\,(G_1) &= \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : G_1(t) \subseteq \alpha - \mathbb{R}_+\} \\ &= \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : (-\infty, t] \subseteq (-\infty, \alpha]\} \\ &= \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : t \leq \alpha\}, \end{aligned}$$

and

$$\begin{aligned} \text{epi}\,(G_2) &= \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : G_2(t) \subseteq \alpha - \mathbb{R}_+\} \\ &= \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : (-\infty, -2t-1] \subseteq (-\infty, \alpha]\} \\ &= \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : -2t-1 \leq \alpha\}. \end{aligned}$$

Since epi (G_1) is a closed convex cone, by using Theorem 2.9 (*iii*), we have

$$T(\text{epi}\,(G_1); (0,0)) = \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : t \leq \alpha\}.$$

In view of Theorem 2.1, we obtain that

$$D_E G_1(0,0)(x) = \min\{y \in \mathbb{R} : (x,y) \in T(\text{epi}\,(G_1); (0,0))\} = \min\{y \in \mathbb{R} : x \leq y\} = x,$$

for all $x \in \mathbb{R}$. We can consider $D_E G_1(0,0)$ as the vector $\mathbf{a}_1 := 1$. Since $(0,0)$ is an element of the interior of the epi (G_2), then, $D_E G_2(0,0)$ does not exist. Therefore, in view of (2.7), we have $C_E(\tilde{t}) = \{\lambda_1 D_E G_1(0,0) : \lambda_1 \geq 0\} = \mathbb{R}_+$, and hence, $(C_E(\tilde{t}))^\circ = \mathbb{R}_-$. On the other hand, one has $T(\tilde{S}; \tilde{t}) = cl(cone(\tilde{S} - \tilde{t})) = \mathbb{R}_-$, and hence, $cl(conv(T(\tilde{S}; \tilde{t}))) = \mathbb{R}_- = (C_E(\tilde{t}))^\circ$, i.e., the generalized Guignard's constraint qualification holds at the point $\tilde{t}$. Since $\tilde{S}$ is also closed, thus, all hypotheses of Theorem 4.2 hold. Now, let $t \in \mathbb{R}$ be such that $t > 0$. Since $\tilde{t} \neq t$, by using (4.7), it follows that $\partial \| \cdot - t \|(\tilde{t}) = \{-1\}$. Also, it is easy to see that $(K - \tilde{t})^\circ = \mathbb{R}_+$. We now choose $\tilde{g}_1 = 0 \in G_1(\tilde{t})$, $\tilde{g}_2 = -1 \in G_2(\tilde{t})$, $\tilde{\lambda}_1 = 1$ and $\tilde{\lambda}_2 = 0$. Since $(0,-1) \in Graph(G_2)$ and epi (G_2) is a closed convex cone, by using Theorem 2.9 (*iii*), we have

$$T(epi(G_2); (0,-1)) = \{(t,\alpha) \in \mathbb{R} \times \mathbb{R} : -2t-1 \leq \alpha\}.$$

In view of Theorem 2.1, one can easily see that

$$D_E G_2(0,-1)(x) = -2x - 1, \ \forall\, x \in \mathbb{R}.$$

We can also consider $D_E G_2(0,-1)$ as the vector $\mathbf{a}_2 := -2$. Thus,

$$\begin{aligned} 0 \in \mathbb{R}_+ &= (K - \tilde{t})^\circ \\ &= -1 + (K - \tilde{t})^\circ + \tilde{\lambda}_1 D_E G_1(0,0) + \tilde{\lambda}_2 D_E G_2(0,-1) \\ &= \partial \| \cdot - t \|(\tilde{t}) + (K - \tilde{t})^\circ + \tilde{\lambda}_1 D_E G_1(0,0) + \tilde{\lambda}_2 D_E G_2(0,-1). \end{aligned}$$

Then, the assertion (*ii*) holds in Theorem 4.2. Therefore, in view of Theorem 4.2 (the implication $[(ii) \Longrightarrow (i)]$), we conclude that $\tilde{t} = P_{\tilde{S}}(t)$ for all $t > 0$.

5. Conclusions and Applications

In this study, by employing contingent epiderivatives and the generalized Guignard's constraint qualification we first gave the dual cone characterizations of the constraint set S with set valued constraints. Next, by using this fact, we established necessary and sufficient conditions for characterizing "perturbation property" of the constrained best approximation. Finally, under the generalized Guignard's constraint qualification we derived the Lagrange multipliers characterizations of the constrained best approximation with set valued constraints. By the numerical examples, we illustrated and clarified the nature of obtained results. Our results extend and solve the constrained best approximation problem with set valued constraints, in a general case, and not only recapture the corresponding known results of [3, 4, 10, 16, 23, 24, 27, 28, 29, 30, 32] and the references therein, but also we can employ the new obtained techniques and methods for solving nonsmooth set optimization problems with a general objective function other than the norm. These results will appear in a forthcoming study.

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