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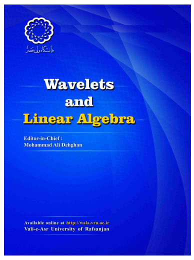

Some Results on Subdifferential of Spectral functions

A. R. Sattarzadeh[a,*], M. H. Daryaei[b]

[a]*Department of Mathematics, Faculty of Sciences and Modern Technologies, Graduate University of Advanced Technology, Kerman, Iran.*
[b]*Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.*

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Abstract

In this paper, we explore the properties of spectral functions from the perspective of convex analysis and monotone operator theory. Specifically, we examine the ε-subdifferential and ε-enlargement of a spectral function. Also, we study representative functions associated with the subdifferential of the spectral function. In addition, the Fitzpatrick function has been studied due to its significance as one of the most important representative functions.

1. Introduction

Optimization problems involving eigenvalues are widely applicable in various mathematical disciplines. These disciplines include structural analysis, combinatorial optimization and many

*Corresponding author
Email addresses: a.sattarzadeh@kgut.ac.ir (A. R. Sattarzadeh), daryaei@uk.ac.ir (M. H. Daryaei)

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others. The problems in these disciplines involve finding optimal solutions based on the eigenvalues of matrices. Eigenvalues are essential quantities that determine the behavior of the system being studied.[5, 10]

The use of real symmetric matrices as variables in optimization problems has become increasingly prevalent in a variety of industries and scientific fields. One reason for this rise in popularity is that such matrices often encode valuable information about the system or process being modeled. For example, in engineering, the eigenvalues of these matrices may provide critical insight into the stability characteristics of a mechanical or electrical system.

Advances in the study of spectral optimization have demonstrated that convex spectral functions can always be expressed as symmetric convex functions of the eigenvalues. This result provides a strong foundation upon which to build more sophisticated techniques for solving optimization problems involving real symmetric matrices. As such, this area of research holds great promise for uncovering innovative ways to tackle challenging problems across a range of domains. (see [5, 10], and references therein).

A function F is said to be spectral when for any matrix A belonging to the set $\mathcal{S}_n$ of all $n \times n$ real symmetric matrices and any matrix U belonging to the set O_n of all real orthogonal matrices, F satisfies the condition $F(U^T AU) = F(A)$.
It is straightforward to verify that every spectral function on $\mathcal{S}_n$ can be represented as the composition of a symmetric function f on $\mathbb{R}^n$ and the eigenvalue mapping $\lambda : \mathcal{S}_n \longrightarrow \mathbb{R}^n$,. i.e.,

$$F(A) = (f \circ \lambda)(A), \ \forall \ A \in \mathcal{S}_n.$$

There is a one-to-one correspondence between the set of spectral functions F defined on the $\mathcal{S}_n$ and the set of symmetric functions f defined on the real vector space $\mathbb{R}^n$. Recently, it has been shown that many results in mathematics concern this relationship: understanding properties of f can be used to understand properties of F, and conversely, understanding properties of the spectral function F can be used to understand properties of the symmetric function f. This work has become an active area of research in recent years[5, 10, 12, 14].

The maximal monotone operators play a key role in optimization because which include both the subdifferential operator of a convex, lower semicontinuous and proper function, and any continuous linear positive operator. Recently, it was found that many fundamental results on maximal monotone operators can be proven more easily using Fitzpatrick functions[5, 7, 9].

Next, we move on to defining some important mathematical objects and notation. We consider the Euclidean space $\mathbb{R}^n$ with the inner product $\langle ., . \rangle$ and the induced norm $\|.\|$. The set of all proper lower semi-continuous (l.s.c) and convex functions defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ is denoted by $\Gamma_0(\mathbb{R}^n)$. The Fenchel-Moreau conjugate of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by $f^* : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$f^*(x) := \sup_{y \in \mathbb{R}^n} \{\langle x, y \rangle - f(y)\}, \qquad \forall \ x \in \mathbb{R}^n.$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function and $x_0 \in dom(f)$. Recall [1] that the subdifferential of f is the set valued mapping $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial f(x_0) := \{u \in \mathbb{R}^n : \langle u, x - x_0\rangle \leq f(x) - f(x_0), \quad \forall\, x \in \mathbb{R}^n\}.$$

and for given $\varepsilon \geq 0$, the ε-subdifferential of f is the set valued mapping defined by

$$\partial_\varepsilon f(x_0) := \{u \in \mathbb{R}^n : \langle u, x - x_0\rangle \leq f(x) - f(x_0) + \varepsilon, \quad \forall\, x \in \mathbb{R}^n\},$$

The Fenchel-Young function $f^{FY} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ associated with f, is defined by

$$f^{FY}(x, u) := f(x) + f^*(u), \qquad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Moreover, f^{FY} also completely characterizes the graph of the subdifferential and ε-subdifferential of f. Namely,

$$u \in \partial f(x) \Longleftrightarrow f^{FY}(x, u) = \langle x, u\rangle. \tag{1.1}$$

Also, for $\varepsilon > 0$ we have

$$u \in \partial_\varepsilon f(x) \Longleftrightarrow f^{FY}(x, u) \leq \langle x, u\rangle + \varepsilon. \tag{1.2}$$

For set valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we consider the graph of T by

$$G(T) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : u \in Tx\},$$

and T is called monotone, if

$$\langle x - y, u - v\rangle \geq 0, \quad \forall\, (x, u) \in G(T), \quad \forall\, (y, v) \in G(T),$$

A set valued mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called maximal monotone, if T is monotone and $T = T'$ for any monotone mapping $T' : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ such that $G(T) \subseteq G(T')$.

Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be monotone. We say that the convex function $h : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ represents T if

$$h(x, u) \geqslant \langle x, u\rangle, \qquad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^n.$$

with equality holds when $u \in Tx$.

The set of all lower semi-continuous convex functions which represent T is denoted by $\mathcal{H}(T)$. Fitzpatrick in [9] proved that if T is the maximal monotone operator, then the family $H(T)$ is non-empty. Indeed, in Correspondence to the maximal monotone operator T, he defined the function $\mathcal{F}_T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$\mathcal{F}_T(x, u) = \sup_{(y,v)\in G(T)} \{\langle x, v\rangle + \langle y, u\rangle - \langle y, v\rangle\}, \quad \forall\, (x, u) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{1.3}$$

and showed that $\mathcal{F}_T \in \mathcal{H}(T)$.

The following theorems are well-known in convex analysis and monotone operator theory.

Theorem 1.1. [6, 8] Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be maximal monotone operator. Then $\mathcal{F}_T$ is the smallest element in $\mathcal{H}(T)$ and the conjugate of $\mathcal{F}_T$ is the biggest element in $\mathcal{H}(T)$. In other words, for every $h \in \mathcal{H}(T)$, we have

$$\mathcal{F}_T(x, u) \leq h(x, u) \leq (\mathcal{F}_T)^*(x, u), \qquad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Theorem 1.2. [6, 8] Let $f \in \Gamma_0(\mathbb{R}^n)$. Then, $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone operator. Also, we have $\mathcal{F}_{\partial f}, \mathcal{F}^*_{\partial f} \in \mathcal{H}(T)$, and

$$\mathcal{F}_{\partial f}(x,u) \leq f^{FY}(x,u) \leq (\mathcal{F}_{\partial f})^*(x,u), \qquad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1.4}$$

Theorem 1.3. [6, 7, 13] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a lower semi-continuous and sublinear function. Then, $\mathcal{H}(\partial f)$ is singleton and we have

$$\mathcal{F}_{\partial f}(x,u) = f^{FY}(x,u) = (\mathcal{F}_{\partial f})^*(x,u), \qquad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Suppose that $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the maximal monotone operator such that $\mathcal{H}(T)$ is a singleton set. Then, we have $\mathcal{F}_T = \mathcal{F}^*_T$. In this case, we say that $\mathcal{F}_T$ is autoconjugate. Also, If $\mathcal{F}_T$ is autoconjugate, Then we can conclude $\mathcal{H}(T)$ is singleton. In view of theorem 1.3, sublinear functions have autoconjugate Fitzpatrick function associated with their subdifferential.The autoconjugate representative function of the maximal monotone operator has been studied in [3, 7, 8].

In the following, we will express the definition and some properties of the spectral function.
We define an inner product on the set $\mathcal{S}_n$ of $n \times n$ real symmetric matrices as follows: the inner product of two matrices A and B is given by

$$\langle A, B \rangle := tr(AB), \ \forall \ A, B \in \mathcal{S}_n,$$

where tr denotes the trace of a matrix. This inner product gives rise to the Frobenius norm of a matrix A, which is denoted by $\|A\|_F = \sqrt{tr(A^2)}$.
For any vector $x \in \mathbb{R}^n$, we define the diagonal matrix $Diag(x)$ whose entries on the diagonal are the components of x and zeros elsewhere. For a scalar value $x \in \mathbb{R}^n$, we define the column vector $x^\dagger \in \mathbb{R}^n$, obtained by stacking copies of x on top of each other, that is $x^\dagger := (x, x, \cdots, x)$.
We introduce the eigenvalue function $\lambda : \mathcal{S}_n \longrightarrow \mathbb{R}^n$, mapping each element $A \in \mathcal{S}_n$, to an ordered tuple of its eigenvalues $\lambda_1(A), \cdots, \lambda_n(A)$ arranged in decreasing order. The following two inequalities involving eigenvalue mappings are quite well-known [5, 10]

$$\|\lambda(A) - \lambda(B)\| \leq \|A - B\|_F, \quad \forall A, B \in \mathcal{S}_n,$$

and

$$\langle A, B \rangle \leq \langle \lambda(A), \lambda(B) \rangle, \quad \forall A, B \in \mathcal{S}_n. \tag{1.5}$$

furthermore, In (1.5) equality holds if and only if A and B are simultaneously diagonalizable.
Every element in the symmetric matrices $\mathcal{S}_n$ has a spectral decomposition given by the formula $A = UDiag(\lambda(A))U^T$, where U belong to the orthogonal matrices O_n. Given an element $A \in \mathcal{S}_n$, we can define its corresponding orthogonal group O_A as follow

$$O_A := \{U \in O_n : U^T A U = Diag(\lambda(A))\}.$$

It is easy to see that O_A is non-empty for each $A \in \mathcal{S}_n$.
It is worth noting that $A, B \in \mathcal{S}_n$ is said to be simultaneously diagonalizable whenever $O_A \cap O_B \neq \emptyset$.

Previously, we discussed the significance of spectral functions in this article. Having established the necessary background knowledge, we now turn our attention towards defining spectral functions in greater detail.
A function $F : \mathcal{S}_n \longrightarrow \bar{\mathbb{R}}$ is called spectral if F is $\mathcal{O}_n$-invariant, i.e.,[5, 10]

$$F(U^T AU) = F(A), \ \forall \ A \in dom(F), \ \forall \ U \in \mathcal{O}_n.$$

By examining the properties of spectral functions, we observe that any such function F defined on $\mathcal{S}_n$ can be expressed as a composition of f and the eigenvalue mapping λ, provided that f is a symmetric function on $\mathbb{R}^n$. Let us mention that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called symmetric if $f(x) = f(Px)$ for all permutation matrices P and for all $x \in \mathbb{R}^n$.
To illustrate how spectral functions relate to symmetric functions, consider the following example. Let $A \in \mathcal{S}_n$ be arbitrary. It can be easily proved

$$||A||_F^2 = \sum_{i=1}^{n} [\lambda_i(A)]^2 = ||\lambda(A)||^2,$$

that is

$$||A||_F = (||.|| \circ \lambda)(A).$$

The above equality implies that the Frobenius norm represents a spectral function associated with the Euclidean norm on $\mathbb{R}^n$.

The following theorems present some properties of spectral functions in point of view convex analysis.

Theorem 1.4. [1, 5, 10] Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. Then, $f \in \Gamma_0(\mathbb{R}^n)$ if and only if $f \circ \lambda \in \Gamma_0(\mathcal{S}_n)$. Also, one has

$$(f \circ \lambda)^*(A) = f^* \circ \lambda(A), \ \forall \ A \in \mathcal{S}_n. \tag{1.6}$$

Remark 1.1. In view of theorem 1.4, one has if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a symmetric function. Then the Fenchel-young function associated with f is equal to the Fenchel-Young function associated with $f \circ \lambda$, that is

$$f^{FY}(\lambda(A), \lambda(B)) = (f \circ \lambda)^{FY}(A, B), \qquad \forall A, B \in \mathcal{S}_n.$$

Theorem 1.5. [1, 5, 10] Let $f \in \Gamma_0(\mathbb{R}^n)$ be symmetric function. Let $A \in \mathcal{S}_n$ be arbitrary. Then,

$$\partial(f \circ \lambda)(A) = \{UDiag(v)U^T : v \in \partial f(\lambda(A)), \quad U \in \mathcal{O}_A\}.$$

Also, if $B \in \partial(f \circ \lambda)(A)$, Then A and B are simultaneously diagonalizable.

2. Fitzpatrick function of subdifferential of the spectral function

We first present some properties of subdifferential of the spectral function. These properties are an immediate consequence of Theorem 1.5.

Lemma 2.1. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. Let $A, B \in \mathcal{S}_n$. Then the following assertions are true:

1) If $B \in \partial(f \circ \lambda)(A)$, then $\lambda(B) \in \partial f(\lambda(A))$.

2) If $B \in \partial(f \circ \lambda)(A)$ and $U \in \mathcal{O}_n$, then $UBU^T \in \partial(f \circ \lambda)(UAU^T)$.

3) $y \in \partial f(x)$ if and only if $Diag(y) \in \partial(f \circ \lambda)(Diag(x))$.

4) Let $v \in \partial f(\lambda(A))$ and $U \in \mathcal{O}_A$. Then, $UDiag(v)U^T \in \partial(f \circ \lambda)(A)$.

5) Suppose that $\lambda(B) \in \partial f(\lambda(A))$, and A, B are simultaneously diagonalizable. Then, $B \in \partial(f \circ \lambda)(A)$.

Proof. 1)Let $B \in \partial(f \circ \lambda)(A)$. In view of (1.1) one has $(f \circ \lambda)^{FY}(A, B) = \langle A, B\rangle$. Thus from Fenchel-Young inequality, (1.5) and (1.6) we have

$$\begin{aligned}\langle\lambda(A), \lambda(B)\rangle &\leq f(\lambda(A)) + f^*(\lambda(B)) = f^{FY}(\lambda(A), \lambda(B))\\ &= (f \circ \lambda)^{FY}(A, B) = \langle A, B\rangle \leq \langle\lambda(A), \lambda(B)\rangle.\end{aligned}$$

Hence $f^{FY}(\lambda(A), \lambda(B)) = \langle\lambda(A), \lambda(B)\rangle$. Therefore in view of (1.1) the proof is complete.

2) Let $B \in \partial(f \circ \lambda)(A)$ and $U \in \mathcal{O}_n$ be arbitrary. Therefore

$$\begin{aligned}\langle UAU^T, UBU^T\rangle &= \langle A, B\rangle = (f \circ \lambda)^{FY}(A, B) = f^{FY}(\lambda(A), \lambda(B))\\ &= f^{FY}(\lambda(UAU^T), \lambda(UBU^T)) = (f \circ \lambda)^{FY}(UAU^T, UBU^T).\end{aligned}$$

Which completes the proof.

3)Since $x = \lambda(Diag(x))$, therefore in view of (1.1) the proof is clear.

4) Let $v \in \partial f(\lambda(A))$ and $U \in \mathcal{O}_A$. Therefore $f^{FY}(\lambda(A), v) = \langle\lambda(A), v\rangle$ and $U^TAU = Diag(\lambda(A))$. Hence

$$\begin{aligned}\langle A, UDiag(v)U^T\rangle &\leqslant (f \circ \lambda)^{FY}(A, UDiag(v)U^T)\\ &= f^{FY}(\lambda(A), v) = \langle\lambda(A), v\rangle\\ &= \langle U^TAU, Diag(v)\rangle = \langle A, UDiag(v)U^T\rangle.\end{aligned}$$

Therefore $UDiag(v)U^T \in \partial(f \circ \lambda)(A)$.

5) Since A and B are simultaneously diagonalizable, hence equality holds in (1.5). Also, we have $f^{FY}(\lambda(A), \lambda(B)) = (f \circ \lambda)^{FY}(A, B)$. In view of (1.1) the proof is clear. □

The following lemma is an immediate consequence of Theorem 1.2 and Theorem 1.4.

Lemma 2.2. Let $f \in \Gamma_0(\mathbb{R}^n)$ be a symmetric function. Then, $\partial(f \circ \lambda)$ is a maximal monotone operator on $\mathcal{S}_n$.

Now, we investigate the Fitzpatrick function of the subdifferential of the spectral function. First, it is worth noting that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a symmetric function, then the Fitzpatrick function associated with the subdifferential of spectral function is so spectral. In other words, It is easy to see that $\mathcal{F}_{\partial(f\circ\lambda)}$ and $\mathcal{F}^*_{\partial(f\circ\lambda)}$ are spectral, in the following sense:

$$\mathcal{F}_{\partial(f\circ\lambda)}(UAU^T, UBU^T) = \mathcal{F}_{\partial(f\circ\lambda)}(A, B), \quad \forall A, B \in \mathcal{S}_n, \quad \forall U \in \mathcal{O}_n. \tag{2.1}$$

Similarly (2.1) holds for $\mathcal{F}^*_{\partial(f\circ\lambda)}$.

Theorem 2.1. Let $f \in \Gamma_0(\mathbb{R}^n)$ be a symmetric function. Let $A, B \in \mathcal{S}_n$ be arbitrary. Then

$$\mathcal{F}_{\partial(f\circ\lambda)}(A, B) \leq \mathcal{F}_{\partial f}(\lambda(A), \lambda(B)). \tag{2.2}$$

Furthermore, suppose that one of the following assertions holds:

(i) A and B are simultaneously diagonalizable.

(ii) $G(\partial f) = \{(x^\dagger, y^\dagger) : x, y \in \mathbb{R}\}$.

Then, equality holds in (2.2).

Proof. First, note that it follows from (1.3) that

$$\mathcal{F}_{\partial(f\circ\lambda)}(A, B) = \sup_{Y\in\partial(f\circ\lambda)(X)} \{\langle A, Y\rangle + \langle X, B\rangle - \langle X, Y\rangle\},$$

and

$$\mathcal{F}_{\partial f}(\lambda(A), \lambda(B)) = \sup_{y\in\partial f(x)} \{\langle x, \lambda(B)\rangle + \langle \lambda(A), y\rangle - \langle x, y\rangle\}.$$

Let $Y \in \partial(f \circ \lambda)(X)$ be arbitrary. Theorem 1.5 implies that there exists $U \in \mathcal{O}_X \cap \mathcal{O}_Y$ such that

$$X = UDiag(\lambda(X))U^T, \qquad Y = UDiag(\lambda(Y))U^T.$$

Now, consider

$$\begin{aligned}
\langle A, Y\rangle + \langle X, B\rangle - \langle X, Y\rangle &= \langle A, Y\rangle + \langle X, B\rangle - \langle UDiag(\lambda(X))U^T, UDiag(\lambda(Y))U^T\rangle \\
&= \langle A, Y\rangle + \langle X, B\rangle - \langle Diag(\lambda(X)), Diag(\lambda(Y))\rangle \\
&\leq \langle \lambda(A), \lambda(Y)\rangle + \langle \lambda(X), \lambda(B)\rangle - \langle \lambda(X), \lambda(Y)\rangle \\
&\leq \mathcal{F}_{\partial f}(\lambda(A), \lambda(B)).
\end{aligned}$$

By taking supremum over all $(X, Y) \in G(\partial(f \circ \lambda))$, we get

$$\mathcal{F}_{\partial(f\circ\lambda)}(A, B) \leq \mathcal{F}_{\partial f}(\lambda(A), \lambda(B)). \tag{2.3}$$

Now, suppose that assertion (i) holds. Let $U \in \mathcal{O}_n$ be such that

$$A = UDiag(\lambda(A))U^T, \qquad B = UDiag(\lambda(B))U^T.$$

Let $y \in \partial f(x)$ be arbitrary. Consider

$$\begin{aligned}
&\langle x, \lambda(B)\rangle + \langle \lambda(A), y\rangle - \langle x, y\rangle \\
&= \langle Diag(x), Diag(\lambda(B))\rangle + \langle Diag(\lambda(A)), Diag(y)\rangle - \langle Diag(x), Diag(y)\rangle \\
&= \langle Diag(x), U^T BU\rangle + \langle U^T AU, Diag(y)\rangle - \langle Diag(x), Diag(y)\rangle \\
&= \langle UDiag(x)U^T, B\rangle + \langle A, UDiag(y)U^T\rangle - \langle UDiag(x)U^T, UDiag(y)U^T\rangle \\
&\leq \mathcal{F}_{\partial(f\circ\lambda)}(A, B).
\end{aligned}$$

Taking supremum over all $(x, y) \in G(\partial f)$. We conclude that the reverse of the inequality (2.2) holds.
Now, assume that assertion (ii) holds. Let $(x^\dagger, y^\dagger) \in G(\partial f)$ be arbitrary. Since $\sum_{i=1}^n c_{ii} = \sum_{i=1}^n \lambda_i(C)$, for each $C = (c_{ij}) \in \mathcal{S}_n$. Hence

$$\begin{aligned}
&\langle x^\dagger, \lambda(B)\rangle + \langle \lambda(A), y^\dagger\rangle - \langle x^\dagger, y^\dagger\rangle \\
&= \langle Diag(x^\dagger), Diag(\lambda(B))\rangle + \langle Diag(\lambda(A)), Diag(y^\dagger)\rangle - \langle Diag(x^\dagger), Diag(y^\dagger)\rangle \\
&= \langle Diag(x^\dagger), B\rangle + \langle A, Diag(y^\dagger)\rangle - \langle Diag(x^\dagger), Diag(y^\dagger)\rangle \\
&\leq \mathcal{F}_{\partial(f\circ\lambda)}(A, B).
\end{aligned}$$

Now, by taking supremum over all $(x^\dagger, y^\dagger) \in G(\partial f)$, we have

$$\mathcal{F}_{\partial f}(\lambda(A), \lambda(B)) \leq \mathcal{F}_{\partial(f\circ\lambda)}(A, B),$$

which completes the proof. □

Corollary 2.1. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric and sublinear function. Let A and B be simultaneously diagonalizable. Then,

$$\mathcal{F}_{\partial(f\circ\lambda)}(A, B) = (f \circ \lambda)(A) + (f \circ \lambda)^*(B).$$

Proof. Since A and B are simultaneously diagonalizable, In view of theorem 1.3 and 2.1 one has

$$\mathcal{F}_{\partial(f\circ\lambda)}(A, B) = \mathcal{F}_{\partial f}(\lambda(A), \lambda(B)) = f^{FY}(\lambda(A), \lambda(B)) = (f \circ \lambda)^{FY}(A, B).$$

□

Theorem 2.2. Let $f \in \Gamma_0(\mathbb{R}^n)$ be a symmetric function. Then,

$$\mathcal{F}^*_{\partial f}(x, y) \leq \mathcal{F}^*_{\partial(f\circ\lambda)}(Diag(x), Diag(y)), \qquad \forall x, y \in \mathbb{R}^n$$

Also, suppose that $A, B \in \mathcal{S}_n$ are simultaneously diagonalizable. Then,

$$\mathcal{F}^*_{\partial f}(\lambda(A), \lambda(B)) \leqslant \mathcal{F}^*_{\partial(f\circ\lambda)}(A, B).$$

Proof. First note that, in view of theorem 2.1 one has

$$\mathcal{F}_{\partial(f\circ\lambda)}(Diag(x), Diag(y)) \leq \mathcal{F}_{\partial f}(\lambda(Diag(x)), \lambda(Diag(y))) \leq \mathcal{F}_{\partial f}(x, y), \quad \forall x, y \in \mathbb{R}^n$$

Let $(x, y), (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ be arbitrary. Consider

$$\begin{aligned}
&\langle (x, y), (u, v)\rangle - \mathcal{F}_{\partial f}(u, v) = \langle x, u\rangle + \langle y, v\rangle - \mathcal{F}_{\partial f}(u, v) \\
&\leq \langle Diag(x), Diag(u)\rangle + \langle Diag(y), Diag(v)\rangle - \mathcal{F}_{\partial(f\circ\lambda)}(Diag(u), Diag(v)) \\
&= \sup_{(U,V)\in\mathcal{S}_n\times\mathcal{S}_n} \{\langle Diag(x), U\rangle + \langle Diag(y), V\rangle - \mathcal{F}_{\partial(f\circ\lambda)}(U, V)\} \\
&= \mathcal{F}^*_{\partial(f\circ\lambda)}(Diag(x), Diag(y)).
\end{aligned}$$

Now, by taking supremum over all $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, we conclude that

$$\mathcal{F}^*_{\partial f}(x, y) \leq \mathcal{F}^*_{\partial(f\circ\lambda)}(Diag(x), Diag(y)).$$

For the second part, since A and B are simultaneously diagonalizable, we consider $U \in \mathcal{O}_A \cap \mathcal{O}_B$. Therefore

$$\begin{aligned}
&\mathcal{F}^*_{\partial f}(\lambda(A), \lambda(B)) \leq \mathcal{F}^*_{\partial(f\circ\lambda)}(Diag(\lambda(A), Diag(\lambda(B))) \\
&= \mathcal{F}^*_{\partial(f\circ\lambda)}(UAU^T, UBU^T) = \mathcal{F}^*_{\partial(f\circ\lambda)}(A, B).
\end{aligned}$$

Hence the proof is complete. □

Example 2.1. Consider symmetric function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by $f(x) = \frac{1}{2}\|x\|^2$, for all $x \in \mathbb{R}^n$. It can be easily concluded the spectral function corresponding to f, is $F(A) = \frac{1}{2}\|A\|_F^2$, for all $A \in \mathcal{S}_n$. It is easy to see that $\partial f = \{\nabla f\} = I_{\mathbb{R}^n}$ and $\partial F = \{\nabla F\} = I_{\mathcal{S}^n}$, where $I_{\mathbb{R}^n}$ and $I_{\mathcal{S}^n}$ are identity matrices on $\mathbb{R}^n$ and $\mathcal{S}_n$, respectively.
From ([2],example 3.10), we have

$$\mathcal{F}_{I_{\mathbb{R}^n}}(x, u) = \frac{1}{4}\|x + u\|^2, \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^n,$$

and similarly

$$\mathcal{F}_{I_{\mathcal{S}_n}}(A, B) = \frac{1}{4}\|A + B\|_F^2, \quad \forall (A, B) \in \mathcal{S}_n \times \mathcal{S}_n.$$

In view of (2.2) one has

$$\|A + B\|_F^2 \leq \|\lambda(A) + \lambda(B)\|^2, \quad \forall (A, B) \in \mathcal{S}_n \times \mathcal{S}_n,$$

and equality holds whenever A and B are simultaneously diagonalizable.

3. ε-subdifferential and ε-enlargement of spectral functions

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the lower semi-continuous convex function. From theorem 1.2, $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the maximal monotone operator. Let $h \in \mathcal{H}(\partial f)$ be arbitrary. For $x \in \mathbb{R}^n$ and $\varepsilon > 0$, the ε-enlargement of f at x is defined by(see [6, 8])

$$\mathbb{E}^h(x,\varepsilon) := \{u \in \mathbb{R}^n : h(x,u) \leq \langle x,u\rangle + \varepsilon\}.$$

Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. From (1.2), we have $\partial_\varepsilon f(x) = \mathbb{E}^{f^{FY}}(x,\varepsilon)$. Also, in view of (1.4) one has

$$\mathbb{E}^{\mathcal{F}^*_{\partial f}}(x,\varepsilon) \subseteq \partial_\varepsilon f(x) \subseteq \mathbb{E}^{\mathcal{F}_{\partial f}}(x,\varepsilon).$$

If $\mathcal{F}_{\partial f}$ is an autoconjugate function. From theorem 1.3, we conclude

$$\mathbb{E}^{\mathcal{F}^*_{\partial f}}(x,\varepsilon) = \partial_\varepsilon f(x) = \mathbb{E}^{\mathcal{F}_{\partial f}}(x,\varepsilon), \quad \forall x \in \mathbb{R}^n, \forall \varepsilon > 0.$$

It might be interesting to check the properties of ε-subdifferential and ε-enlargement of the spectral function $f \circ \lambda$ and its relation to the ε-subdifferential and ε-enlargement of symmetric function f. The following theorem states the properties of the ε-subdifferential of the spectral function.

Theorem 3.1. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. Let $A, B \in \mathcal{S}_n$. Then the following assertions hold:

1) If $B \in \partial_\varepsilon (f \circ \lambda)(A)$, then $\lambda(B) \in \partial_\varepsilon f(\lambda(A))$.
2) Let $v \in \partial_\varepsilon f(\lambda(A))$ and $U \in \mathcal{O}_A$. Then, $UDiag(v)U^T \in \partial_\varepsilon (f \circ \lambda)(A)$.
3) Suppose that $\lambda(B) \in \partial_\varepsilon f(\lambda(A))$, and A, B are simultaneously diagonalizable. Then, $B \in \partial_\varepsilon (f \circ \lambda)(A)$.

Proof. 1) Let $B \in \partial_\varepsilon (f \circ \lambda)(A)$ be arbitrary. From (1.2) and (1.5), one has

$$f^{FY}(\lambda(A),\lambda(B)) = (f \circ \lambda)^{FY}(A,B) \leq \langle A,B\rangle + \varepsilon \leq \langle \lambda(A),\lambda(B)\rangle + \varepsilon.$$

Hence, the proof is complete.

2) Let $v \in \partial_\varepsilon f(\lambda(A))$ and $U \in \mathcal{O}_A$. Hence $f^{FY}(\lambda(A),v) \leq \langle \lambda(A),v\rangle + \varepsilon$ and $U^TAU = Diag(\lambda(A))$. Therefore

$$(f \circ \lambda)^{FY}(A, UDiag(v)U^T) = f^{FY}(\lambda(A),v) \leq \langle \lambda(A),v\rangle + \varepsilon = \langle U^TAU, Diag(v)\rangle + \varepsilon.$$

Since $\lambda(Diag(v)) = \lambda(UDiag(v)U^T)$, Therefore $UDiag(v)U^T \in \partial_\varepsilon (f \circ \lambda)(A)$.

3) Let $\lambda(B) \in \partial_\varepsilon f(\lambda(A))$. Since A and B are simultaneously diagonalizable, therefore in (1.5) equality holds. Hence

$$(f \circ \lambda)^{FY}(A,B) = f^{FY}(\lambda(A),\lambda(B)) \leq \langle \lambda(A),\lambda(B)\rangle + \varepsilon = \langle A,B\rangle + \varepsilon.$$

which completes the proof.

□

The following theorem expresses properties of the ε-enlargement of the spectral function.

Theorem 3.2. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric function. Let $A, B \in \mathcal{S}_n$. Then the following assertions hold:

1) Let $u \in \mathbb{E}^{\mathcal{F}_{\partial f}}(x, \varepsilon)$. Then $Diag(u) \in \mathbb{E}^{\mathcal{F}_{\partial(f\circ\lambda)}}(Diag(x), \varepsilon)$.

2) Let A, B be simultaneously diagonalizable. Then, $B \in \mathbb{E}^{\mathcal{F}_{\partial(f\circ\lambda)}}(A, \varepsilon)$ if and only if $\lambda(B) \in \mathbb{E}^{\mathcal{F}_{\partial f}}(\lambda(A), \varepsilon)$.

Proof.

1) Since $u \in \mathbb{E}^{\mathcal{F}_{\partial f}}(x, \varepsilon)$, in view of (2.2) one has

$$\mathcal{F}_{\partial(f\circ\lambda)}(Diag(x), Diag(u)) \le \mathcal{F}_{\partial f}(x, u) \le \langle x, u\rangle + \varepsilon = \langle Diag(x), Diag(u)\rangle + \varepsilon.$$

Hence $Diag(u) \in \mathbb{E}^{\mathcal{F}_{\partial(f\circ\lambda)}}(Diag(x), \varepsilon)$.

2) Since A and B are simultaneously diagonalizable. Therefore

$$\begin{aligned} &B \in \mathbb{E}^{\mathcal{F}_{\partial(f\circ\lambda)}}(A, \varepsilon) \Leftrightarrow \mathcal{F}_{\partial(f\circ\lambda)}(A, B) \le \langle A, B\rangle + \varepsilon \\ &\Leftrightarrow \mathcal{F}_{\partial f}(\lambda(A), \lambda(B)) \le \langle \lambda(A), \lambda(B)\rangle + \varepsilon \Leftrightarrow \lambda(B) \in \mathbb{E}^{\mathcal{F}_{\partial f}}(\lambda(A), \varepsilon). \end{aligned}$$

□

Corollary 3.1. Let $\mathcal{D}_n$ be a set of all simultaneously diagonalizable matrices in $\mathcal{S}_n$. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a symmetric and sublinear function. Then for each $\varepsilon > 0$, the graph of $\partial_\varepsilon(f \circ \lambda)$ coincides with to graph of $\mathbb{E}^{\mathcal{F}_{\partial(f\circ\lambda)}}(., \varepsilon)$ on $\mathcal{D}_n$. That is

$$B \in \partial_\varepsilon(f \circ \lambda)(A) \Longleftrightarrow B \in \mathbb{E}^{\mathcal{F}_{\partial(f\circ\lambda)}}(A, \varepsilon), \quad \forall A, B \in \mathcal{D}_n, \quad \forall \varepsilon > 0.$$

Example 3.1. Consider symmetric function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined by $f(x) = \|x\|$, for all $x \in \mathbb{R}^n$. It is clear that the spectral function corresponding to f, is $F(A) = f \circ \lambda(A) = \|A\|_F$, for all $A \in \mathcal{S}_n$. From ([2], example 3.3), we have

$$\mathcal{F}_{\partial F}(A, B) = F^{FY}(A, B) = F(A) + F^*(B) = \begin{cases} \|A\|_F, & if \quad \|B\|_F \leqslant 1 \\ +\infty, & otherwise. \end{cases} \tag{3.1}$$

Let $\varepsilon > 0$ and $A \in \mathcal{S}_n$ be given. In view of (1.2) and (3.1) one has

$$\mathbb{E}^{\mathcal{F}_{\partial F}}(A, \varepsilon) = \partial_\varepsilon F(A) = \{B \in \mathcal{S}_n : \|B\|_F \leqslant 1, \quad \|A\|_F \leqslant tr(AB) + \varepsilon\}. \tag{3.2}$$

Let $\varepsilon = \frac{1}{2}$ and

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

From (3.2), it is clear that $B \in \partial_\varepsilon F(A)$. In view of theorem 3.1 (1), we have $\lambda(B) \in \partial_\varepsilon f(\lambda(A))$. Let $v = (\frac{1}{2}, 0) \in \mathbb{R}^2$. It is clear that $v \in \partial_\varepsilon f(\lambda(A))$. Consider $U \in O_A$, from theorem 3.1 (2), one has $UDiag(v)U^T \in \partial_\varepsilon F(A)$.

Example 3.2. In the setting of example 2.1, let $A, B \in \mathcal{S}_n$ be arbitrary. Then $B \in \partial_\varepsilon F(A)$ if and only if

$$\frac{1}{2}\|A\|_F^2 + \frac{1}{2}\|B\|_F^2 \leqslant tr(AB) + \varepsilon. \tag{3.3}$$

Also, $B \in \mathbb{E}^{\mathcal{F}_{\partial F}}(A, \varepsilon)$ if and only if

$$\frac{1}{4}\|A + B\|_F^2 \leqslant tr(AB) + \varepsilon. \tag{3.4}$$

Therefore (3.4) is equivalent to the following equation

$$\frac{1}{4}\|A\|_F^2 + \frac{1}{4}\|B\|_F^2 \leqslant \frac{1}{2}tr(AB) + \varepsilon. \tag{3.5}$$

Let $\varepsilon = 7$ and

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}, \qquad B = A^{-1}.$$

It is easy to see that A and B are simultaneously diagonalizable. Also from (3.3) and (3.5) we have $B \in \mathbb{E}^{\mathcal{F}_{\partial F}}(A, \varepsilon)$, but $B \notin \partial_\varepsilon F(A)$. Therefore in this case

$$\partial_\varepsilon F(A) \subsetneqq \mathbb{E}^{\mathcal{F}_{\partial F}}(A, \varepsilon).$$

It is worth noting that $f(x) = \frac{1}{2}\|x\|^2$ is a convex function on $\mathbb{R}^n$, but f is not sublinear. Hence, the assumption of sublinearity in collorary 3.1 is necessary.

Optimizing the difference of convex functions is a significant application of studying ε-subdifferentials. The following theorem is known in the topic of optimization of the difference of convex functions.

Theorem 3.3. [11] Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two convex functions. $x_0 \in \mathbb{R}^n$ is global minimum of $f - g$ if and only if $\partial_\varepsilon g(x_0) \subseteq \partial_\varepsilon f(x_0)$, for all $\varepsilon > 0$.

Remark 3.3. For every convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we have $\bigcap_{\varepsilon>0} \partial_\varepsilon f(x_0) = \partial f(x_0)$. Therefore, we can state the above theorem as follows:

$$x_0 \in \underset{\mathbb{R}^n}{\operatorname{argmin}}(f - g) \Longleftrightarrow \partial g(x_0) \subseteq \partial f(x_0). \tag{3.6}$$

In the following theorem, we investigate the optimization of the difference between spectral convex functions and its connection to optimizing the difference in the corresponding symmetric functions

Theorem 3.4. Let $f, g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be two convex symmetric functions. Then, the following assertions hold:

1) Let $A \in \mathcal{S}_n$ and $A \in \operatorname{argmin}_{\mathcal{S}_n}(f \circ \lambda - g \circ \lambda)$. Then, $\lambda(A) \in \operatorname{argmin}_{\mathbb{R}^n}(f - g)$.

2) Let $x_0 \in \mathbb{R}^n$ and $x_0 \in \operatorname{argmin}_{\mathbb{R}^n}(f - g)$. Then, $Diag(x_0) \in \operatorname{argmin}_{\mathcal{S}_n}(f \circ \lambda - g \circ \lambda)$.

3) Let $A \in \mathcal{S}_n$ and $A \in \operatorname{argmin}_{\mathcal{S}_n}(f\circ\lambda - g\circ\lambda)$. Let $U \in \mathcal{O}_n$. Then, $UAU^T \in \operatorname{argmin}_{\mathcal{S}_n}(f\circ\lambda - g\circ\lambda)$.

Proof. 1)Let $A \in \operatorname{argmin}_{\mathcal{S}_n}(f \circ \lambda - g \circ \lambda)$. In view of (3.6) one has $\partial(g \circ \lambda)(A) \subseteq \partial(f \circ \lambda)(A)$. We must show that $\partial g(\lambda(A)) \subseteq \partial f(\lambda(A))$. To this end, Let $v \in \partial g(\lambda(A))$ be arbitrary and let $U \in \mathcal{O}_A$. From lemma 2.1(4) we have $UDiag(v)U^T \in \partial(g \circ \lambda)(A)$. Therefore $UDiag(v)U^T \in \partial(f \circ \lambda)(A)$. Now by theorem 2.1(1) we have $\lambda(UDiag(v)U^T) \in \partial f(\lambda(A))$. Hence the proof is complete.

2)Let $x_0 \in \operatorname{argmin}_{\mathbb{R}^n}(f - g)$. From (3.6) we have $\partial g(x_0) \subseteq \partial f(x_0)$. Now, we show that $\partial(g \circ \lambda)(Diag(x_0)) \subseteq \partial(f \circ \lambda)(Diag(x_0))$. Suppose $B \in \partial(g \circ \lambda)(Diag(x_0))$. Hence from lemma 2.1(1) we have $\lambda(B) \in \partial g(\lambda(Diag(x_0)))$. Also, in view of throem 1.5 B and $Diag(x_0)$ are simultaneously diagonalizable. Therefore $\lambda(B) \in \partial f(\lambda(Diag(x_0)))$. Now, from lemma 2.1(5) we conclude $B \in \partial(f \circ \lambda)(Diag(x_0))$, which completes the proof.

3) Since $A \in \operatorname{argmin}_{\mathcal{S}_n}(f\circ\lambda - g\circ\lambda)$, hence $\partial(g\circ\lambda)(A) \subseteq \partial(f\circ\lambda)(A)$. Now, Let $B \in \partial(g\circ\lambda)(UAU^T)$. It is easy to see that $U^TBU \in \partial(g\circ\lambda)(A)$. Therefore $U^TBU \in \partial(f\circ\lambda)(A)$. Thus $B \in \partial(f\circ\lambda)(UAU^T)$ and the proof is complete. □

Example 3.4. Consider the following optimization problem

$$\min_{A\in\mathcal{S}_n}\left\{\|A\|_F^2 - tr(A)\right\}.$$

Let $F(A) = \|A\|_F^2$ and $G(A) = tr(A)$, for each $A \in \mathcal{S}_n$. It is clear that F and G are convex spectral functions on $\mathcal{S}_n$. For each $x \in \mathbb{R}^n$, put $f(x) = \|x\|^2 = \sum_{i=1}^n x_i^2$ and $g(x) = \sum_{i=1}^n x_i$. It is clear that f and g are convex symmetric functions on $\mathbb{R}^n$ corresponding to F and G, respectively. That is $F(A) = f \circ \lambda(A)$ and $G(A) = g \circ \lambda(A)$, for each $A \in \mathcal{S}_n$.
Let $\xi = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$. It can be shown by simple calculations $\xi \in \operatorname{argmin}_{\mathbb{R}^n}(f - g)$. Therefore, in view of theorem 3.4(2) $Diag(\xi) \in \operatorname{argmin}_{\mathcal{S}_n}(F - G)$. Also, from theorem 3.4(3), we have $UDiag(\xi)U^T \in \operatorname{argmin}_{\mathcal{S}_n}(F - G)$, for each $U \in \mathcal{O}_n$.

Example 3.5. Consider the following optimization problem

$$\min_{A\in\mathcal{S}_2}\left\{e^{tr(A)} - tr(A)\right\}.$$

Let $f(x_1, x_2) = e^{x_1+x_2}$ and $g(x_1, x_2) = x_1 + x_2$, for each $(x_1, x_2) \in \mathbb{R}^2$. It is clear that f and g are convex symmetric functions on $\mathbb{R}^2$ corresponding to $F(A) = e^{tr(A)}$ and $G(A) = tr(A)$, respectively. After solving the optimization problem $\min_{\mathbb{R}^2}\{f - g\}$, we get

$$\{(\xi, -\xi) : \xi \in \mathbb{R}\} \subseteq \operatorname*{argmin}_{\mathbb{R}^2}\{f - g\}.$$

Hence in view of theorem 3.4(2),(3), we conclude

$$\left\{UDiag(v)U^T : v = (\xi, -\xi) \in \mathbb{R}^2, \quad \xi \in \mathbb{R}, \quad U \in \mathcal{O}_2\right\} \subseteq \operatorname*{argmin}_{\mathcal{S}_2}\{F - G\}.$$

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