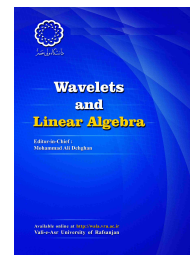


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## A class of Gabor frames with bounded compactly supported generator function

Mohammad Ali Hasankhani Fard\*<sup>a</sup>

<sup>a</sup>Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.

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### ABSTRACT

In this paper a class of Gabor frames with time shift parameter  $a > 0$ , frequency shift parameter  $b > 0$  and bounded compactly supported generator function  $g$  such that  $\text{supp } g \subseteq \left[ (k+2)a - \frac{2}{b}, ka + \frac{1}{b} \right]$  or  $\text{supp } g \subseteq \left[ (k+1)a - \frac{1}{b}, ka + \frac{1}{b} \right]$ , where  $k$  is an integer number is introduced. In particular, a sufficient condition on a function  $g \in C_c^+(\mathbb{R})$  with  $\text{supp } g \subseteq \left[ (k+2)a - \frac{2}{b}, ka + \frac{1}{b} \right]$  and positive decreasing derivative  $g'$  on  $\left( ka - \frac{1}{b}, (k+2)a \right)$ , that make  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  into a Gabor frame, is given.

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### 1. Introduction

Frames were first introduced by Duffin and Schaeffer [3] in the study of nonharmonic Fourier series in 1952. Frames have very important and interesting properties which make them very useful in the characterization of function spaces, signal processing and many other fields. A frame is a family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements [2].

\*Corresponding author

Email address: m.hasankhani@vru.ac.ir (Mohammad Ali Hasankhani Fard\*)

**Definition 1.1.** [2] Given a separable Hilbert space  $\mathcal{H}$  with inner product  $\langle \cdot, \cdot \rangle$ , a sequence  $\{f_k\}_{k=1}^\infty$  is called a frame for  $\mathcal{H}$  if there exist constants  $A > 0, B < \infty$  such that for all  $f \in \mathcal{H}$ ,

$$A \|f\|^2 \leq \sum_{k=1}^\infty |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \tag{1.1}$$

where  $A, B$  are the lower and upper frame bounds, respectively.

The second inequality of the frame condition (1.1) is also known as the Bessel condition for  $\{f_k\}_{k=1}^\infty$ . If  $A = B$ , then  $\{f_k\}_{k=1}^\infty$  is called a tight frame. For more information concerning frames, we refer the reader to [2, 6, 9].

For any  $x, y \in \mathbb{R}$  the translation operator  $T_x$  and modulation operator  $E_y$  on  $L_2(\mathbb{R})$  are defined by

$$(T_x g)(t) = g(t - x), \quad (E_y g)(t) = e^{2\pi i y t} g(t).$$

**Definition 1.2.** [2] A Gabor system with window function  $g \in L_2(\mathbb{R})$ , time shift parameter  $a > 0$  and frequency shift parameter  $b > 0$  is the sequence  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ . A Gabor system  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  is called a Gabor frame if it is a frame for  $L_2(\mathbb{R})$ , i.e., if there exist constants  $0 < A \leq B < \infty$  such that for all  $f \in L_2(\mathbb{R})$ ,

$$A \|f\|^2 \leq \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb} T_{na} g \rangle|^2 \leq B \|f\|^2, \tag{1.2}$$

where  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  denote the standard norm and inner product of  $L_2(\mathbb{R})$ .

A sufficient and necessary condition on a Gabor system  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ , that make  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  into a Gabor frame, is given by Ron-Shen theorem.

**Theorem 1.3.** [4, 8] A Gabor system  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame with bounds  $0 < A \leq B < \infty$  if and only if

$$bAI \leq M_g(t) M_g^*(t) \leq bBI, \quad a.e. \quad t \in \mathbb{R},$$

where  $I$  denote the identity operator on  $\ell_2(\mathbb{Z})$  and  $M_g(t)$  is the bi-infinite matrix defined by

$$M_g(t) = \left( g \left( t + na - \frac{k}{b} \right) \right)_{k,n \in \mathbb{Z}}, \quad a.e. \quad t \in \mathbb{R},$$

where  $k$  is the row index and  $n$  is the column index.

Two Gabor frames  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb} T_{na} h\}_{m,n \in \mathbb{Z}}$  form dual frames for  $L_2(\mathbb{R})$  if

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb} T_{na} g \rangle E_{mb} T_{na} h,$$

for all  $f \in L_2(\mathbb{R})$ [2]. For more information concerning Gabor frames, we refer the reader to [1, 4, 5].

In this paper we give some sufficient conditions on a bounded compactly supported generator function  $g$  such that  $supp \ g \subseteq \left[ (k+2)a - \frac{2}{b}, ka + \frac{1}{b} \right]$  or  $supp \ g \subseteq \left[ (k+1)a - \frac{1}{b}, ka + \frac{1}{b} \right]$ ,  $k \in \mathbb{Z}$ , for which  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

## 2. Main results

The duality condition for a pair of Gabor systems  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$  is presented in [7].

**Lemma 2.1.** [7] *Two Bessel sequences  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$  form dual frames for  $L_2(\mathbb{R})$  if and only if*

$$\sum_{m \in \mathbb{Z}} g\left(x + ma - \frac{n}{b}\right) \overline{h(x + ma)} = b\delta_{n,0}, \quad \text{a.e. } x \in [0, a], \quad \forall n \in \mathbb{Z}.$$

It is well known that the Gabor system  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is not a frame if  $a$  and  $b$  are real number such that  $ab > 1$  and the Gabor system  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is Bessel sequence, if  $g$  is a bounded function with compact support.

In the next theorem, a sufficient condition on a bounded compactly supported generator function  $g$  such that  $\text{supp } g \subseteq \left[(k + 2)a - \frac{2}{b}, ka + \frac{1}{b}\right]$ ,  $k \in \mathbb{Z}$ , for which  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame, is given.

**Theorem 2.2.** *Let  $a, b$  be positive numbers with  $ab \leq 1$  and  $k \in \mathbb{Z}$ . Assume that  $g$  is a bounded function with compact support such that  $\text{supp } g \subseteq \left[(k + 2)a - \frac{2}{b}, ka + \frac{1}{b}\right]$  and  $\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| > 0$ , where  $\Lambda_g(x) = g(x)g\left(x + a - \frac{1}{b}\right) - g\left(x + a\right)g\left(x - \frac{1}{b}\right)$ . Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.*

*Proof.* Let  $h$  be the complex function defined on  $\mathbb{R}$  by

$$h(x) := \begin{cases} \frac{bg\left(x - \frac{1}{b} + a\right)}{\Lambda_g(x)} & ka < x < (k + 1)a, \\ \frac{-bg\left(x - \frac{1}{b} - a\right)}{\Lambda_g(x - a)} & (k + 1)a < x < (k + 2)a, \\ 0 & \text{o.w.} \end{cases}$$

Then  $h \in L_2(\mathbb{R})$  since  $\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| > 0$  and  $g$  is bounded. Also  $h$  is bounded and compactly supported with  $\text{supp } h \subseteq [ka, (k + 1)a]$ . Using,  $\text{supp } g \subseteq \left[(k + 2)a - \frac{2}{b}, ka + \frac{1}{b}\right]$  and  $\text{supp } h \subseteq [ka, (k + 1)a]$  we have

$$\begin{aligned} \sum_{m \in \mathbb{Z}} g\left(x + ma - \frac{n}{b}\right) \overline{h(x + ma)} &= \sum_{m=k}^{k+1} g\left(x + ma - \frac{n}{b}\right) \overline{h(x + ma)} \\ &= b\delta_{n,0}, \end{aligned}$$

for all  $x \in (0, a)$  and for all  $n \in \mathbb{Z}$  and hence for a.e.  $x \in [0, a]$  and for all  $n \in \mathbb{Z}$ , since the set  $\{0, a\}$  has zero Lebesgue measure.

Thus by Lemma 2.1,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$  are dual Gabor frames. In particular  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.  $\square$

**Corollary 2.3.** Let  $a, b$  be positive numbers with  $ab \leq 1$  and  $k \in \mathbb{Z}$ . Assume that  $g$  is a bounded function with compact support such that  $\text{supp } g \subseteq I_k := \left[ (k+1)a - \frac{1}{b}, (k+1)a \right]$  and  $\inf_{x \in I_k} |g(x)| > 0$ . Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

In Particular, if  $\text{supp } g \subseteq \left[ a - \frac{1}{b}, a \right]$  and  $\inf_{x \in [a-\frac{1}{b}, a]} |g(x)| > 0$ , then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

*Proof.* Using,  $\text{supp } g \subseteq \left[ (k+1)a - \frac{1}{b}, (k+1)a \right]$  we have

$$\begin{aligned} |\Lambda_g(x)| &= \left| g(x) g\left(x + a - \frac{1}{b}\right) \right| \\ &\geq \left( \inf_{x \in I_k} |g(x)| \right)^2, \end{aligned}$$

for all  $x \in (ka, (k+1)a)$ . Thus  $\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| > 0$ . By Theorem 2.2, the result obtains.  $\square$

**Corollary 2.4.** Let  $a, b$  be positive numbers with  $ab \leq \frac{1}{2}$  and  $k \in \mathbb{Z}$ . Assume that  $g$  is a bounded function with compact support such that  $\text{supp } g \subseteq I_k := \left[ ka - \frac{1}{b}, (k+1)a - \frac{1}{b} \right] \cup \left[ (k+1)a, (k+2)a \right]$  and  $\inf_{x \in I_k} |g(x)| > 0$ . Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

*Proof.* The inequality  $ab \leq \frac{1}{2}$  implies that  $\text{supp } g \subseteq \left[ (k+2)a - \frac{2}{b}, ka + \frac{1}{b} \right]$ . Using,  $\text{supp } g \subseteq I_k$  we have

$$\begin{aligned} |\Lambda_g(x)| &= \left| g(x+a) g\left(x - \frac{1}{b}\right) \right| \\ &\geq \left( \inf_{x \in I_k} |g(x)| \right)^2, \end{aligned}$$

for all  $x \in (ka, (k+1)a)$ . Thus  $\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| > 0$ . Now Theorem 2.2 implies that  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.  $\square$

**Corollary 2.5.** Let  $a, b$  be positive numbers with  $ab \leq 1$  and  $k \in \mathbb{Z}$ . Assume that  $g$  is a bounded function with compact support such that  $\text{supp } g \subseteq \left[ (k+2)a - \frac{2}{b}, ka + \frac{1}{b} \right]$ . If either

$$0 < \inf_{x \in I_k} |g(x)| \leq \sup_{x \in I_k} |g(x)| \leq \frac{1}{\sqrt{2}} \inf_{x \in J_k} |g(x)| \tag{2.1}$$

or

$$0 < \inf_{x \in J_k} |g(x)| \leq \sup_{x \in J_k} |g(x)| \leq \frac{1}{\sqrt{2}} \inf_{x \in I_k} |g(x)|, \tag{2.2}$$

where  $I_k = \left[ (k+1)a - \frac{1}{b}, (k+1)a \right]$  and  $J_k = \left[ ka - \frac{1}{b}, (k+2)a \right]$ , then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

*Proof.* If inequality (2.1) holds, then for all  $x \in (ka, (k + 1) a)$ , we have

$$\begin{aligned} |\Lambda_g(x)|^2 &= \left| g(x) g\left(x + a - \frac{1}{b}\right) - g(x + a) g\left(x - \frac{1}{b}\right) \right|^2 \\ &\geq |g(x)|^2 \left| g\left(x + a - \frac{1}{b}\right) \right|^2 \\ &\quad + |g(x + a)| \left| g\left(x - \frac{1}{b}\right) \right| \left( |g(x + a)| \left| g\left(x - \frac{1}{b}\right) \right| - 2 |g(x)| \left| g\left(x + a - \frac{1}{b}\right) \right| \right) \\ &\geq \left[ \inf_{x \in I_k} |g(x)| \right]^2 + \left[ \inf_{x \in J_k} |g(x)| \right]^2 \left( \left[ \inf_{x \in J_k} |g(x)| \right]^2 - 2 \left[ \sup_{x \in I_k} |g(x)| \right]^2 \right) \\ &\geq \left[ \inf_{x \in I_k} |g(x)| \right]^2 \end{aligned}$$

and hence  $\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| > 0$ . Thus by Theorem 2.2,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame. Similarly,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame, if inequality (2.2) holds.  $\square$

**Proposition 2.6.** *Let  $a, b$  be positive numbers with  $ab \leq 1$  and  $k \in \mathbb{Z}$ . Assume that  $g \in C_c(\mathbb{R})$  such that  $\text{supp } g \subseteq \left[ (k + 2) a - \frac{2}{b}, ka + \frac{1}{b} \right]$  and  $\Lambda_g(x) \neq 0$ , for any  $x \in [ka, (k + 1) a]$ , where  $\Lambda_g(x) = g(x) g\left(x + a - \frac{1}{b}\right) - g(x + a) g\left(x - \frac{1}{b}\right)$ . Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.*

*Proof.* The continuity of  $g$  implies that  $|\Lambda_g|$  is continuous and hence  $\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| \geq \inf_{x \in [ka, (k+1)a]} |\Lambda_g(x)| > 0$  by  $\Lambda_g(x) \neq 0$ , for any  $x \in [ka, (k + 1) a]$ . By Theorem 2.2, the result obtains.  $\square$

**Corollary 2.7.** *Let  $a, b$  be positive numbers with  $ab \leq \frac{1}{2}$  and  $k \in \mathbb{Z}$ . Assume that  $g \in C_c^+(\mathbb{R})$  such that  $\text{supp } g \subseteq \left[ (k + 2) a - \frac{2}{b}, ka + \frac{1}{b} \right]$ ,  $g$  is strictly increasing (strictly decreasing) on  $\left[ ka - \frac{1}{b}, (k + 2) a - \frac{1}{b} \right]$  and  $g$  is strictly decreasing (strictly increasing) on  $[ka, (k + 2) a]$ . Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.*

*Proof.* The hypothesis in theorem implies that  $\Lambda_g(x) > 0$  (or  $\Lambda_g(x) < 0$ ), for any  $x \in [ka, (k + 1) a]$  and hence by Proposition 2.6,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.  $\square$

In the next theorem, we give a sufficient condition on a function  $g \in C_c^+(\mathbb{R}) := \{g \in C_c(\mathbb{R}) : g(x) \geq 0, \forall x \in \mathbb{R}\}$  with positive decreasing derivative  $g'$  on  $\left(ka - \frac{1}{b}, (k + 2) a\right)$ , for which  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

**Theorem 2.8.** *Let  $a, b$  be positive numbers with  $ab \leq \frac{1}{2}$  and  $k \in \mathbb{Z}$ . Assume that  $g \in C_c^+(\mathbb{R})$  such that  $\text{supp } g \subseteq \left[ (k + 2) a - \frac{2}{b}, ka + \frac{1}{b} \right]$  and  $g$  is differentiable on  $\left(ka - \frac{1}{b}, (k + 2) a\right)$  with positive decreasing derivative  $g'$ . Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.*

*Proof.* For any  $x \in [ka, (k + 1)a]$ , using the mean value theorem there exist  $x_1 \in (x, x + a)$ ,  $x_2 \in (x - \frac{1}{b}, x + a - \frac{1}{b})$  and  $x_3 \in (x + a - \frac{1}{b}, x + a)$  we have

$$\begin{aligned} \Lambda_g(x) &= g(x)g\left(x + a - \frac{1}{b}\right) - g(x + a)g\left(x - \frac{1}{b}\right) \\ &= g(x)g\left(x + a - \frac{1}{b}\right) - g(x + a)g\left(x + a - \frac{1}{b}\right) \\ &\quad + g(x + a)g\left(x + a - \frac{1}{b}\right) - g(x + a)g\left(x - \frac{1}{b}\right) \\ &= g(x + a)\left(g\left(x + a - \frac{1}{b}\right) - g\left(x - \frac{1}{b}\right)\right) \\ &\quad - g\left(x + a - \frac{1}{b}\right)(g(x + a) - g(x)) \\ &= a\left[g(x + a)g'(x_2) - g\left(x + a - \frac{1}{b}\right)g'(x_1)\right] \\ &\geq ag'(x_1)\left[g(x + a) - g\left(x + a - \frac{1}{b}\right)\right] \\ &= \frac{a}{b}g'(x_1)g'(x_3) \\ &> 0. \end{aligned}$$

Now Proposition 2.6 implies the result. □

In the next theorem, a sufficient condition on a bounded compactly supported generator function  $g$  such that  $\text{supp } g \subseteq \left[(k + 1)a - \frac{1}{b}, ka + \frac{1}{b}\right]$ ,  $k \in \mathbb{Z}$ , for which make  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  into a Gabor frame, is given.

**Theorem 2.9.** *Let  $a, b$  be positive numbers with  $ab \leq 1$  and  $k \in \mathbb{Z}$ . Assume that  $g$  is a bounded function with compact support such that  $\text{supp } g \subseteq \left[(k + 1)a - \frac{1}{b}, ka + \frac{1}{b}\right]$  and  $\inf_{x \in (ka, (k+1)a)} |g(x)| > 0$ . Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.*

*In particular if  $\text{supp } g \subseteq \left[a - \frac{1}{b}, \frac{1}{b}\right]$  and  $\inf_{x \in (0,a)} |g(x)| > 0$ , then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.*

*Proof.* Let  $h$  be the complex function defined on  $\mathbb{R}$  by

$$h(x) := \begin{cases} \frac{b}{g(x)} & ka < x < (k + 1)a, \\ 0 & \text{o.w.} \end{cases}$$

Then  $h \in \mathcal{L}_2(\mathbb{R})$  by  $\inf_{x \in (ka, (k+1)a)} |g(x)| > 0$ . Also  $h$  is bounded and compactly supported with  $\text{supp } h \subseteq [ka, (k + 1)a]$ . Using,  $\text{supp } g \subseteq \left[(k + 1)a - \frac{1}{b}, ka + \frac{1}{b}\right]$  and  $\text{supp } h \subseteq [ka, (k + 1)a]$  we have

$$\begin{aligned} \sum_{m \in \mathbb{Z}} g\left(x + ma - \frac{n}{b}\right) \overline{h(x + ma)} &= g\left(x + ka - \frac{n}{b}\right) \overline{h(x + ka)} \\ &= b\delta_{n,0}, \end{aligned}$$

for all  $x \in (0, a)$  and for all  $n \in \mathbb{Z}$  and hence for a.e.  $x \in [0, a]$  and for all  $n \in \mathbb{Z}$ , since the set  $\{0, a\}$  has zero Lebesgue measure.

Thus by Lemma 2.1,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$  are dual Gabor frames. In particular  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.  $\square$

### 3. Examples

**Example 3.1.** Let  $a, b$  be positive numbers with  $ab \leq 1$  and  $k \in \mathbb{Z}$ . Assume that  $g$  is a complex function on  $\mathbb{R}$ . Then  $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$  is a Gabor frame, where  $h$  is the function defined on  $\mathbb{R}$  by

$$h(x) := \begin{cases} 2 + \frac{1}{1+|g(x)|} & x \in (ka, (k+1)a), \\ 1 & x \in \left[ (k+2)a - \frac{2}{b}, ka + \frac{1}{b} \right] \setminus (ka, (k+1)a), \\ 0 & o.w. \end{cases}$$

Indeed, the function  $h$  is bounded with  $supp h = \left[ (k+2)a - \frac{2}{b}, ka + \frac{1}{b} \right]$  and  $\inf_{x \in (ka, (k+1)a)} |\Lambda_h(x)| \geq 1$ . By Theorem 2.2, the result obtains.

**Example 3.2.** Let  $a, b$  be positive numbers with  $ab \leq \frac{1}{2}$ ,  $a \neq n\pi$  and  $b \neq \frac{1}{n\pi}$ , for all  $n \in \mathbb{N}$ . Assume  $k \in \mathbb{Z}$  and  $g$  is the function defined on  $\mathbb{R}$  by

$$g(x) := \begin{cases} \sin x & (k+2)a - \frac{2}{b} \leq x \leq ka + \frac{1}{b}, \\ 0 & o.w. \end{cases}$$

For any  $x \in (ka, (k+1)a)$  we have

$$\begin{aligned} |\Lambda_g(x)| &= \left| \sin x \sin \left( x + a - \frac{1}{b} \right) - \sin \left( x + a \right) \sin \left( x - \frac{1}{b} \right) \right| \\ &= \left| \sin a \sin \frac{1}{b} \right|. \end{aligned}$$

Now

$$\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| = \left| \sin a \sin \frac{1}{b} \right| > 0.$$

Then by Theorem 2.2,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

**Example 3.3.** Let  $a, b$  be positive numbers with  $ab \leq \frac{1}{2}$ ,  $c \in \mathbb{R} \setminus \{0\}$  and  $d \in \mathbb{R}$ . Assume  $k \in \mathbb{Z}$  and  $g$  is the function defined on  $\mathbb{R}$  by

$$g(x) := \begin{cases} cx + d & (k+2)a - \frac{2}{b} \leq x \leq ka + \frac{1}{b}, \\ 0 & o.w. \end{cases}$$

We have

$$\begin{aligned} |\Lambda_g(x)| &= \left| (cx + d) \left[ c \left( x + a - \frac{1}{b} \right) + d \right] - [c(x + a) + d] \left[ c \left( x - \frac{1}{b} \right) + d \right] \right| \\ &= \frac{c^2 a}{b} \end{aligned}$$

for any  $x \in (ka, (k + 1) a)$ . Now

$$\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| = \frac{c^2 a}{b} > 0$$

and hence by Theorem 2.2,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

**Example 3.4.** Let  $a, b$  be positive numbers with  $ab \leq \frac{1}{2}$ . Assume  $c > 0, k \in \mathbb{Z}$  such that  $k \leq \frac{3-2ab+bc}{2ab-2}$  and  $g$  is the function defined on  $\mathbb{R}$  by

$$g(x) := \begin{cases} x^2 & (k + 2) a - \frac{2}{b} \leq x \leq ka + \frac{1}{b}, \\ 0 & o.w. \end{cases}$$

Using the inequality  $k \leq \frac{3-2ab+bc}{2ab-2}$ , we have

$$\begin{aligned} \Lambda_g(x) &= x^2 \left( x + a - \frac{1}{b} \right)^2 - (x + a)^2 \left( x - \frac{1}{b} \right)^2 \\ &= \frac{a}{b} \left[ 2x^2 + 2 \left( a - \frac{1}{b} \right) x - \frac{a}{b} \right] \\ &\geq \frac{a}{b} \left[ 2 \left( a - \frac{1}{b} \right) x - \frac{a}{b} \right] \\ &> \frac{a}{b} \left[ 2 \left( a - \frac{1}{b} \right) (k + 1) a - \frac{a}{b} \right] \\ &\geq \frac{ca^2}{b} \end{aligned}$$

for any  $x \in (ka, (k + 1) a)$ . Thus

$$\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| \geq \frac{ca^2}{b} > 0.$$

Then Theorem 2.2 implies that  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

**Example 3.5.** Let  $a, b$  be positive numbers with  $ab \leq \frac{1}{2}$ . Assume  $k \in \mathbb{Z}$  and  $g$  is the function defined on  $\mathbb{R}$  by

$$g(x) := \begin{cases} e^{-x} & x \in \left[ ka - \frac{1}{b}, (k + 1) a - \frac{1}{b} \right) \cup [ka, (k + 1) a), \\ e^x & x \in \left[ (k + 1) a - \frac{1}{b}, (k + 2) a - \frac{1}{b} \right) \cup [(k + 1) a, (k + 2) a), \\ 0 & o.w. \end{cases}$$

Then  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame. Indeed, for any  $x \in (ka, (k + 1)a)$  we have

$$\begin{aligned} |\Lambda_g(x)| &= \left| e^{-x} e^{(x+a-\frac{1}{b})} - e^{(x+a)} e^{-(x-\frac{1}{b})} \right| \\ &= \left| e^{(a-\frac{1}{b})} - e^{(a+\frac{1}{b})} \right|. \end{aligned}$$

Now we have

$$\inf_{x \in (ka, (k+1)a)} |\Lambda_g(x)| = \left| e^{(a-\frac{1}{b})} - e^{(a+\frac{1}{b})} \right| > 0.$$

By Theorem 2.2, the result obtains.

**Example 3.6.** Let  $a, b$  and  $c$  be positive numbers and  $k \in \mathbb{Z}$  with  $\frac{1+bc}{k} < ab \leq \frac{1}{2}$ . Assume  $g$  is the function defined on  $\mathbb{R}$  by

$$g(x) := \begin{cases} \ln(x+1) & x \in (ka - \frac{1}{b}, (k+2)a), \\ h(x) & x \in [(k+2)a - \frac{2}{b}, ka - \frac{1}{b}), \\ k(x) & x \in ((k+2)a, ka + \frac{1}{b}), \\ 0 & o.w. \end{cases},$$

where  $h$  and  $k$  are nonnegative continuous function which make  $g$  into a continuous function on  $\mathbb{R}$ . It is easy to show that  $g \in C_c^+(\mathbb{R})$  with  $supp g \subseteq [(k+2)a - \frac{2}{b}, ka + \frac{1}{b}]$  and  $g$  is differentiable on  $(ka - \frac{1}{b}, (k+2)a)$  with positive decreasing derivative  $g'$ . Thus by Theorem 2.8,  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame.

**Example 3.7.** Let  $a, b$  be positive numbers with  $ab \leq 1$ . Assume  $g_1, g_2, g_3$  and  $g_4$  are the functions defined on  $\mathbb{R}$  by

$$g_1(x) := \begin{cases} \sin x & (k+1)a - \frac{1}{b} \leq x \leq ka + \frac{1}{b}, \\ 0 & o.w. \end{cases},$$

where  $k \in \mathbb{Z} \cap (\frac{n\pi}{a}, \frac{(n+1)\pi}{a} - 1)$ ,

$$g_2(x) := \begin{cases} \cos x & (k+1)a - \frac{1}{b} \leq x \leq ka + \frac{1}{b}, \\ 0 & o.w. \end{cases},$$

where  $k \in \mathbb{Z} \cap (\frac{(2n+1)\pi}{2a}, \frac{(2n+3)\pi}{2a} - 1)$ ,

$$g_3(x) := \begin{cases} cx^2 + dx + e & (k+1)a - \frac{1}{b} \leq x \leq ka + \frac{1}{b}, \\ 0 & o.w. \end{cases},$$

where  $c \neq 0$  and  $d^2 < 4ec$ ,

$$g_4(x) := \begin{cases} e^x & (k+1)a - \frac{1}{b} \leq x \leq ka + \frac{1}{b}, \\ 0 & \text{o.w.} \end{cases}.$$

For  $i \in \{1, 2, 3, 4\}$  we have

$$\inf_{(ka, (k+1)a)} |g_i(x)| > 0.$$

Thus by Theorem 2.9,  $\{E_{mb}T_{na}g_i\}_{m,n \in \mathbb{Z}}$  is a Gabor frame for  $i \in \{1, 2, 3, 4\}$ .

## References

- [1] A.A. Arefijamaal and E. Zekae. Signal processing by alternate dual Gabor frames. *Appl. Comput. Harmon. Anal.*, 35: 535–540, 2013.
- [2] O. Christensen. *An Introduction to Frames and Riesz Bases*. Birkhäuser., Boston, Basel, Berlin, 2002.
- [3] R.J. Duffin and A.C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72: 341–366, 1952.
- [4] H.G. Feichtinger and T. Strohmer, Eds. *Gabor Analysis and Algorithms-Theory and Applications*. Birkhäuser., Boston, 1998.
- [5] K. Gröchenig. *Aspects of Gabor analysis on locally compact Abelian groups, Gabor analysis and Algorithms*. ANHA, Birkhäuser., Boston MA, 211–231, 1998.
- [6] C. Heil, D. Walnut. Continuous and discrete wavelet transform. *SIAM Rev.*, 31: 628–666, 1969.
- [7] A.J.E.M. Janssen. *The duality condition for Weyl-Heisenberg frames*. In Gabor analysis: theory and application (eds. H.G. Feichtinger and T. Strhmer). Birkhäuser., Boston, 1998.
- [8] A. Ron, Z. Shen. Weyl-Heisenberg systems and Riesz bases in  $L_2(\mathbb{R}^d)$ . *Duke. Math. J.*, 89: 237–282, 1997.
- [9] R. Young. *An Introduction to Nonharmonic Fourier Series*. Academic Press., New York, 1980.