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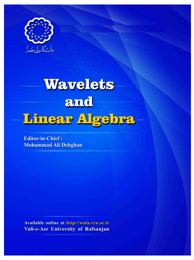

An equivalent condition for linear preservers of multivariate group majorization on matrices

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Abstract

T. Ando characterized linear preservers of majorization in [Linear Algebra Appl. 118 (1989) 163-248]. In this note, we present a method to state a simple proof of Ando's theorem. By using this method, we state an equivalent condition for matrix representations of linear preservers of G-majorization on matrices, where G is a finite subgroup of orthogonal group $O(\mathbb{R}^n)$. Moreover, we introduce reflective majorization and characterize all its linear preservers.

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1. Introduction

Let M_n be the set of all $n \times n$ real matrices, $\mathbb{R}^n$ be the set of all $n \times 1$ (column) real vectors, $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, $e = (1, \ldots, 1)^t$, $\mathbb{P}_n$ be the set of all $n \times n$ permutation matrices and J_n be the $n \times n$ matrix with all entries equal to one.

For $x, y \in \mathbb{R}^n$, we say that y majorizes x and write $x \prec y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \leqslant \sum_{i=1}^{k} x_i^{\downarrow}$$

for $k = 1, \ldots, n-1$ and equality holds for $k = n$, where $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$ is arrangement of x in non-increasing order. Notation $x \sim y$ means that $x \prec y$ and $y \prec x$. It is easy to see that $x \sim y$ if and only if there exists $P \in \mathbb{P}_n$ such that $x = Py$. We say that a linear operator $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ preservers majorization, if $Ax \prec Ay$ whenever $x \prec y$. The following theorem has an essential role to characterize linear preservers of majorization, see [1].

Theorem 1.1. *[1, Theorem 2.6] Let A be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then the following conditions are mutually equivalent:*

(i) *A preserves majorization.*

(ii) *$Ax \sim Ay$ whenever $x \sim y$.*

(iii) *For any permutation matrix $\Pi \in M_n$ there exists a permutation matrix $\widehat{\Pi} \in M_m$ such that $\widehat{\Pi}A = A\Pi$.*

The following theorem characterizes all linear preservers of majorization.

Theorem 1.2. *[1, Corollary 2.7] Any linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ preserving majorization has one of the following forms:*
(a) $A = \boldsymbol{a}e^t$ for some $\boldsymbol{a} \in \mathbb{R}^n$.
(b) $A = \alpha\Pi + \beta J_n$ for some $\alpha, \beta \in \mathbb{R}$ and $\Pi \in \mathbb{P}_n$.

A matrix $D \in M_n$ is called doubly stochastic if $De = e$ and $D^t e = e$. We know that $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix D. Birkhoff theorem [3, Theorem II.2.3] says that the set of all $n \times n$ doubly stochastic matrices is the convex hull of $\mathbb{P}_n$. On the other word, $x \prec y$ if and only if $x \in \operatorname{conv}\{Px : P \in \mathbb{P}_n\}$. By replacing $\mathbb{P}_n$ with any subgroup of orthogonal group $O(\mathbb{R}^n)$, we can define a new concept of majorization on $\mathbb{R}^n$ which is called group majorization induced by G. More details and examples of group majorization available in [9].

Definition 1.3. Let V be a finite dimensional inner product space and G be a subgroup of orthogonal group $O(V)$. We say that x is group majorized by y, write $x \prec_G y$, if $x \in \operatorname{conv}\{gy : g \in G\}$.

In section 2, we present a method to state a simple proof of Theorem 1.1 and by using this method, we state an equivalent condition for matrix representations of linear preservers $T : M_{n,m} \to M_{n,m}$ of G-majorizations, where G is a finite subgroup of $O(\mathbb{R}^n)$. Also, we improve some known results on matrix majorizations. In section 3, a new concept of majorization on $\mathbb{R}^2$ will be introduced and extended for $2 \times m$ matrices. Then we will characterize all its linear preservers.

2. Multivariate group majorization and its linear preservers

In this section, we present a method which has an essential role to characterize linear preservers of various types of majorizations. In the following theorem, we state our method to prove $(ii) \to (iii)$ of Theorem 1.1. Note that the cases $(i) \to (ii)$ and $(iii) \to (i)$ are obvious.

Theorem 2.1. *Let A be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. If A preserves $\sim$, then for any permutation matrix $\Pi \in M_n$ there exists a permutation matrix $\widehat{\Pi} \in M_m$ such that $\widehat{\Pi} A = A\Pi$.*

Proof. Let $\Pi \in \mathbb{P}_n$ be arbitrary. We define $\Delta(A,\Pi) := \min_{\widehat{\Pi}\in\mathbb{P}_m} \min\{\|(\widehat{\Pi}A - A\Pi)e_i\|_2 : (\widehat{\Pi}A - A\Pi)e_i \neq 0, i = 1,\dots,n\}$ and $\Delta(A,\Pi) = 0$ when $\widehat{\Pi}A = A\Pi$ for some $\widehat{\Pi} \in \mathbb{P}_m$.
On the contrary let $\Delta(A,\Pi) \neq 0$. Suppose that $x = \sum_{i=1}^n \lambda^{i-1} e_i$ where $\lambda \in \left(0, \frac{\Delta(A,\Pi)}{2n\|A\|_2}\right)$. Since A preserves $\sim$ and $\Pi x \sim x$, there exists $\widehat{\Pi} \in \mathbb{P}_m$ such that $\widehat{\Pi}Ax = A\Pi x$. hence

$$\sum_{i=1}^{n} \lambda^{i-1}(\widehat{\Pi}A - A\Pi)e_i = 0. \tag{2.1}$$

Since $\Delta(A,\Pi) \neq 0$, there exists i such that $(\widehat{\Pi}A - A\Pi)e_i \neq 0$. Let i be the first integer with this property. By equation (2.1), $(\widehat{\Pi}A - A\Pi)e_i = \lambda \sum_{j=i+1}^n \lambda^{j-i-1}(\widehat{\Pi}A - A\Pi)e_j$. So $\Delta(A,\Pi) \leq \|(\widehat{\Pi}A - A\Pi)e_i\|_2 \leq \lambda \sum_{j=i+1}^n \lambda^{j-i-1}\|(\widehat{\Pi}A - A\Pi)e_j\|_2 \leq 2n\lambda\|A\|_2$. Then $\lambda \geq \dfrac{\Delta(A,\Pi)}{2n\|A\|_2}$, a contradiction. Therefore, $\Delta(A,\Pi) = 0$. □

In the following, we talk about matrix majorization and define a class of group majorization on $M_{n,m}$. By our method, we are able to find an equivalent condition for linear preservers of group majorization on $M_{n,m}$, see Theorem 2.7.
The concept of matrix majorization is defined by multivariate majorization [2] or directional majorization [6] as follows:

Definition 2.2. For $X, Y \in M_{n,m}$, we say that X is multivariate majorized by Y and write $X \prec_m Y$ if there exists doubly stochastic matrix $D \in M_n$ such that $X = DY$.

Definition 2.3. For $X, Y \in M_{n,m}$, we say that X is directional majorized by Y and write $X \prec_d Y$ if $Xv \prec Yv$ for every $v \in \mathbb{R}^m$.

It is clear that $X \prec_m Y$ implies that $X \prec_d Y$. In the following theorem, by using our method (as in the proof of Theorem 2.1), we show that $X \sim_d Y$ ($X \prec_d Y$ and $Y \prec_d X$) if and only if $X \sim_m Y$ ($X \prec_m Y$ and $Y \prec_m X$). For $X, Y \in M_{n,m}$, we define $\Gamma(X,Y) = 0$ when $X = Y = \alpha J_{n,m}$ and otherwise

$$\Gamma(X,Y) = \min\{|x_{ij} - y_{st}| : x_{ij} \neq y_{st}, 1 \leq i,s \leq n, 1 \leq j,t \leq m\}.$$

Theorem 2.4. *Let $X, Y \in M_{n,m}$. The following statements are equivalent:*
(i) $X \sim_d Y$.
(ii) $X = PY$ for some $P \in \mathbb{P}_n$.
(iii) $X \sim_m Y$.

Proof. $(ii) \to (iii)$ and $(iii) \to (i)$ are obvious. Now, we prove $(i) \to (ii)$. If $X = Y$ the assertion holds. Otherwise, let $\lambda \in \left(0, \frac{\Gamma(X,Y)}{n(\|X\|_2 + \|Y\|_2)}\right)$ and $v = e_1 + \lambda e_2 + \cdots + \lambda^{n-1} e_n$. By the hypothesis, $Xv \sim Yv$ and there exists $P \in \mathbb{P}_n$ such that $Xv = PYv$. So

$$x_1 + \lambda x_2 + \cdots + \lambda^{n-1} x_n = Py_1 + \lambda Py_2 + \cdots + \lambda^{n-1} Py_n,$$

where x_i, y_i are the i^{th} columns of X, Y, respectively. With the same argument as in the proof of Theorem 2.1 we have

$$\|x_1 - Py_1\|_2 = \lambda\|x_2 - Py_2\|_2 + \cdots + \lambda^{n-1}\|x_n - Py_n\|_2 \leq (n-1)\lambda(\|X\|_2 + \|Y\|_2).$$

If $x_1 \neq Py_1$, then $\lambda \geq \frac{\Gamma(X,Y)}{(n-1)(\|X\|_2 + \|Y\|_2)}$ a contradiction. Then $x_1 = Py_1$ and by the same argument we obtain $x_i = Py_i$ for $i = 2, \ldots, m$. Therefore, $X = PY$. □

A class of group majorizations of matrices can be defined as follows.

Definition 2.5. For $X, Y \in M_{n,m}$, X is said to be multivariate group majorized by Y (written as $X \prec_{mg} Y$), if $X = \sum_{i=1}^{k} c_i g_i Y$ where $g_i \in G$, $c_i \geq 0$, $\sum_{i=1}^{k} c_i = 1$ and G is a subgroup of $O(\mathbb{R}^n)$.

By using the method as in the proof of Theorem 2.1, we prove an equivalent condition for linear preservers of multivariate group majorization. To do this, we need some preliminaries. For every $A = (a_{ij}) \in M_{n,m}$, we associate the vector $\mathrm{vec}(A) \in \mathbb{R}^{nm}$ defined by

$$\mathrm{vec}(A) = [a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{1m}, \ldots, a_{nm}]^t.$$

Let $\mathcal{B} = \{E_{11}, \ldots, E_{n1}, E_{12}, \ldots, E_{n2}, \ldots, E_{1m}, \ldots, E_{nm}\}$ be the standard basis of $M_{n,m}$ and $[T]_{\mathcal{B}}$ be representation of T with respect to $\mathcal{B}$. Then

$$[T]_{\mathcal{B}} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{pmatrix}, \tag{2.2}$$

where each $B_{ij} \in M_n$ and $\mathrm{vec}(T(X)) = [T]_{\mathcal{B}}(\mathrm{vec}(X))$. Let $A \in M_{n,m}$, $X \in M_{m,p}$, $B \in M_{p,q}$ and $C \in M_{n,q}$. By [5, Lemma 4.3.1], $AXB = C$ if and only if

$$\mathrm{vec}(C) = \mathrm{vec}(AXB) = (B^t \otimes A)\mathrm{vec}(X). \tag{2.3}$$

To verify linear preservers of multivariate group majorization, we deal with $x \sim_{mg} y$ means $x \prec_{mg} y$ and $y \prec_{mg} x$. The following theorem gives an equivalent condition for $\sim_{mg}$.

Theorem 2.6. *Let $X, Y \in M_{n,m}$. Then $X \sim_{mg} Y$ if and only if $X = gY$ for some $g \in G$.*

Proof. By the definition of multivariate group majorization, $X \prec_{mg} Y$ means that $X = \sum_{t=1}^{k} \alpha_t g_t Y$. Since $g_t \in O(\mathbb{R}^n)$,

$$\|X\|_2 = \| \sum_{t=1}^{k} \alpha_t g_t Y\|_2 \leq \sum_{t=1}^{k} \alpha_t \|g_t Y\|_2 = \sum_{t=1}^{k} \alpha_t \|Y\|_2 = \|Y\|_2. \tag{2.4}$$

On the other hand, $Y \prec_{mg} X$ and then $\|Y\| \leq \|X\|$. Hence, equality holds in (2.4). If $\alpha_{t'} \neq 0$ for some $1 \leq t' \leq k$, then

$$\|\alpha_{t'} g_{t'} Y + Z\|_2 = \|\alpha_{t'} g_{t'} Y\|_2 + \|Z\|_2,$$

where $Z = \sum_{t=1, t\neq t'}^{k} \alpha_t g_t Y$. Since equality holds in triangle inequality(cauchy-schwarz inequality), $Z = \lambda \alpha_{t'} g_{t'} Y$ for some $\lambda \in \mathbb{R}$. Therefore, $X = (1+\lambda)\alpha_{t'} g_{t'} Y$. Since $\|X\|_2 = \|Y\|_2$, $(1+\lambda)\alpha_{t'} = 1$. □

The following theorem states an equivalent condition for matrix representations of linear operator $T : M_{n,m} \to M_{n,m}$ which preserves multivariate group majorization, where G is a finite subgroup of $O(\mathbb{R}^n)$.

Theorem 2.7. *Let G be a finite subgroup of $O(\mathbb{R}^n)$, $T : M_{n,m} \to M_{n,m}$ be a linear operator and $[T]_{\mathcal{B}}$ be as (3.1). Then T preserves $\sim_{mg}$ if and only if for every $g \in G$ there exists a matrix $\widehat{g} \in G$ such that $\widehat{g} B_{ij} = B_{ij} g$ for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$.*

Proof. For necessity, fix $g \in G$. We define $\Delta_{min}(T, g) := \min_{g' \in G} \min\{\|(g' B_{ij} - B_{ij} g) e_k\|_2 : \|(g' B_{ij} - B_{ij} g) e_k\|_2 \neq 0, 1 \leq k \leq n, 1 \leq i \leq n, 1 \leq j \leq m\}$ and $\Delta_{\min}(T, g) = 0$ when there exists $g' \in G$ such that $g' B_{ij} - B_{ij} g = 0$ for every $1 \leq i \leq n, 1 \leq j \leq m$.
On the contrary let $\Delta_{\min}(T, g) \neq 0$. Define

$$X = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda^{n(j-1)+(i-1)} E_{ij} \in M_{n,m}, \quad \text{where} \quad \lambda \in \left(0, \frac{\Delta_{\min}(T, g)}{2mn\|T\|_2}\right).$$

Since T preserves $\sim_{mg}$, for every $g \in G$ there exists $\widehat{g} \in G$ such that $T(gX) = \widehat{g} T(X)$. By (2.3),

$$[T]_{\mathcal{B}} (I_m \otimes g) \mathrm{vec}(X) = (I_m \otimes \widehat{g}) [T]_{\mathcal{B}} \mathrm{vec}(X). \tag{2.5}$$

Now, by the definition of X,

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda^{n(j-1)+(i-1)} \left([T]_{\mathcal{B}} (I_m \otimes g) - (I_m \otimes \widehat{g}) [T]_{\mathcal{B}}\right) \mathrm{vec}(E_{ij}) = 0. \tag{2.6}$$

Since $\Delta_{\min}(T, g) \neq 0$, there exists i, j such that $(g' B_{ij} - B_{ij} g) e_k \neq 0$ and this means

$$\left([T]_{\mathcal{B}} (I_m \otimes g) - (I_m \otimes \widehat{g}) [T]_{\mathcal{B}}\right) \mathrm{vec}(E_{ij}) \neq 0. \tag{2.7}$$

Let E_{ts} be the first element of ordered basis $\mathcal{B}$ such that (2.7) holds. By the definition of $\Delta_{\min}(T, g)$ and (2.6),

$$\begin{aligned} \Delta_{\min}(T, g) &\leq \| \left([T]_{\mathcal{B}} (I_m \otimes g) - (I_m \otimes \widehat{g}) [T]_{\mathcal{B}}\right) \mathrm{vec}(E_{ts})\|_2 \\ &\leq \sum_{i=t+1}^{n} \lambda^{i-t} \| \left((I_m \otimes \widehat{g}) [T]_{\mathcal{B}} - [T]_{\mathcal{B}} (I_m \otimes g)\right) \mathrm{vec}(E_{is})\|_2 \\ &+ \sum_{j=s+1}^{m} \sum_{i=1}^{n} \lambda^{n(j-s)+(i-t)} \| \left((I_m \otimes \widehat{g}) [T]_{\mathcal{B}} - [T]_{\mathcal{B}} (I_m \otimes g)\right) \mathrm{vec}(E_{ij})\|_2 \end{aligned}$$

Therefore $\Delta_{\min}(T,g) \le 2mn\lambda\|T\|_2$ a contradiction. Hence $\Delta_{\min}(T,g) = 0$ and by the definition of $\Delta_{\min}(T,g)$ the assertion holds.
conversely, for every $g \in G$ there exists a matrix $\widehat{g} \in G$ such that $\widehat{g}B_{ij} = B_{ij}g$, for each $i = 1,\dots,n$ and $j = 1,\dots,m$. Then (2.5) holds and T is a linear preserver of $\sim_{mg}$. □

Now by using this argument, we are able to improve some results of [6]. For that, we need to prove two lemmas.

Lemma 2.8. *Let $n \ge 3$. If there exists a permutation $\Pi \in \mathbb{P}_n$ such that $\Pi P = P\Pi$ for all $P \in \mathbb{P}_n$, then $\Pi = I_n$.*

Proof. Assume if possible $\Pi \ne I_n$. Then there exists $i \ne j$ such that $\Pi e_i = e_j$. Since $n \ge 3$, there exists $k \in \{1,\dots,n\} \setminus \{i,j\}$. Choose $P \in \mathbb{P}_n$ such that $Pe_i = e_i$ and $Pe_j = e_k$. Then $\Pi P e_i = e_j$ and $P\Pi e_i = e_k$ and hence $\Pi P \ne P\Pi$, a contradiction. □

Lemma 2.9. *Let T_1, T_2 be $n\times n$ matrices and for every $\Pi \in \mathbb{P}_n$ there exists $\widehat{\Pi} \in \mathbb{P}_n$ such that $\widehat{\Pi}T_i = T_i\Pi$, $i = 1,2$. Then T_1 and T_2 have the same structure, which means that one of the following statements hold.*
(a) $T_1 = \boldsymbol{a}e^t, T_2 = \boldsymbol{b}e^t$ for some $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^n$.
(b) There exists $P \in \mathbb{P}_n$ such that $T_1 = \alpha_1 P + \beta_1 J_n$ and $T_2 = \alpha_2 P + \beta_2 J_n$ for some $\alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{R}$.

Proof. By Theorem 1.1, T_1 and T_2 are linear preservers of majorization and By Theorem 1.2, T_1 and T_2 should satisfy (a) or (b). If T_1 and T_2 satisfy (a), the result holds. Without loss of generality, assume that $T_1 = \alpha_1 P + \beta_1 J_n$. By assumptions, $P\Pi = \widehat{\Pi}P$ and $\widehat{\Pi}T_2 = T_2\Pi$. This implies that for every $\Pi \in \mathbb{P}_n$

$$P\Pi P^t T_2 = T_2\Pi. \tag{2.8}$$

Let T_2 satisfies (a). Then $T_2 = \mathbf{a}e^t$, for some $\mathbf{a} \in \mathbb{R}^n$. Since T_2 has the same columns, $T_2\Pi = T_2$, for every $\Pi \in \mathbb{P}_n$. Hence by (2.8), for every $\Pi \in \mathbb{P}_n$, $\Pi T_2 = T_2$. This means that T_2 has the same rows and hence $T_2 = \beta J_n$, the result holds. Now, let T_2 satisfies (b). We consider two cases: Let $n = 2$. Since $\mathbb{P}_2 = \{I_2, J_2 - I_2\}$, $T_2 = \alpha I_2 + \beta J_2 = -\alpha(J_2 - I_2) + (\alpha+\beta)J_2$ and the result holds. Now, let $n \ge 3$ and $T_2 = \alpha_2 Q + \beta_2 J_n$ for some $\alpha_2 \ne 0$ and $Q \in \mathbb{P}_n$. By equation (2.8), we obtain that $P\Pi P^t Q = Q\Pi$, and hence $\Pi(P^tQ) = (P^tQ)\Pi$ for every $\Pi \in \mathbb{P}_n$. Then the permutation P^tQ commutes with all permutations $\Pi \in \mathbb{P}_n$. Since $n \ge 3$, by Lemma 2.8, $P^tQ = I_n$ and hence $T_2(x) = \alpha_2 Px + \beta_2 J_n x$. □

In the following, we will prove [6, Theorem 2] as a result of Theorem 2.7.

Corollary 2.10. *Let T be a linear operator on $M_{n,m}$. The following are equivalent:*
(i) T preserves multivariate majorization.
(ii) T preserves directional majorization.
(iii) $TX \prec_d TY$ whenever $X \prec_m Y$.
(iv) $TX \sim_d TY$ whenever $X \sim_d Y$.
(v) $TX \sim_m TY$ whenever $X \sim_m Y$.
(vi) One of the following holds :
(a) There exist $R, S \in M_m$ and $P \in \mathbb{P}_n$ such that $T(X) = PXR + J_n XS$.
(b) There exist $A_1,\dots,A_m \in M_{n,m}$ such that $T(X) = \sum_{j=1}^m tr(x_j)A_j$.

Proof. By the definition of multivariate majorization and directional majorization, $(i) \rightarrow (ii)$ and $(ii) \rightarrow (iii)$ are clear. Theorem 2.4 implies $(iii) \rightarrow (iv)$ and $(iv) \rightarrow (v)$.
$(v) \longrightarrow (vi)$ By Theorem 2.7 for every $P \in \mathbb{P}_n$ there exist $Q \in \mathbb{P}_n$ such that $B_{ij}P = QB_{ij}$ for each i, j. By Lemma 2.9 two cases may occur. First, let there exists $\Pi \in \mathbb{P}_n$ such that $B_{ij} = \alpha_{ij}\Pi + \beta_{ij}J_n$, $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$. Then $[T]_{\mathcal{B}} = A \otimes \Pi + B \otimes J_n$, where $A = (\alpha_{ij}), B = (\beta_{ij})$. By equation 2.3, $T(X) = \Pi XR + J_n XS$, where $R = A^t, S = B^t$. Now, let there exist $\mathbf{b}_{ij} \in \mathbb{R}^n$ such that $B_{ij} = \mathbf{b}_{ij}e^t \in M_n$. So $T(X) = \left(\sum_{j=1}^m \mathbf{b}_{1j}e^t x_j | \cdots | \sum_{j=1}^m \mathbf{b}_{mj}e^t x_j\right)$ and hence $T(X) = \sum_{j=1}^m tr(x_j)A_j$, where $A_j := \left(\mathbf{b}_{1j} | \cdots | \mathbf{b}_{mj}\right)$.
$(vi) \longrightarrow (i)$ Let $P \in \mathbb{P}_n$. Then for every $Q \in \mathbb{P}_n$, there exists $Q' \in \mathbb{P}_n$ such that $PQ = Q'P$ and $J_n Q = Q' J_n$. Therefore, for every doubly stochastic matrix D, There exists a doubly stochastic matrix D' such that $T(DX) = D'T(X)$ and hence $(vi)(a)$ implies (i). Also, $tr(x_j) = tr(Dx_j)$, for every doubly stochastic matrix D. Then $(vi)(b)$ implies (i). □

By Theorem 2.7 and the above argument, we are able to prove [4, Theorem 4.3], [7, Theorem 2.5] and [8, Theorems 3.4, 3.6].

3. Reflective majorization

In this section, we define a class of group majorization on $O(\mathbb{R}^2)$ and characterize its linear preservers. Let $B_\theta \in M_2$ be the reflection about the line passing through the origin that forms an angle $\frac{\theta}{2}$ with the positive x-axis in $O(\mathbb{R}^2)$. In other words

$$B_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}. \tag{3.1}$$

It is easy to see that $B_\theta \in O(\mathbb{R}^2)$ and $G_\theta = \{B_\theta, I\}$ is a subgroup of $O(\mathbb{R}^2)$.

Definition 3.1. The group majorization induced by G_θ is called reflective majorization associated to θ and denoted by $\prec_\theta$. On the other word, $x \prec_\theta y$ means that $x = \lambda y + (1-\lambda)B_\theta y$ for some $\lambda \in (0, 1)$.

The reflective majorization associated to $\dfrac{\pi}{2}$ is the same as majorization. Figure 1 shows the difference between majorization and a reflective majorization.

In the following theorem, we will characterize linear preservers of reflective majorization. The linear preservers of $\prec_{\frac{\pi}{2}}$ must be as same as Theorem 1.2.

Theorem 3.2. *Let $\theta \neq k\pi$. If A is a linear preserver of $\prec_\theta$, then A has one of the following form:*

(i) $A = \begin{pmatrix} \alpha\cos(\frac{\theta}{2}) & \alpha\sin(\frac{\theta}{2}) \\ \beta\cos(\frac{\theta}{2}) & \beta\sin(\frac{\theta}{2}) \end{pmatrix}$ *for some* $\alpha, \beta \in \mathbb{R}$.

(ii) $A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha - 2\beta\cot(\theta) \end{pmatrix}$ *for some* $\alpha, \beta \in \mathbb{R}$.

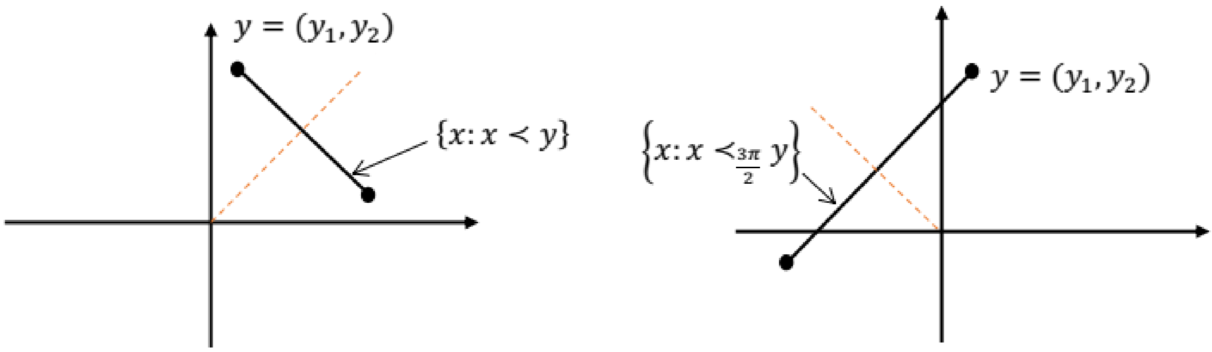

Figure 1: majorization and reflective majorization associated to $\frac{3\pi}{2}$

Proof. By Theorem 2.7, for every $g \in G_\theta$ there exists a matrix $\widehat{g} \in G_\theta$ such that $\widehat{g}A = Ag$. If $g = I$, we can choose $\widehat{g} = I$. Now assume that

$$g = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $G_\theta = \{B_\theta, I\}$, two cases can be occurred:
Case1) Assume that $\widehat{g} = I$. So

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It means that $(a, b), (c, d)$ are placed on the line passing through the origin that forms an angle $\frac{\theta}{2}$ with the positive x-axis. Therefore there exists α, β such that

$$(a \quad b) = \left(\alpha \cos\left(\frac{\theta}{2}\right) \quad \alpha \sin\left(\frac{\theta}{2}\right)\right), (c \quad d) = \left(\beta \cos\left(\frac{\theta}{2}\right) \quad \beta \sin\left(\frac{\theta}{2}\right)\right)$$

Case2) Now, let $B_\theta A = AB_\theta$. So

$$a\cos\theta + c\sin\theta = a\cos\theta + b\sin\theta \tag{3.2}$$
$$b\cos\theta + d\sin\theta = a\sin\theta - b\cos\theta \tag{3.3}$$
$$a\sin\theta - c\cos\theta = c\cos\theta + d\sin\theta \tag{3.4}$$
$$b\sin\theta - d\cos\theta = c\sin\theta - d\cos\theta \tag{3.5}$$

Since $\theta \neq k\pi$, equation (3.2) implies that $b = c$ and equation (3.4) implies that $d = a - 2c\cot\theta$. Therefore A has form (ii). □

For $\theta = (2k+1)\pi$, we have $B_{(2k+1)\pi} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and the linear preservers of $\prec_{(2k+1)\pi}$ has the form

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix} \; or \; A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Also the linear preservers of reflective majorization associated to $2k\pi$ has the form

$$A = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix} \; or \; A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

By choosing $\beta = \gamma \sin(\theta)$ in the Theorem 3.2 (ii), we know that a linear preservers of reflective majorization associated to θ has one of the following forms:

$$A = \begin{pmatrix} \alpha\cos(\frac{\theta}{2}) & \alpha\sin(\frac{\theta}{2}) \\ \beta\cos(\frac{\theta}{2}) & \beta\sin(\frac{\theta}{2}) \end{pmatrix}, A = \alpha I + \beta \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & -2\cos\theta \end{pmatrix}, \alpha, \beta \in \mathbb{R}.$$

In the proof of Theorem 3.2, we see that if $A = AB_\theta$ then A has form (i) and if $B_\theta A = AB_\theta$ then A has form (ii). By this fact, we have the following theorem that is used to characterize linear preservers of matrix majorization associated to θ.

Theorem 3.3. *Let B_1, B_2 be linear preservers of reflective majorization associated to θ with the property that for every $g \in B_\theta$ there exist $\widehat{g} \in B_\theta$ such that $\widehat{g}B_1 = B_1 g$ and $\widehat{g}B_2 = B_2 g$. Then*

$$B_1 = \begin{pmatrix} \alpha_1\cos(\frac{\theta}{2}) & \alpha_1\sin(\frac{\theta}{2}) \\ \beta_1\cos(\frac{\theta}{2}) & \beta_1\sin(\frac{\theta}{2}) \end{pmatrix}, B_2 = \begin{pmatrix} \alpha_2\cos(\frac{\theta}{2}) & \alpha_2\sin(\frac{\theta}{2}) \\ \beta_2\cos(\frac{\theta}{2}) & \beta_2\sin(\frac{\theta}{2}) \end{pmatrix}$$

for some $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ or

$$B_1 = \lambda_1 I + \gamma_1 \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & -2\cos\theta \end{pmatrix}, B_2 = \lambda_2 I + \gamma_2 \begin{pmatrix} 0 & \sin\theta \\ \sin\theta & -2\cos\theta \end{pmatrix}$$

for some $\lambda_1, \gamma_1, \lambda_2, \gamma_2 \in \mathbb{R}$.

In the following, we will characterize linear preservers of multivariare reflective majorization associated to θ on $2 \times m$ matrices. This is an application of Theorem 2.7.

Theorem 3.4. *Let T be an operator on $M_{2,m}$. Then T is a linear preserver of reflective matrix majorization associated to θ if and only if it has one of the following form:*

(i) $T(X) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ 0 & 0 \end{pmatrix} XA + \begin{pmatrix} 0 & 0 \\ \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \end{pmatrix} XB$, *for some $A, B \in M_m$.*

(ii) $T(X) = XC + \begin{pmatrix} 0 & \sin(\theta) \\ \sin(\theta) & -2\cos(\theta) \end{pmatrix} XD$, *for some $C, D \in M_m$.*

Proof. Let $[T]_{\mathcal{B}}$ be the representation of T with respect to standard basis of $M_{2,m}$. Then $[T]_{\mathcal{B}}$ is the block matrix as in (3.1) and each B_{ij} is 2×2 matrix. Theorem 2.7 implies that for every $g \in B_\theta$ there exists $\widehat{g} \in B_\theta$ such that $B_\theta g = \widehat{g}B_\theta$ for every $i, j = 1, \ldots, m$. By Theorem 3.3 two cases can be occurred.

Case1) If $B_{ij} = \begin{pmatrix} \alpha_{ij}\cos(\frac{\theta}{2}) & \alpha_{ij}\sin(\frac{\theta}{2}) \\ \beta_{ij}\cos(\frac{\theta}{2}) & \beta_{ij}\sin(\frac{\theta}{2}) \end{pmatrix}$ for every $i, j = 1, \ldots, m$, then

$$[T]_{\mathcal{B}} = \begin{pmatrix} C_{11} & \cdots & C_{1m} \\ \vdots & & \vdots \\ C_{m1} & \cdots & C_{mm} \end{pmatrix} + \begin{pmatrix} D_{11} & \cdots & D_{1m} \\ \vdots & & \vdots \\ D_{m1} & \cdots & D_{mm} \end{pmatrix},$$

where $C_{ij} = \alpha_{ij}\begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ 0 & 0 \end{pmatrix}$ and $D_{ij} = \alpha_{ij}\begin{pmatrix} 0 & 0 \\ \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \end{pmatrix}$. Therefore

$$\mathrm{vec}(T(X)) = \left(A \otimes \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ 0 & 0 \end{pmatrix} + B \otimes \begin{pmatrix} 0 & 0 \\ \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \end{pmatrix}\right) \mathrm{vec}(X),$$

where $A = (\alpha_{ij}), B = (\beta_{ij}) \in M_m$. By equation (2.3), we have

$$T(X) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ 0 & 0 \end{pmatrix} XA^t + \begin{pmatrix} 0 & 0 \\ \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \end{pmatrix} XB^t.$$

Case2) If $B_{ij} = \lambda_{ij} I + \gamma_{ij}\begin{pmatrix} 0 & \sin\theta \\ \sin\theta & -2\cos\theta \end{pmatrix}$ for every $i, j = 1, \ldots, m$, then

$$\mathrm{vec}(T(X)) = \left(\Lambda \otimes I_2 + \Gamma \otimes \begin{pmatrix} 0 & \sin(\theta) \\ \sin(\theta) & -2\cos(\theta) \end{pmatrix}\right)(\mathrm{vec}(X))$$

where $\Lambda = (\lambda_{ij}), \Gamma = (\gamma_{ij}) \in M_m$. By the same argument as in above, we have $T(X) = IXC + \begin{pmatrix} 0 & \sin(\theta) \\ \sin(\theta) & -2\cos(\theta) \end{pmatrix} XD$, where $C = \Lambda^t$ and $D = \Gamma^t$. □

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