Wavelets and Linear Algebra 10(2) (2023) 63- 69

Vali-e-Asr University of Rafsanjan

Wavelets and Linear Algebra

http://wala.vru.ac.ir

A study on the continuity of some classes of E-$\mathbb{Q}$-convex functions

Zohreh Heydarpour[a], Masoumeh Aghajani[b,*]

[a] *Department of Mathematics, Payame Noor University (PNU), P. O. Box 19395-4697, Tehran, Iran.*
[b] *Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, P.O. Box 16785-136, Tehran, Iran.*

Article Info

Article history:
Received 29 May 2023
Accepted 14 October 2023
Available online 22 November 2023
Communicated by Ali Armandnejad

Keywords:
Bernstein-Doetsch theorem, E-convex set, Approximately E-convex function, E-$\mathbb{Q}$-convex function, E-$\mathbb{Q}$-convex set.

2010 MSC:
26B25, 26A51, 39B72.

Abstract

As a generalization of convexity, E-convexity has been defined and studied in many publications. In this study, we recall the class of E-$\mathbb{Q}$-convex sets, E-$\mathbb{Q}$-convex and E-additive functions and proved some properties of E-$\mathbb{Q}$-convex functions. Also, we develop the classical theorems of Jensen and Bernstein-Doetsch on E-$\mathbb{Q}$-convex functions when vector spaces are over the rational numbers $\mathbb{Q}$.

*Corresponding author
Email addresses: zohreh.heydarpour@pnu.ac.ir (Zohreh Heydarpour), m.aghajani@sru.ac.ir (Masoumeh Aghajani)

http://doi.org/10.22072/wala.2023.2003060.1422

1. Introduction

The concept of convexity and $\mathbb{Q}$-convexity are important for various branches of mathematical sciences. A question that received much interest is the following one: under what conditions a $\mathbb{Q}$-convex function is continuous? In 1905, Jensen proved that every $\mathbb{Q}$-convex function $f : (a, b) \longrightarrow \mathbb{R}$ which is bounded on interval (a, b) is continuous. In 1915, F. Bernstein and Doetsch proved that every $\mathbb{Q}$-convex function $f : (a, b) \longrightarrow \mathbb{R}$ which is bounded above on some open subinterval of (a, b) is continuous. Results concerning various conditions for representation and continuity of $\mathbb{Q}$-convex functions and their generalizations have been obtained in a number of papers (see '[1, 3, 4, 6, 11]').

Let us fix our notation and terminology. As we know a subset U of a vector space X is $\mathbb{Q}$-convex if $\lambda x + (1 - \lambda)y \in U$ for each $x, y \in U$ and $\lambda \in (0, 1) \cap \mathbb{Q}$ also U is $\mathbb{Q}$-radial at a point $a \in U$ if for every $x \in X$ there exists a number $r_x > 0$ such that $a + rx \in U$ whenever $r \in \mathbb{Q} \bigcap (0, r_x)$. A subset B of a vector space X is said to be balanced if $\alpha B \subseteq B$ for every $\alpha \in \mathbb{F}$ with $| \alpha | \leq 1$. Also B is said to be symmetric if $B = -B$.

E-convexity is one of the generalizations of convexity, introduced by Youness '[16]'. However, as pointed out by Yang '[15]', some results and proofs in Youness '[16]' seems to be incorrect. Youness '[17]' also discussed optimality criteria for E-convex programming problems. Subsequently, some mathematicians have investigated various aspects of this concept and its generalization, and the reader can refer to the papers for more information see '[3, 4, 6, 7, 11, 12, 13, 17]'.

Let X be a vector space. A set $U \subseteq X$ is said to be an E-convex if there exists a map $E : X \to X$ such that $\lambda E(x) + (1 - \lambda)E(y) \in U$ for each $x, y \in U$ and $0 \leqslant \lambda \leqslant 1$. Clearly, every convex set is E-convex if E is the identity map. A function $\varphi : X \to \mathbb{R}$ is called an E-convex on a set $U \subset X$ if there exists a map $E : X \to X$ such that U is an E-convex, also for each $x, y \in U$ and $0 \leqslant \lambda \leqslant 1$, $\varphi(\lambda E(x) + (1 - \lambda)E(y)) \leqslant \lambda\varphi(E(x)) + (1 - \lambda)\varphi(E(y))$. It is clear that every convex function is E-convex if E is the identity map.

In the present paper, we recall the class of E-$\mathbb{Q}$-convex sets, E-$\mathbb{Q}$-convex and E-additive functions. Then, we extend Theorem 1.1 by M. Kuczma '[5]', for an E-additive function.

M. Kuczma '[5]' proved the following theorem.

Theorem 1.1. *Let X be a vector space over $\mathbb{Q}$, Y be a subspace of X and $S \subseteq X$ be a $\mathbb{Q}$-convex and $\mathbb{Q}$-radial at 0. If $\varphi : Y \to \mathbb{R}$ is an additive function with $\varphi |_{Y \cap S} \leqslant 1$, then there exists an additive function $\phi : X \to \mathbb{R}$ such that $\phi |_Y = \varphi$ and $\phi |_S \leqslant 1$.*

Recently, Mirzapour et al.'[10]' proved the classical theorems of Jensen, Bernstein-Doetsch, Blumberg-Sierpinski, and Ostrowski for E-$\mathbb{Q}$-convex function when vector spaces are over the real numbers $\mathbb{R}$.

Inspired by the above research, we extend such theorems over $\mathbb{Q}$ and prove them.

2. Main results

Throughout this paper, X is a vector space and all vector spaces are over the field $\mathbb{Q}$ of rational numbers. The notion of an E-$\mathbb{Q}$-convex is a natural generalization of that of an E-convex arising under the replacement of the field of scalars $\mathbb{R}$ by $\mathbb{Q}$.

Definition 2.1. A subset U of X is an E-$\mathbb{Q}$-convex if there exists a map $E : X \to X$ such that $\lambda E(x) + (1-\lambda)E(y) \in U$ for all $x, y \in U$ and every rational number $0 \leqslant \lambda \leqslant 1$.

Definition 2.2. A function $\varphi : X \to \mathbb{R}$ is said to be an E-additive if there exists a map $E : X \to X$ such that $\varphi(E(x) + E(y)) = \varphi(E(x)) + \varphi(E(y))$, for all $x, y \in X$.

Definition 2.3. A function $\varphi : X \to \mathbb{R}$ is said to be an E-$\mathbb{Q}$-convex on a set $U \subset X$ if there exists a map $E : X \to X$ such that U is an E-$\mathbb{Q}$-convex also for all $x, y \in U$ and every rational number $0 \leqslant \lambda \leqslant 1$, $\varphi(\lambda E(x) + (1-\lambda)E(y)) \leqslant \lambda\varphi(E(x)) + (1-\lambda)\varphi(E(y))$.

It is evident that each E-convex function is an E-$\mathbb{Q}$-convex function, but the converse is not true.

Example 2.4. Let $f, E : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = E(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 2 & x \notin \mathbb{Q}. \end{cases}$$

Then, f is E-$\mathbb{Q}$-convex on $\mathbb{R}$ but it is not E-convex.

Definition 2.5. '[5]' A subset A of X is a convex cone if for all $x, y \in A$ and the real number $\lambda > 0$, $\lambda x + y \in A$.

Lemma 2.6. *Let $E : X \to X$ be a map such that $E(K)$ is a convex cone of K for every subspace $K \subseteq X$ and let Y and Z be two subspaces of X with $Z = Y + \mathbb{Q}z_0$ for some $z_0 \in X \setminus Y$. If $S \subseteq X$ is a $\mathbb{Q}$-convex and $\mathbb{Q}$-radial at 0, $E(Z) \subseteq E(Y) + \mathbb{Q}z_0$ and $\varphi : Y \to \mathbb{R}$ is an E-additive function with $\varphi\,|_{Y\cap S} \leqslant 1$, then there exists an E-additive function $\phi : Z \to \mathbb{R}$ such that $\phi\,|_Y = \varphi$ and $\phi\,|_{E(Z)\cap S} \leqslant 1$.*

Proof. Consider the following sets $A, B \subset Y \times \mathbb{Q}$:

$$A := \left\{ (y, r) : y \in Y, r > 0, \frac{E(y) - z_0}{r} \in S \right\},$$

$$B := \left\{ (y, r) : y \in Y, r > 0, \frac{E(y) + z_0}{r} \in S \right\}.$$

Since S is $\mathbb{Q}$-radial at 0, there exists a rational number $r > 0$ such that $\frac{E(y) \pm z_0}{r} \in S$. Consequently $A, B \neq \emptyset$. Put

$$a = sup\{\varphi(E(y)) - r : (y, r) \in A\}$$

$$b = inf\{r - \varphi(E(y)) : (y, r) \in B\}.$$

We show that $a \leq b$ and therefore $-\infty < a \leq b < \infty$. Assume towards a contradiction that $a > b$. Then there exist $(y, r) \in A$ and $(y', s) \in B$ such that $\varphi(E(y)) - r > b$ and $\varphi(E(y)) - r > s - \varphi(E(y'))$. So $\varphi(E(y)) + \varphi(E(y')) > r + s$. On the other hand, we have

$$\frac{E(y) + E(y')}{r + s} = \frac{r}{r + s}\frac{E(y) - z_0}{r} + \frac{s}{r + s}\frac{E(y') + z_0}{s}.$$

Hence

$$\frac{E(y)+E(y')}{r+s} \in S.$$

Since $E(Y)$ is a convex cone of Y, we get $\dfrac{E(y)+E(y')}{r+s} \in Y \cap S$, and so

$$\varphi\left(\frac{E(y)+E(y')}{r+s}\right) \leqslant 1.$$

Since $r+s \in \mathbb{Q}$ and φ is an E-additive function, $\varphi(E(y))+\varphi(E(y')) \leqslant r+s$. This is a contradiction, so $a \leq b$.

Set $m \in [a,b]$ and define $\phi : Z \to \mathbb{R}$ by

$$\phi(z) = \begin{cases} \varphi(z) & z \in Y \\ m & z = z_0 \\ \varphi(y) + qm & z \in Y + qz_0. \end{cases}$$

We claim that $\phi\,|_{E(Z)\cap S} \leqslant 1$. Let $E(z) \in E(Z) \cap S$. By the hypothesis, $E(z) = E(y) + qz_0$ for some $y \in Y$ and $q \in \mathbb{Q}$. If $q > 0$, then

$$E(z) = \frac{\frac{1}{q}E(y) + z_0}{\frac{1}{q}} \in S.$$

Since $E(Y)$ is a cone, $\dfrac{1}{q}E(y) = E(y')$ for some $y' \in Y$, and so $\left(y', \dfrac{1}{q}\right) \in B$. Thus

$$\begin{aligned} \frac{1}{q} - \varphi(E(y')) \geqslant b > m &\Rightarrow 1 - q\varphi(E(y')) > qm \\ &\Rightarrow q\varphi(E(y')) + qm < 1 \\ &\Rightarrow \varphi(qE(y')) + qm < 1 \\ &\Rightarrow \varphi(E(y)) + qm < 1 \\ &\Rightarrow \phi(E(y) + qz_0) < 1. \end{aligned}$$

If $q < 0$, then

$$\frac{-\frac{1}{q}E(y) - z_0}{-\frac{1}{q}} \in S.$$

A similar argument shows that

$$\varphi\left(\frac{-E(y)}{q}\right) + \frac{1}{q} \leqslant a < m,$$

and so $\varphi(E(y)) + qm < 1$. This establishes the claim. □

Theorem 2.7. *Let $E : X \to X$ be a map such that, $E(K)$ is a convex cone of K for every subspace $K \subseteq X$ and let Y and Z be two subspaces of X with $Z = Y + \mathbb{Q}z_0$ for some $z_0 \in X \setminus Y$. If $S \subseteq X$ is $\mathbb{Q}$-convex and $\mathbb{Q}$-radial at 0, $E(Z) \subseteq E(Y) + \mathbb{Q}z_0$ and $\varphi : Y \to \mathbb{R}$ is an E-additive function with $\varphi |_{E(Y)\cap S} \leqslant 1$, then there exists an E-additive function $\phi : X \to \mathbb{R}$ such that $\phi |_Y = \varphi$ and $\phi |_{E(X)\cap S} \leqslant 1$.*

Proof. Let F be the set of all pairs (Y', φ'), where Y' is a subspace of Y, and $\varphi' : Y' \to \mathbb{R}$ is an E-additive function with $\varphi' |_Y = \varphi$ and $\varphi' |_{E(Y')\cap S} \leqslant 1$. It is immediately that we obtain a nonempty poset, indeed $(Y, \varphi) \in F$. Suppose that $F_0 = \{(Y_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is a chain in F. Then we set $\tilde{Y} := \cup_{\alpha\in I} Y_\alpha$ and define $\tilde{\varphi} : \tilde{Y} \to \mathbb{R}$ by $\tilde{\varphi} |_{Y_\alpha} = \varphi_\alpha$ for all $\alpha \in I$. Then $(\tilde{Y}, \tilde{\varphi}) \in F$ and $(\tilde{Y}, \tilde{\varphi})$ is clearly an upper bound for the chain F_0. By Zorns lemma, F has a maximal element, say (Y_{max}, φ_{max}). Assume towards a contradiction that $Y_{max} \neq X$. Then choose any $y_0 \in X \setminus Y_{max}$ and put $Y = Y_{max} + \mathbb{Q}y_0$. So Y satisfies the condition of the preceding lemma with Y_{max} for Y and Y for Z. Then there exists $\psi : Y \to \mathbb{R}$ such that $\psi|_{Y_{max}} = \varphi_{max}$ and $\psi|_{E(Y_{max})\cap S} \leqslant 1$. Thus $(Y, \psi) \in F$ and this contradicts the maximality claimed for (Y_{max}, φ_{max}). Hence $Y_{max} = X$ and so by taking $\phi = \varphi_{max}$ the result follows. □

Proposition 2.8. *Let S be a $\mathbb{Q}$-convex subset of $\mathbb{R}$ and $\mathbb{Q}$-radial at some point in S. Then S is an interval or there exists a discontinuous E-additive function $\phi : \mathbb{R} \to \mathbb{R}$ such that ϕ is upper bounded on S, where $E : \mathbb{R} \to \mathbb{R}$ is a function.*

Proof. By [5, Theorem 2], S is an interval or there exists a discontinuous additive function $\phi : \mathbb{R} \to \mathbb{R}$ where ϕ is upper bounded on S. Since every additive function is E-additive, the proof is completed. □

Similar to the above theorem, by [5, Theorem 3], the following theorem is now immediate.

Theorem 2.9. *Let $S \subseteq \mathbb{R}^n$ be a $\mathbb{Q}$-convex set and $\mathbb{Q}$-radial at some point in S. Then either S contains a ball or there exists a discontinuous E-additive function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that ϕ is upper bounded on S, where $E : \mathbb{R}^n \to \mathbb{R}^n$ is a function.*

Theorem 2.10. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be an open map with $0 \in E(\mathbb{R}^n)$, $S \subseteq E(\Delta)$ where Δ is an open, E-$\mathbb{Q}$-convex subset of $\mathbb{R}^n$. Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is an E-$\mathbb{Q}$-convex function and upper bounded on S. If S contains a ball, then φ is continuous at every interior point of S.*

Proof. Suppose that $E(x_0)$ is an interior point of S. We can find an open neighborhood W of 0 such that $E(x_0) + (W \cap E(\mathbb{R}^n)) \subset S$. Put $W \cap E(\mathbb{R}^n) := E(V)$. We can assume that $E(V)$ is symmetric and balanced. Since φ is upper bounded on S, there exists $M \in \mathbb{R}$ with $\varphi(y) < \varphi(E(x_0)) + M$ for any $y \in E(x_0) + E(V)$. Take $\varepsilon > 0$. Then there exists $\delta \in (0, 1) \cap \mathbb{Q}$, $\delta M < \varepsilon$. For any $E(z) \in E(V)$, we have

$$E(x_0) + \delta E(z) = (1 - \delta)E(x_0) + \delta(E(x_0) + E(z)).$$

Since φ is an E-$\mathbb{Q}$-convex function, we obtain

$$\varphi(E(x_0) + \delta E(z)) \leqslant (1 - \delta)\varphi(E(x_0)) + \delta\varphi(E(x_0) + E(z)).$$

So

$$\varphi(E(x_0) + \delta E(z)) - \varphi(E(x_0)) \leqslant \delta\left(\varphi(E(x_0) + E(z)) - \varphi(E(x_0))\right). \tag{2.1}$$

By replacing $E(z)$ with $-E(z)$, we have

$$\varphi(E(x_0) - \delta E(z)) - \varphi(E(x_0)) \leqslant \delta\left(\varphi(E(x_0) - E(z)) - \varphi(E(x_0))\right). \tag{2.2}$$

On the other hand,

$$E(x_0) = \frac{1}{2}\left[E(x_0) + \delta E(z) + E(x_0) - \delta E(z)\right],$$

and since $E(V)$ is symmetric and balanced, we obtain

$$\varphi(E(x_0)) \leqslant \frac{\varphi(E(x_0) + \delta E(z))}{2} + \frac{\varphi(E(x_0) - \delta E(z))}{2}.$$

Therefore

$$\varphi(E(x_0)) - \varphi(E(x_0) + \delta E(z)) \leqslant \varphi(E(x_0) - \delta E(z)) - \varphi(E(x_0)). \tag{2.3}$$

By '(2.2)' and '(2.3)' we have

$$\varphi(E(x_0)) - \varphi(E(x_0) + \delta E(z)) \leqslant \delta(\varphi(E(x_0) - E(z)) - \varphi(E(x_0))). \tag{2.4}$$

Now, by '(2.1)' and '(2.4)' we have

$$\begin{aligned}
&|\ \varphi(E(x_0)) - \varphi(E(x_0) + \delta E(z))\ | \\
&\leqslant \delta max\{\varphi(E(x_0) + E(z)) - \varphi(E(x_0)), \varphi(E(x_0) - E(z)) - \varphi(E(x_0))\} \\
&< \frac{\varepsilon}{M}.M = \varepsilon.
\end{aligned}$$

Therefore φ is continuous at $E(x_0)$. □

Corollary 2.11. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be an open map with $0 \in E(\mathbb{R}^n)$, $\Delta \subseteq \mathbb{R}^n$ be open, E-$\mathbb{Q}$-convex. Suppose that $\varphi : \Delta \to \mathbb{R}$ is an E-$\mathbb{Q}$-convex function and upper bounded on $E(\Delta)$. Then φ is continuous on $E(\Delta)$.*

Proof. If we take $S = E(\Delta)$ in the previous theorem, then the result follows. □

Theorem 2.12. *Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be an open map with $0 \in E(\mathbb{R}^n)$, $\Delta \subset \mathbb{R}^n$ be open and E-$\mathbb{Q}$-convex set and let $E(\mathbb{R}^n)$ be a convex cone. Suppose that $T \subseteq E(\Delta)$ and $\varphi : \Delta \to \mathbb{R}$ is an E-$\mathbb{Q}$-convex function, and upper bounded on $E(T)$. If φ is discontinuous on $E(\Delta)$, then there exists a discontinuous E-additive function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that ψ is upper bounded on $E(T)$.*

Proof. Let $\varphi\ |_{E(T)} \leqslant M$ and $S = \{E(x) : x \in \Delta, \varphi(E(x)) < M\}$. It is easy to see that $E(T) \subseteq S$. Suppose that $E(x), E(y) \in S$ and $\lambda \in [0,1] \cap \mathbb{Q}$. Then

$$\varphi(\lambda E(x) + (1-\lambda)E(y)) \leqslant \lambda\varphi(E(x)) + (1-\lambda)\varphi(E(y)) < M.$$

So S is $\mathbb{Q}$-convex. Since φ is discontinuous, it follows from Corollary 2.11 that S contains no ball. Now, we show that S is $\mathbb{Q}$-radial at some point $E(x_0) \in S$. Let $x \in \mathbb{R}^n$. Since E is open and $E(\mathbb{R}^n)$ is convex cone with nonempty interior, $\mathbb{R}^n = E(\mathbb{R}^n) - E(\mathbb{R}^n)$. Then there exist $y_1, y_2 \in \mathbb{R}^n$, $x = E(y_1) - E(y_2)$. Also Δ is open, thus there exists a rational number $\gamma > 0$ such that $E(x_0) + \gamma(E(y_1) - E(y_2)) \in E(\Delta)$. Then

$$\begin{aligned}\varphi(E(x_0) + \gamma(E(y_1) - E(y_2))) =& \varphi\left[\left(1 - \frac{\alpha}{\gamma}\right)E(x_0) + \frac{\alpha}{\gamma}(E(x_0) + \gamma(E(y_1) - E(y_2)))\right] \\ &\leqslant \left(1 - \frac{\alpha}{\gamma}\right)\varphi(E(x_0) + \frac{\alpha}{\gamma}\varphi\left[E(x_0) + \gamma((E(y_1) - E(y_2))\right],\end{aligned}$$

for any $\alpha \in (0, \gamma) \cap \mathbb{Q}$. So

$$\lim_{\alpha\to 0} sup\varphi(E(x_0) + \gamma(E(y_1) - E(y_2))) \leqslant \varphi(E(x_0)) < M.$$

So for a small rational number α, $\varphi(E(x_0) + \alpha(E(y_1) - E(y_2))) < M$, hence S is $\mathbb{Q}$-radial at $E(x_0)$. Since S contains no ball, we conclude from Theorem 2.9 that there is a discontinuous E-additive function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that ψ is upper bounded on $E(T)$. □

References

[1] M. Aghajani and K. Nourouzi, The continuity of $\mathbb{Q}_+$- homogeneous superadditive correspondences, *J. Nonlinear Convex Anal.*, **16** (2015), 1899–1904.

[2] X. Chen, Some properties of semi-E-convex functions, *J. Math. Anal. Appl.*, **275**(1) (2002), 251–262.

[3] A. Hussain and A. Iqbal, Quasi strongly -convex functions with applications, *Non-linear Funct. Anal. Appl.*, **26** (2021), 1077–1089.

[4] A. Iqbal, and I. Ahmad, Strong geodesic convex functions of order m, *Numer. Funct. Anal. Optim.*, **40** (2019), 1840–1846.

[5] M.E. Kuczma, On discontinuous additive function, *Fund. Math.*, **66** (1970), 383–392.

[6] S.N. Majeed and M.I. Abd Al-Majeed, On convex functions, E-convex functions and their generalizations: applications to non-linear optimization problems, *Int. J. Pure Appl. Math.*, **116** (2017), 655–673.

[7] S.N. Majeed, On strongly E-convex sets and strongly E-convex cone sets, *J. AL-Qadisiyah Comput. Sci. Math.*, **11** (2019), 52–59.

[8] G.G. Magaril-Ilyaev and V.M. Tikhomirov, *Convex Analysis: Theory and Applications*, AMS, Providence, R.I., Transl. of Math. Monographs, 2003.

[9] M.R. Mehdi, On convex functions, *J. London Math. Soc.*, **39** (1964), 321–326.

[10] F. Mirzapour, A. Mirzapour and M. Meghdadi, Generalization of some important theorems to E-midconvex functions, *Appl. Math. Lett.*, **24**(8) (2011), 1384–1388.

[11] P. Najmadi and M. Aghajani, Some families of sublinear correspondences, *J. Appl. Anal.*, **25**(1) (2019), 91–95.

[12] W. Saleh, HermiteHadamard type inequality for (E, F)-convex functions and geodesic (E, F)-convex functions, *Rairo-Oper. Res.*, **56** (2022), 4181–4189.

[13] M. Soleimani-damaneh, E-convexity and its generalizations, *Int. J. Comput. Math.*, **88** (2011), 3335–3349.

[14] Y.R. Syau and E.S. Lee, Some properties of E-convex functions, *Appl. Math. Lett.*, **18** (2005), 1074–1080.

[15] X.M. Yang, E-convex sets, E-convex functions and E-convex programming, *J. Optim. Theory Appl.*, **109** (2001), 699–704.

[16] E.A. Youness, E-convex functions and E-convex programming, *J. Optim. Theory Appl.*, **102**(2) (1999), 439–450.

[17] E.A. Youness, Optimality criteria in E-convex programming, *Chaos Solitons Fractals*, **12** (2001), 1737–1745.

[18] B.C. Joshi and Pankaj, Mathematical programs involving E-convex functions, *Scientific Bulletin. upb. ro.*, **83**(2) (2011), 77–86.