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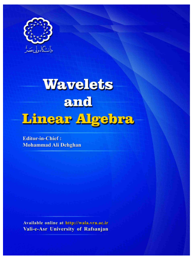

Localization Operators on Sobolev Spaces

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Abstract

In this paper, we discuss some generalizations coming from wavelet transform on Sobolev spaces. In particular, we introduce the bounded localization operators on Sobolev spaces which are related to multi-dimensional wavelet transform on Sobolev spaces. Moreover, we propose the localization operators on Sobolev spaces are in p-Schatten class and they are compact. Finally, we give the boundedness and compactness of localization operators on Sobolev spaces with two admissible wavelets.

1. Introduction and preliminaries

Wavelet analysis is a universal tool with a very rich mathematical content and great potential for applications in various scientific fields. The localization operators were introduced and studied

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by Daubechies [3, 4]. This class of operators occurs in various branches of applied and pure mathematics and has been studied by many authors. Localization operators are recognized as an important new mathematical tool and have found many applications to the theory of differential equations, quantum mechanics, time-frequency analysis, signal processing (see [9, 5]). The multi-dimensional continuous wavelet transform on Sobolev space has been studied in [7] whereas in this paper, we give a systematic study of localization operators on Sobolev space which related to the multi-dimensional continuous wavelet transform.

The space of infinity differentiable functions on $\mathbb{R}^n$ denoted by $C^\infty(\mathbb{R}^n)$ accompanied by uniform convergence on compact sets of functional sequences and all their derivatives. Let $C_c^\infty(\mathbb{R}^n)$ be the space of infinitely differentiable function on $\mathbb{R}^n$ with compact support. A tempered distribution is a continuous linear functional on Schwartz spaces $\mathcal{S}$. We recall that $\mathcal{S}$ is a Fréchet space with topology defined by the norm:

$$||\phi||_{[N,\alpha]} = sup_{x\in\mathbb{R}^n}(1+|x|)^N \partial^\alpha \phi(x),$$

for multi-index α and $N \in \mathbb{N}$. The space of tempered distributions is denoted by $\mathcal{S}'$. This space is identified with the set of distributions that extend continuously from C_c^∞ to $\mathcal{S}$. A locally integrable function is said to be tempered if it is tempered as a distribution. Note that every compactly supported distribution is tempered (for more details see [6].)

For the reader's convenience, we review the definition of Sobolev spaces and mention some properties of them. Suppose that $k \in \mathbb{N}$ and let H_k be the space of all $f \in L^2(\mathbb{R}^n)$ whose distribution derivatives $\partial^\alpha f$ are L^2-functions, for multi index α with $|\alpha| \le k$. It has been shown that H_k is a Hilbert space with the inner product

$$\prec f, g \succ_k = \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} (\partial^\alpha f)(x)(\overline{\partial^\alpha g})(x)dx,$$

for $f, g \in H_k$. It is more convenient to use an equivalent inner product defined in terms of the Fourier transform. One can check that $f \in H_k$ if and only if $(1+|\xi|^2)^{k/2}\widehat{f} \in L^2(\mathbb{R}^n)$ and that the norms $f \mapsto (\sum_{|\alpha|\le k} ||\partial^\alpha f||_2^2)^{1/2}$ and $f \mapsto ||(1+|\xi|^2)^{k/2}\widehat{f}||_2$ are equivalent. The latter norm makes sense for any $k \in \mathbb{R}$, the definition of $\mathbb{H}_k$ can be extended to all real k. Furthermore, for any $s \in \mathbb{R}$, the function $\xi \mapsto (1+|\xi|^2)^{s/2}$ is in $C^\infty(\mathbb{R}^n)$ and slowly increasing, so the map J_s defined by $J_s f = (1+|\xi|^2)^{s/2}\hat{f})^\vee$ is a continuous linear operator on $\mathcal{S}'$, in which $\hat{f}$ is the Fourier transform of f and $f^\vee$ is the inversion of the Fourier transform. Note that J_s is isomorphism, since $J_s^{-1} = J_{-s}$. Now, for any real s, by $\mathbb{H}_s(\mathbb{R}^n)$ we denote the Sobolev space and define as:

$$\mathbb{H}_s(\mathbb{R}^n) = \{f \in \mathcal{S}'; J_s f \in L^2(\mathbb{R}^n)\}.$$

The inner product and norm on $\mathbb{H}_s(\mathbb{R}^n)$ are given as:

$$\prec f, g \succ_s = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

and

$$||f||_s = \left[\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right]^{1/2}.$$

The following properties of Sobolev spaces are simple consequences of the definitions:

i (i) The Fourier transform is a unitary isomorphism from $\mathbb{H}_s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n, \mu_s)$ where $d\mu_s(\xi) = (1 + |\xi|^2)^s d\xi$. In particular $\mathbb{H}_s(\mathbb{R}^n)$ is a Hilbert space.

ii (ii) Schwartz space $\mathcal{S}$ is a dense subspace of $\mathbb{H}_s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

iii (iii) J_t is a unitary isomorphism from $\mathbb{H}_s$ to $\mathbb{H}_{s-t}$ for all $s, t \in \mathbb{R}$.

iv (iv) $\mathbb{H}_0 = L^2$ and $\|.\|_0 = \|.\|_2$ (see [6, 8]).

2. Main results

In this section, the multi-dimensional wavelet transform on Sobolev space $H_s(\mathbb{R}^n)$ is introduced. It is shown that the multi-dimensional wavelet transform of Sobolev space $H_s(\mathbb{R}^n)$ into $H_{0,s}(\mathbb{R}^n \times \mathbb{R}_0^+ \times S^{n-1})$ is an isometry, for arbitrary real s. Also, the boundedness localization operators regarding this transform on Sobolev space are obtained.
Multi-dimensional wavelets may be derived from the similitude group of $\mathbb{R}^n (n > 1)$ denoted by $SIM(n) = \mathbb{R}^n \times (\mathbb{R}_0^+ \times SO(n))$, consisting of dilations, rotations and translations. This group has the following natural action on a n-dimensional signal

$$f_{b,a,R}(x) = [\pi(b, a, R)f](x) = a^{-n/2} f(a^{-1}R^{-1}(x - b)), \tag{2.1}$$

for all $(b, a, R) \in SIM(n)$. In [1, Theorem 14.2.1], it has been shown that the operator defined in (2.1) is a unitary irreducible representation of $SIM(n)$ in $L^2(\mathbb{R}^n)$. Also, this representation is square integrable. A vector $0 \neq \psi \in L^2(\mathbb{R}^n)$ is called admissible if $\prec \psi_{b,a,R}, \psi \succ_{L^2(\mathbb{R}^n)}$ is in $L^2(SIM(n))$. Moreover, one can check a vector $\psi \in L^2(\mathbb{R}^n)$ is admissible if and only if, it satisfies

$$c_\psi = (2\pi)^n A_{n-1} \int_{\mathbb{R}^n} |\widehat{\psi}(k)|^2 \frac{d^n k}{|k|^n} < \infty, \tag{2.2}$$

where $A_{n-1} = \prod_{k=2}^{n-1} \frac{2\pi^{k/2}}{\Gamma(k/2)}$ is the volume of $SO(n-1)$. The admissible vector ψ is called an admissible wavelet if $\|\psi\| = 1$. The continuous wavelet transform corresponding to the wavelet $\psi \in L^2(\mathbb{R}^n)$ is defined as

$$W_\psi f(b, a, R) = c_\psi^{-1/2} \prec \psi_{b,a,R}, f \succ, \tag{2.3}$$

for all $f \in L^2(\mathbb{R}^n)$. Note that if the wavelet ψ is axially symmetric i.e. $SO(n-1)$−invariant, then one can replace everywhere $SO(n)$ by $\frac{SO(n)}{SO(n-1)} \simeq S^{n-1}$, the unit sphere in $\mathbb{R}^n$. The rotation R becomes $R \equiv R(\varpi), \varpi \in SO(n-1)$, and continuous wavelet transform leads to $W_\psi f(b, a, \varpi) \in L^2(X, d\upsilon)$, in which

$$X = \frac{SIM(n)}{SO(n-1)} = \mathbb{R}^n \times (\mathbb{R}_0^+ \times S^{n-1}),$$

and $d\upsilon = \frac{da}{a^{n+1}} d\varpi db$, is $SIM(n)$−invariant measure for X (for more details see [1]).
In the following, we assume that $\psi \in L^2(\mathbb{R}^n)$ is an admissible wavelet and integrable. The multi-dimensional continuous wavelet transform defined on the similitude group $SIM(n)$ as:

$$W_\psi f(b, a, \varpi) = c_\psi^{-1/2} \prec \psi_{b,a,\varpi}, f \succ, \tag{2.4}$$

where $f \in L^2(\mathbb{R}^n)$ and $\psi_{b,a,\varpi}, c_\psi$ are as in (2.1) and (2.2), respectively. We now prove that for $f \in H_s(\mathbb{R}^n)$, $W_\psi f$ is in $L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}} d\varpi), H_s(\mathbb{R}^n))$ in which,

$$L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}} d\varpi), H_s(\mathbb{R}^n)) =$$

$$\{f \in H_s(\mathbb{R}^n), \int_{\mathbb{R}_0^+ \times S^{n-1}} \|f(., a, \varpi)\|_s^2 \frac{da}{a^{n+1}} d\varpi < \infty\}.$$

To this end, let $\psi \in L^2(\mathbb{R}^n)$ be an admissible wavelet. For $f \in H_s(\mathbb{R}^n)$ we have,

$$\begin{aligned} W_\psi f(b, a, \varpi) &= c_\psi^{-1/2} < \psi_{b,a,\varpi}, f > \\ &= c_\psi^{-1/2} \int_{\mathbb{R}^n} \psi_{b,a,\varpi}(x) \overline{f}(x) dx \\ &= c_\psi^{-1/2} \int_{\mathbb{R}^n} a^{-n/2} \psi(a^{-1} \varpi^{-1}(x - b)) \overline{f}(x) dx \\ &= c_\psi^{-1/2} \int_{\mathbb{R}^n} D_{-a} L_\varpi \psi(b - x) \overline{f}(x) dx \\ &= c_\psi^{-1/2} (D_{-a} L_\varpi \psi * \overline{f})(b). \end{aligned}$$

And then

$$\begin{aligned} (W_\psi f(., a, \varpi))\widehat{\ }(\xi) &= c_\psi^{-1/2} (D_{-a} L_\varpi \psi * \overline{f})\widehat{\ }(\xi) \\ &= c_\psi^{-1/2} (D_{-a} L_\varpi \psi)\widehat{\ }(\xi) \widehat{\overline{f}}(\xi) \\ &= c_\psi^{-1/2} a^{n/2} \widehat{\psi}(-a\varpi\xi) \widehat{\overline{f}}(\xi), \end{aligned}$$

in which the operators $D_a\psi(x) := \frac{1}{\sqrt{a^n}} \psi(a^{-1}x)$ and $L_\varpi \psi(x) := \psi(\varpi^{-1}x)$ are dilation and rotation operators, respectively. So we have

$$(W_\psi f(., a, \varpi))\widehat{\ }(\xi) = c_\psi^{-1/2} a^{n/2} \widehat{\psi}(-a\varpi\xi) \widehat{\overline{f}}(\xi). \tag{2.5}$$

Moreover, in [7], we have shown that, for an admissible and integrable wavelet $\psi \in L^2(\mathbb{R}^n)$, the continuous wavelet transform, $W_\psi f$ is in $L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}} d\varpi), H_s(\mathbb{R}^n))$, for $f \in H_s(\mathbb{R}^n)$. You can see more details of the following proposition in [Proposition2.4, [7]].

Proposition 2.1. *Let $\psi \in L^2(\mathbb{R}^n)$ be admissible and integrable. For $f \in H_s(\mathbb{R}^n)$, the continuous wavelet transform, $W_\psi f$ is in $L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}} d\varpi), H_s(\mathbb{R}^n))$.*

Note that the inner product and the norm on $L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}} d\varpi), H_s(\mathbb{R}^n))$ are denoted by $<<, >>$ and $|||, |||$ defined as following:

$$<< \varphi, \psi >>= \int_{\mathbb{R}_0^+ \times S^{n-1}} < \varphi(., a, \varpi), \psi(., a, \varpi) >_s \frac{da}{a^{n+1}} d\varpi,$$

and

$$|||\varphi||| = \int_{\mathbb{R}_0^+ \times S^{n-1}} \|\varphi(., a, \varpi)\|_s^2 \frac{da}{a^{n+1}} d\varpi,$$

for φ, ψ in $L^2((\mathbb{R}_0^+ \times S^{n-1}, \frac{da}{a^{n+1}} d\varpi), H^s(\mathbb{R}^n))$.

The inversion formula for the wavelet transform on the Sobolev spaces is given as follows. For an admissible and integrable vector $\psi \in L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} << W_\psi f, W_\psi g >> &= \frac{1}{c_\psi} \int_{SIM(n)} (1 + |\xi|^2)^s \widehat{f}(\xi) \overline{\widehat{g}(\xi)} |\widehat{\psi}(a\xi\varpi)|^2 \frac{da}{a^{n+1}} d\varpi) d\xi \\ &= \frac{1}{c_\psi} \int_{\mathbb{R}^n} \int_{\mathbb{R}_0^+ \times S^{n-1}} |\widehat{\psi}(a\varpi\xi)|^2 (1 + |\xi|^2)^s \widehat{f}(\xi) \overline{\widehat{g}(\xi)} \frac{da}{a^{n+1}} d\varpi) d\xi \\ &= \frac{1}{c_\psi} (\int_{\mathbb{R}^n} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|^n} d\xi) \prec f, g \succ_s \\ &= \prec f, g \succ_s, \end{aligned}$$

in which, $f, g \in H_s(\mathbb{R}^n)$. As an important consequence of the inversion formula is that, the continuous multi-dimensional wavelet transform is an isometry from $H_s(\mathbb{R}^n)$ into $H_{0,s}(\mathbb{R}^n \times \mathbb{R}_0^+ \times S^{n-1})$.

Corollary 2.2. *The multi-dimensional continuous wavelet transform is an isometry from $H_s(\mathbb{R}^n)$ into $H_{0,s}(\mathbb{R}^n \times \mathbb{R}_0^+ \times S^{n-1})$. In particular, $|||W_\psi f||| = \|f\|_s$.*

Proposition 2.3 (Proposition2.7, [7])**.** *Let $\varphi, \psi \in L^2(\mathbb{R}^n)$ be two admissible wavelets. For $f, g \in H_s(\mathbb{R}^n)$, we have,*

$$<< W_\varphi f, W_\psi g >>= \frac{c_{\varphi,\psi}}{\sqrt{c_\varphi c_\psi}} \prec f, g \succ_s,$$

in which $c_{\varphi,\psi} = \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)||\widehat{\psi}(\xi)| \frac{d\xi}{|\xi|^n}$.

In the sequel, we extend the localization operators related to continuous wavelet transform which defined on $L^2(\mathbb{R}^n)$ to Sobolev space $H_s(\mathbb{R}^n)$ and we study their boundary properties and compaction. For admissible wavelet ψ in $L^2(\mathbb{R}^n)$, a class of bounded linear operators $T_{\psi,F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ has been studied in [4] for all $F \in L^p(\mathbb{R}^n \times \mathbb{R}^n), 1 \le p \le \infty$. The localization operators $T_{\psi,F}$ are defined by

$$\prec T_{\psi,F} f, g \succ = \prec F.W_\psi f, W_\psi g \succ,$$

for $f, g \in L^2(\mathbb{R}^n)$. In this direction, some results for localization operators associated with the representations of locally compact group G and Hilbert space $\mathcal{H}$ can be found in [9]. It has been shown that for $F \in L^p(G)$, $1 \le p \le \infty$, the operator $T_{\psi,F}$ is bounded and

$$\|T_{\psi,F}\| \le (\frac{1}{c_\psi})^{1/p} \|F\|_p.$$

Moreover, the localization operator $T_{\psi,F}$ is in p-Schatten class S_p and

$$\|T_{\psi,F}\|_{S_p} \leq (\frac{4}{c_\psi})^{1/p}\|F\|_p,$$

for $1 \leq p \leq \infty$. The constant $\frac{4}{c_\psi}$ can be improved to $\frac{1}{c_\psi}$ and obtain a lower bound for the norm $\|T_{\psi,F}\|_{S_1}$ of localization operator $T_{\psi,F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, i.e

$$\|T_{\psi,F}\|_{S_1} \leq (\frac{1}{c_\psi})\|F\|_1.$$

Now, let ψ be an admissible wavelet and $F \in L^p(SIM(n)), 1 \leq p \leq \infty$. The localization operator $\tilde{T}_{\psi,F}$ on Sobolev spaces is given through $\tilde{T}_{\psi,F} := J_{-s}T_{\psi,F}J_s$, in which J_s, J_{-s} are unitary isomorphisms $J_s : H_s \to H_0$ and $J_-s : H_0 \to H_s$, respectively. Therefore, we can introduce the localization operator on Sobolev space as follows:

$$\begin{aligned} \prec \tilde{T}_{\psi,F}f, g \succ_s &= \prec J_s\tilde{T}_{\psi,F}f, J_sg \succ_2 \\ &= \prec T_{\psi,F}J_sf, J_sg \succ_2 \\ &= \frac{1}{c_\psi}\int_{SIM(n)} F(b,a,\varpi) \prec \psi_{b,a,\varpi}, J_sf \succ_2 \prec J_sg, \psi_{b,a,\varpi} \succ_2 db\frac{da}{a^{n+1}}d\varpi. \end{aligned}$$

In the following proposition we show that the localization operator $\tilde{T}_{\psi,F} : H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is bounded.

Proposition 2.4. *Assume that ψ is an admissible wavelet and $F \in L^p(SIM(n))$. Then $\tilde{T}_{\psi,F}$ on $H_s(\mathbb{R}^n)$ is a bounded linear operator and*

$$\|\tilde{T}_{\psi,F}f\|_s \leq (\frac{1}{c_\psi})^{1/p}\|F\|_p\|f\|_s.$$

Proof. Consider ψ is an admissible wavelet and $T_{F,\psi}$ is the localization opeator on $L^2(\mathbb{R}^n)$. For $f \in H_s(\mathbb{R}^n)$, we have

$$\begin{aligned} \|\tilde{T}_{\psi,F}f\|_s &= \|J_s(\tilde{T}_{\psi,F})f\|_2 \\ &= \|J_sJ_{-s}T_{\psi,F}J_sf\|_2 \\ &= \|T_{\psi,F}J_sf\|_2 \\ &\leq \|T_{\psi,F}\|\|J_sf\|_2 \\ &\leq \|T_{\psi,F}\|\|f\|_s \\ &\leq (\frac{1}{c_\psi})^{1/p}\|F\|_p\|f\|_s. \end{aligned}$$

Therefore, the linear operator $\tilde{T}_{\psi,F} : H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is bounded. □

Proposition 2.5. *Let ψ be an admissible wavelet and $F \in L^1(SIM(n))$. Then $\tilde{T}_{\psi,F} : H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is in S^1 and*

$$\|\tilde{T}_{\psi,F}\|_{S^1} \leq \frac{4}{c_\psi}\|F\|_1.$$

Proof. Let $\{\varphi_k\}$ be an orthonormal basis for $H_s(\mathbb{R}^n)$. Then we have

$$\begin{aligned}
\|\tilde{T}_{\psi,F}\|_{S^1} &= \sum_{k=1}^{\infty} \prec \tilde{T}_{\psi,F}\varphi_k, \varphi_k \succ_s \\
&= \sum_{k=1}^{\infty} \prec J_s\tilde{T}_{\psi,F}\varphi_k, J_s\varphi_k \succ_2 \\
&= \sum_{k=1}^{\infty} \prec J_sJ_{-s}T_{\psi,F}J_s\varphi_k, J_s\varphi_k \succ_2 \\
&= \sum_{k=1}^{\infty} \prec T_{\psi,F}J_s\varphi_k, J_s\varphi_k \succ_2 \\
&= \|T_{\psi,F}\|_{S^1} \\
&\leq \frac{4}{c_\psi}\|F\|_1.
\end{aligned}$$

□

Remark 2.6. Since J_s is an unitary isometry, so $J_s\varphi_k$ is an orthonormal basis for $H_0 = L^2$. Moreover, by interpolation theorem, for $F \in L^p(SIM(n))$, $1 \leq p < \infty$, we have

$$\|\tilde{T}_{\psi,F}\|_{S^p} \leq (\frac{4}{c_\psi})^{1/p}\|F\|_p.$$

That is $\tilde{T}_{\psi,F}$ is in Schatten P-class.

Theorem 2.7. *Let $F \in L^p(SIM(n))$, $1 \leq p < \infty$. Then the localization operator $\tilde{T}_{\psi,F} : H_s(\mathbb{R}^n) \to H_s(\mathbb{R}^n)$ is compact.*

Proof. Since the localization operator $T_{\psi,F}$ on $L^2(\mathbb{R}^n)$ is compact and J_s is unitary isomorphism, so $\tilde{T}_{\psi,F}$ is a compact operator. □

It turns out that it is possible and indeed natural to consider more general localization operators $\tilde{T}_{F,\varphi,\psi}$ on Sobolev space $H_s(\mathbb{R}^n)$ associated with the functions F in $L^p(SIM(n))$, $1 \leq p \leq \infty$ and two admissible wavelets φ, ψ. We follow Wong in [9], suppose φ, ψ are two admissible wavelets, $T_{F,\psi,\varphi}$ is the Daubecheis operator on $L^2(\mathbb{R}^n)$. Then, we define the two-wavelet localization operator $\tilde{T}_{F,\varphi,\psi}$ on $H_s(\mathbb{R}^n)$ by

$$\begin{aligned}
\prec \tilde{T}_{F,\varphi,\psi}f, g \succ_s &= \prec T_{F,\varphi,\psi}J_sf, J_sg \succ_s \\
&= \frac{1}{c_{\psi,\varphi}} \int_{SIM(n)} F(b, a, \varpi) \prec \psi_{b,a,\varpi}, J_sf \succ\prec J_sg, \varphi_{b,a,\varpi} \succ db\frac{da}{a^{n+1}}d\varpi,
\end{aligned}$$

in which $c_{\varphi,\psi} = \int_{\mathbb{R}^n} |\widehat{\varphi}(\xi)||\widehat{\psi}(\xi)|\frac{d\xi}{|\xi|^n}$.

Proposition 2.8. *Let F be a symbol in $L^1(SIM(n))$ and $\varphi \in L^\infty(\mathbb{R}^n), \psi \in L^1(\mathbb{R}^n)$. Then $\tilde{T}_{F,\varphi,\psi} : \mathbb{H}_s(\mathbb{R}^n) \to \mathbb{H}_s(\mathbb{R}^n)$ is a bounded linear operator and*

$$\|\tilde{T}_{F,\varphi,\psi}\| \leq \|\varphi\|_\infty \|\psi\|_1 \|F\|_1.$$

Proof. Let φ, ψ be two admissible wavelets. For $f, g \in \mathbb{H}_s(\mathbb{R}^n)$ we have,

$$\begin{aligned} |<\tilde{T}_{F,\varphi,\psi} f, g>_s| &= |<T_{F,\varphi,\psi} J_s f, J_s g>_2| \\ &\leq \|\varphi\|_\infty \|\psi\|_1 \|F\|_1 \|J_s f\|_2 \|J_s g\|_2 \\ &\leq \|\varphi\|_\infty \|\psi\|_1 \|F\|_1 \|f\|_s \|g\|_s. \end{aligned}$$

Therefore,

$$\|\tilde{T}_{F,\varphi,\psi}\| \leq \|\varphi\|_\infty \|\psi\|_1 \|F\|_1.$$

□

Proposition 2.9. *Let $F \in L^1(SIM(n)), \varphi \in L^p(\mathbb{R}^n)$ and $\psi \in L^q(\mathbb{R}^n)$, where q is the conjugate of p, for $1 \leq p \leq \infty$. Then the localization operator $\tilde{T}_{F,\varphi,\psi} : \mathbb{H}_s(\mathbb{R}^n) \to \mathbb{H}_s(\mathbb{R}^n)$ is a bounded linear operator and we have*

$$\|\tilde{T}_{F,\varphi,\psi}\| \leq \|\varphi\|_p \|\psi\|_q \|F\|_1. \tag{2.6}$$

Proof. Let $\varphi \in L^p(\mathbb{R}^n), \psi \in L^q(\mathbb{R}^n)$. For $f, g \in \mathbb{H}_s(\mathbb{R}^n)$ we have,

$$\begin{aligned} |<\tilde{T}_{F,\varphi,\psi} f, g>_s| &= |<T_{F,\varphi,\psi} J_s f, J_s g>_2| \\ &\leq \|\varphi\|_p \|\psi\|_q \|F\|_1 \|J_s f\|_2 \|J_s g\|_2 \\ &\leq \|\varphi\|_p \|\psi\|_q \|F\|_1 \|f\|_s \|g\|_s. \end{aligned}$$

Therefore,

$$\|\tilde{T}_{F,\varphi,\psi}\| \leq \|\varphi\|_p \|\psi\|_q \|F\|_1.$$

□

Theorem 2.10. *Let F be in $L^1(SIM(n))$ and $\varphi, \psi \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then the bounded linear operator $\tilde{T}_{F,\varphi,\psi}$ on $\mathbb{H}_s(\mathbb{R}^n)$ is compact.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $\mathbb{H}_s(\mathbb{R}^n)$ such that $f_n \to 0$ weakly in $\mathbb{H}_s(\mathbb{R}^n)$ as $n \to \infty$. Since J_s is unitary isomorphism, so $J_s f_n \to 0$ weakly in $L^2(\mathbb{R}^n)$. Then $T_{F,\varphi,\psi} J_s f_n \to 0$. Therefore, $\tilde{T}_{F,\varphi,\psi} f_n \to 0$. Thus the proof is complete. □

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