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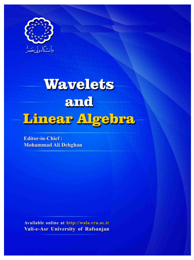

On L-rays of toeplitz matrices

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Abstract

A toeplitz matrix or diagonal-constant matrix is a matrix in which each descending diagonal from left to right is constant. A matrix R is called integral row stochastic, if each row has exactly a nonzero entry, $+1$, and other entries are zero. In this paper, we present L-rays of integral row stochastic toeplitz matrices, and we provide an algorithm for constructing these matrices.

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1. Introduction

Any n-by-n matrix A of the form

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & & \ddots & \vdots \\ \vdots & \ddots & & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \dots & \dots & a_2 & a_1 & a_0 \end{pmatrix}$$

is a toeplitz matrix. If $A = [a_{ij}]$, then we have $a_{ij} = a_{i+1j+1} = a_{i-j}$.

A toeplitz matrix may be defined as a matrix $A = [a_{ij}]$ where $a_{ij} = c_{i-j}$, for constants $c_{1-n}, \dots, c_{n-1}$. The set of n-by-n toeplitz matrices is a subspace of the vector space of n-by-n matrices under matrix addition and scalar multiplication.

Toeplitz matrices are also closely connected with Fourier series, because the multiplication operator by a trigonometric polynomial, compressed to a finite-dimensional space, can be represented by such a matrix. Similarly, one can represent linear convolution as multiplication by a toeplitz marrix.

If each row of a matrix R has exactly a nonzero entry, $+1$, and its other entries zero, R is called integral row stochastic . The collection of all n-by-n integral row stochastic toeplitz matrices is denoted by $\mathcal{TR}(n)$.

For each $1 \leq k \leq n$ define

$$L^{(k)} = \{(k, 1), (k, 2), \dots, (k, k), (k-1, k), \dots, (1, k)\}.$$

Note that $L^{(k)}$ consists of the first k positions in row k and column k. The sets $L^{(k)}$ are shaped like an L (backward). See [5].

For instance, for $n = 5$

$$\begin{aligned} L^{(1)} &= \{(1, 1)\}, \\ L^{(2)} &= \{(2, 1), (2, 2), (1, 2)\}, \\ L^{(3)} &= \{(3, 1), (3, 2), (3, 3), (2, 3), (1, 3)\}, \\ L^{(4)} &= \{(4, 1), (4, 2), (4, 3), (4, 4), (3, 4), (2, 4), (1, 4)\}, \\ L^{(5)} &= \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (4, 5), (3, 5), (2, 5), (1, 5)\}. \end{aligned}$$

Let $A \in \mathbf{M}_n$. For each $1 \leq k \leq n$ define

$$\sigma_k(A) = \sum_{(i,j)\in L^{(k)}} a_{ij},$$

and

$$\sigma(A) = (\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A)).$$

$\sigma(A)$ is called the L-ray of A.

A matrix in $\mathbf{M}_n$ is doubly stochastic if it is componentwise nonnegative and the entries sum of each row and each column is 1. In [4], the author presented descriptions of L-rays of permutation matrices and doubly stochastic matrices. For more information we refer the reader to [1, 2, 3].

In this paper, we investigate the image set $\sigma(\mathcal{TR}(n)) = \{\sigma(A) : A \in \mathcal{TR}(n)\}$ for the class of integral row stochastic toeplitz matrices. We call this the *L-ray problem*. We show that they are associated with majorization theory. The book [6] is a comprehensive study of majorization theory and its applications.

2. *L*-rays of integral row stochastic toeplitz matrices

In this section, we characterize L-rays of integral row stochastic toeplitz matrices.

Example 2.1. Let

$$\mathcal{TR}(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\sigma(\mathcal{TR}(2)) = \{(1,1),(0,2)\}.$$

One can obtain the following proposition directly from the definition of an integral row stochastic toeplitz matrix.

Proposition 2.2. *Let $A \in \mathcal{TR}(n)$. Then $0 \le \sigma_1(A) \le 1$, and for each k $(2 \le k \le n)$ $0 \le \sigma_k(A) \le 2$.*

The notation $x = (x_1, x_2, \ldots, x_n) \in \{0,1,2\}^n$ means that $x_i \in \{0,1,2\}$ for each $1 \le i \le n$. A real vector $x = (x_1, x_2, \ldots, x_n)$ is called monotone whenever $x_1 \le x_2 \le \ldots \le x_n$.

We use the following variation of majorization.
Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then $x \prec^* y$ if $\sum_{i=1}^k x_i \le \sum_{i=1}^k y_i$ for $k < n$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

The following algorithm constructs an integral row stochastic toeplitz matrix $A = (a_{ij})$. Note that $e = (1, 1, \ldots, 1)$.

Algorithm

Input: A monotone vector $x = (x_1, x_2, \ldots, x_n) \in \{0,1,2\}^n$ with $x \prec^* e$.
1. Initialize: Let $A = (a_{ij}) = 0_n$ (the zero matrix).
2. for $k = 1, 2, \ldots, n$ do
 (a) If $x_k = 1$,
  If $k = \min\{1 \le i \le k : x_i \ne 0\}$, let $a_{1k} = 1$.
  Else $s = \max\{1 \le i \le k-1 : x_i \ne 0\}$ and l = the number of obtained rows in the stage $x_s = 1$, let $a_{(l+1)k} = 1$.
 (b) If $x_k = 2$,
  Let m be the number 2's in the vector x.
   for $i = 1, 2, \ldots, m$ do

Let $a_{ki} = a_{(k-m)k} = 1$.
Output: A.

In the following theorem we describe the L-rays of integral row stochastic toeplitz matrices. Remember that card(A) is the number of its elements, where A is a finite set.

Theorem 2.3. $\sigma(\mathcal{TR}(n)) = \{x \in \{0, 1, 2\}^n \mid x \prec^* e, \;\; x \;\text{ is }\; \text{monotone}\}$.

Furthermore, if $x \in \{0, 1, 2\}^n$, $x \prec^* e$ and x is monotone, then Algorithm offers an integral row stochastic toeplitz matrix A with $\sigma(A) = x$.

Proof. First, assume that $x = \sigma(A)$ where $A \in \mathcal{TR}(n)$. Proposition 2.2 ensures that $x \in \{0, 1, 2\}^n$. We see that for each $k \leq n$

$$\begin{aligned} \textstyle\sum_{r=1}^{k} x_r &= \textstyle\sum_{r=1}^{k} \sigma_r(A) \\ &= \textstyle\sum_{i,j \leq k} r_{ij} \leq k, \end{aligned}$$

and

$$\begin{aligned} \textstyle\sum_{r=1}^{n} x_r &= \textstyle\sum_{r=1}^{n} \sigma_r(A) \\ &= \textstyle\sum_{i,j \leq n} r_{ij} \\ &= n. \end{aligned}$$

These imply that $x \prec^* e$. Now, we claim that x is monotone. As A is an integral row stochastic toeplitz matrix, we have two steps.
Step 1. If $A = I_n$, then $x = e$, and so x is monotone.
Step 2. If

$$A = \begin{pmatrix} & & & & 1 & & & \\ & & & & & 1 & 0 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ 1 & & & 0 & & & & \\ & 1 & & & & & & \\ & 0 & \ddots & & & & & \\ & & & & 1 & & & \end{pmatrix},$$

where $a_{1k}, a_{2(k+1)}, \ldots, a_{tn}, a_{(t+1)1}, a_{(t+2)2}, \ldots, a_{n(k-1)} \neq 0$, we consider three cases.
Case 1. $k = t + 1$;

We observe that

$$\sigma_1(A) = \cdots = \sigma_{k-1}(A) = 0,$$

and

$$\sigma_k(A) = \cdots = \sigma_n(A) = 2.$$

That is, $x = (0, \ldots, 0, 2, \ldots, 2)$.
Case 2. $k > t + 1$;

Then

$$\sigma_1(A) = \cdots = \sigma_t(A) = 0,$$

$$\sigma_{t+1}(A) = \cdots = \sigma_{k-1}(A) = 1,$$

and

$$\sigma_k(A) = \cdots = \sigma_n(A) = 2.$$

Thus, $x = (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2)$.
Case 3. $k < t + 1$;

In this case,

$$\sigma_1(A) = \cdots = \sigma_{k-1}(A) = 0,$$

$$\sigma_k(A) = \cdots = \sigma_t(A) = 1,$$

and

$$\sigma_{t+1}(A) = \cdots = \sigma_n(A) = 2.$$

Hence $x = (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2)$.
In any case, we deduce that x is monotone.

For the converse, suppose that $x \in \{0, 1, 2\}^n$, $x \prec^* e$ and x is monotone. We claim that Algorithm constructs some $A \in \mathcal{TR}(n)$ such that $x = \sigma(A)$.

Claim: After each iteration k (of step 2) the present matrix A has the property $\sigma_i(A) = x_i$ for each $i = 1, 2, \ldots, k$.

Proof of Claim: By induction on k we prove it. If $k = 1$ there is nothing to prove. Suppose that $k \leq n$ and the statement holds for $k' < k$. We consider three cases.

Case 1. If $x_k = 0$;

We see A is not modified. So $\sigma_k(A) = 0$, and the induction statement holds in this case.

Case 2. If $x_k = 1$;

If $k = \min\{1 \leq i \leq k : x_i \neq 0\}$ then $a_{1k} = 1$, and so $\sigma_k(A) = x_k$. If not; $a_{l+1k} = 1$ and hence $\sigma_k(A) = x_k$.

Case 3. If $x_k = 2$;

Algorithm ensures that $a_{ki} = a_{(k-m)k} = 1$ for each $1 \leq i \leq m$. This follows that $\sigma_k(A) = x_k$.
So the induction statement holds, and then $\sigma(A) = x$.

It remains to prove that $A \in \mathcal{TR}(n)$.
Define

$$I_0 = \{1 \leq i \leq n : x_i = 0\},$$

$$I_1 = \{1 \leq i \leq n : x_i = 1\},$$

and

$$I_2 = \{1 \leq i \leq n : x_i = 2\}.$$

We see that

$$n = \mathrm{card}(\mathrm{I}_0) + \mathrm{card}(\mathrm{I}_1) + \mathrm{card}(\mathrm{I}_2). \tag{2.1}$$

Step 1. $\mathrm{card}(\mathrm{I}_0) = 0$;

As $x \prec^* e$, we have

$$\begin{aligned} n &= \textstyle\sum_{i=1}^{n} x_i \\ &= \mathrm{card}(\mathrm{I}_1) + 2\mathrm{card}(\mathrm{I}_2). \end{aligned}$$

The relation (2.1) shows that $\mathrm{card}(\mathrm{I}_2) = 0$, and so $x = e$. Algorithm states that $A = I_n$, and then $A \in \mathcal{TR}(n)$.

Step 2. $\mathrm{card}(\mathrm{I}_0) \neq 0$;

If $\mathrm{card}(\mathrm{I}_1) = 0$; then $x = (0, \ldots, 0, 2, \ldots, 2)$. Since $x \prec^* e$, this implies that $n = 2\mathrm{card}(\mathrm{I}_2)$, and then $\mathrm{card}(\mathrm{I}_2) = \mathrm{card}(\mathrm{I}_0) = \frac{\mathrm{n}}{2}$. Now, by applying Algorithm for x, we observe that $A \in \mathcal{TR}(n)$.

If $\mathrm{card}(\mathrm{I}_1) \neq 0$; if $\mathrm{card}(\mathrm{I}_2) = 0$, then, since $x \prec^* e$, we obtain a contradiction. So $\mathrm{card}(\mathrm{I}_2) \neq 0$. By the relation (2.1) and $x \prec^* e$, we have

$$n = \mathrm{card}(\mathrm{I}_1) + 2\mathrm{card}(\mathrm{I}_2) \quad \Rightarrow \quad \mathrm{card}(\mathrm{I}_0) = \mathrm{card}(\mathrm{I}_2).$$

Let m be the number 2's. So $\mathrm{card}(\mathrm{I}_0) = \mathrm{m}$. Algorithm ensures that

$$A = \begin{pmatrix} & & & & 1 & & & \\ & & & & & 1 & 0 & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ 1 & & & 0 & & & & \\ & 1 & & & & & & \\ & 0 & \ddots & & & & & \\ & & & & 1 & & & \end{pmatrix},$$

where the entries of the positions $(1, m+1), (2, m+2), \ldots, (n-2m, n-m), (n-2m, n-m), (n-2m+1, n-m+1), \ldots, (n-m, n), (n-m+1, 1), \ldots, (n, m)$ are 1 and the other entries are zero. This means that $A \in \mathcal{TR}(n)$, as desired. □

Let $x \in \sigma(\mathcal{TR}(n))$. Algorithm constructs an integral row stochastic toeplitz matrix A with $\sigma(A) = x$. This A is shown by $A(x)$ and it is called the canonical integral row stochastic toeplitz matrix with L-ray x.

Example 2.4. (i) Consider $x = (0, 1, 1, 2)$. We observe that $x \prec^* e$. Algorithm constructs the following matrix.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) Let $x = (0, 0, 0, 2, 2, 2)$. Then $x \prec^* e$ and the matrix constructed by Algorithm is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

(iii) Let $x = (0, 0, 0, 1, 1, 2, 2, 2)$. So $x \prec^* e$ and the desired matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(iv) If $x = (1, 2, 0)$, then we can not use the Algorithm.

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