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n-weak amenability of a certain class of function spaces

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Abstract

Let A be a non-zero normed vector space and let φ be a non-zero element of A^* such that $\|\varphi\| \leq 1$. Assume that $K = \overline{B_1^{(0)}}$ is the closed unit ball of A. According to the our recent studies on the spaces of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$, generated by $C^b(K)$ and equipped with a new product " $\cdot$ " and different norms $\|\cdot\|_\infty$ and $\|\cdot\|_\varphi$, the $n-$weak amenability of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ are investigated.

1. Introduction

Assume that B is a Banach algebra and X is a Banach $B-$bimodule. A bounded linear map $D : B \longrightarrow X$ is said to be a derivation if $D(ab) = D(a)b + aD(b)$ for all $a, b \in B$. Clearly the

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mapping $\delta_x : B \longrightarrow X$ defined by $\delta_x(b) = bx - xb, b \in B$ is a derivation for all $x \in X$, that is called an inner derivation. A derivation $D : B \longrightarrow X$ is called inner, if $D = \delta_x$ for some $x \in X$.

Let B be a Banach algebra and let n be a non-negative integer. The $n^{th}-$dual $B^{(n)}$ of B is a Banach $B-$bimodule under the module operations defined inductively by

$$\langle G \cdot b, g\rangle = \langle G, b \cdot g\rangle, \langle b \cdot G, g\rangle = \langle g \cdot b, G\rangle, G \in B^{(n)}, g \in B^{(n-1)}, b \in B = B^{(0)}.$$

Obviously B is a Banach $B-$bimodule under its multiplication.
A Banach algebra B is said to be $n-$weakly amenable if every derivation from B into $B^{(n)}$ is inner. The concept of $n-$weak amenability was initiated and intensively studied in [3]. Of course, $1-$weak amenability and weak amenability are the same notions which was first introduced in [1] for commutative Banach algebras and was followed in [4] for non-commutative case .
A Banach algebra B is said to be amenable if for each Banach $B-$bimodule X, every derivation from B into X^* is inner.

In this paper let A be a non-zero normed vector space and let φ be a non-zero element of A^* such that $\|\varphi\| \leq 1$. Let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. We will consider $C^b(K)$ the space of all complex-valued, bounded and continuous functions on K. Obviously $C^b(K)$ is a unital algebra with respect to the pointwise algebraic operations. We will denote by 1_K the identity of $C^b(K)$. The uniform norm on K is defined by $\|f\|_\infty = \sup\{|f(x)| \mid x \in K\}$ for all $f \in C^b(K)$. Clearly $(C^b(K), \|\cdot\|_\infty)$ is a commutative, unital Banach algebra. It is obvious that $\|\varphi\|_\infty = \|\varphi\|$.

By [Examples 3.2.2 (i), 2], $(C^b(K), \|\cdot\|_\infty)$ is a commutative C^*-algebra. Also it is well-known that every commutative C^*-algebra is amenable [Example 2.3.4, 10]. So $(C^b(K), \|\cdot\|_\infty)$ is an amenable Banach algebra.

Let $f, g \in C^b(K)$ and define $f \cdot g = f\varphi g$. The space $C^b(K)$ equipped with the product " $\cdot$ " make $C^b(K)$ into a new associative algebra that we denote it by $C^{b\varphi}(K)$. In [7] we show that $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is a non-unital, commutative Banach algebra and also we characterize some relations between character spaces of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^b(K), \|\cdot\|_\infty)$. Also miscellaneous algebraic properties of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ are investigate in [7].

In [5] for a Banach algebra A, R. A. Kamyabi-Gol and M. Janfada defined a new product " $\cdot$ " on A by, $a \cdot c = a\varepsilon c$ for all $a, c \in A$, where ε is a fixed element of the closed unit ball $\overline{B_1^{(0)}}$ of A. $(A, \cdot)$ is an associative Banach algebra which is denoted by A_ε. Some properties such as, Arens regularity, amenability and derivations on A_ε are investigated in [5]. Also biflatness, biprojectivity, $\varphi-$amenability and $\varphi-$contractibility of A_ε are investigated in [6]. It is worth pointing out that $(C^{b\varphi}(K), \|\cdot\|_\infty) = (C^b(K), \|\cdot\|_\infty)_\varphi$.

Let $n \in \mathbb{N} \cup \{0\}, \Lambda \in (C^{b\varphi}(K), \|\cdot\|_\infty)^{(n)}$ and $f \in (C^{b\varphi}(K), \|\cdot\|_\infty)$. Since $C^{b\varphi}(K)$ is commutative, the $(C^{b\varphi}(K), \|\cdot\|_\infty)-$module operations on $(C^{b\varphi}(K), \|\cdot\|_\infty)^{(n)}$ are given by $\Lambda \cdot f = f \cdot \Lambda = \Lambda f\varphi$.

The space $C^{b\varphi}(K)$ with the norm $\|f\|_\varphi = \|f\varphi\|_\infty, f \in C^{b\varphi}(K)$ is a non-complete normed algebra [8] and also, $\|f\|_\varphi \leq \|f\|_\infty\|\varphi\|$. Similarly, the $(C^{b\varphi}(K), \|\cdot\|_\varphi)-$module operations on $(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(n)}$ are given by $\Lambda \cdot f = f \cdot \Lambda = \Lambda f\varphi$ for all $\Lambda \in (C^{b\varphi}(K), \|\cdot\|_\varphi)^{(n)}, f \in (C^{b\varphi}(K), \|\cdot\|_\varphi), n \in \mathbb{N} \cup \{0\}$. Clearly $(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(n)}$ is a Banach $(C^{b\varphi}(K), \|\cdot\|_\varphi)-$bimodule for all $n \in \mathbb{N}$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ is a normed $(C^{b\varphi}(K), \|\cdot\|_\varphi)-$bimodule. In [9] we characterize the derivations from

$(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ into $(C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)}$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(1)}$ respectively. Also weak and cyclic amenability of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ are investigated in [9].

The results of this paper concerning the spaces of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ can be applied as a source of examples and counterexamples in the field of amenability and $n-$weak amenability.

2. $n-$weak amenability of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$

In this section we characterize the derivations from $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ into $(C^{b\varphi}(K), \|\cdot\|_\infty)^{(n)}$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(n)}$ respectively and also we investigate the $n-$weak amenability of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ for all $n \in \mathbb{N} \cup \{0\}$.

We set $(C^{b\varphi}(K), \|\cdot\|_\infty)^{(n)}$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(n)}$ as the n^{th} dual spaces of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ with the norms $\|\cdot\|_\infty^{(n)}$ and $\|\cdot\|_\varphi^{(n)}$ respectively, where

$$(C^{b\varphi}(K), \|\cdot\|_\infty)^{(0)} = (C^{b\varphi}(K), \|\cdot\|_\infty), \quad \|\cdot\|_\infty^{(0)} = \|\cdot\|_\infty,$$
$$(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(0)} = (C^{b\varphi}(K), \|\cdot\|_\varphi), \quad \|\cdot\|_\varphi^{(0)} = \|\cdot\|_\varphi.$$

Recall that $\|\varphi\|_\infty = \|\varphi\|$ and also $\|f\|_\varphi \le \|f\|_\infty \|\varphi\|$ for all $f \in C^{b\varphi}(K)$.
The mapping $\hat{x} : C^b(K) \longrightarrow \mathbb{C}$ defined by $\langle \hat{x}, f \rangle = f(x), f \in C^b(K)$ is a linear functional. Clearly $\|\hat{x}\|_\infty^{(1)} \le 1$ for all $x \in K$. Also $\|\hat{x}\|_\varphi^{(1)} \le \frac{1}{|\varphi(x)|}$ for all $x \in K \setminus \ker\varphi$.
The following theorem generalizes Theorem 3.2 of [9].

Theorem 2.1. *Let $n \in \mathbb{N} \cup \{0\}$. Also let $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^{(n)}$ be a bounded linear map. Then D is a derivation if and only if $D(f\varphi) = fD(\varphi) = 2D(f)\varphi$ for all $f \in C^{b\varphi}(K)$.*

Proof. The same proof of Theorem 3.2 given in [9] remains valid. □

Corollary 2.2. *Let $n \in \mathbb{N} \cup \{0\}$. Also let $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^{(n)}$ be a derivation. Then $D(f)\varphi^2 = 0$ for all $f \in C^{b\varphi}(K)$.*

Proof. By Theorem 2.1 we have,

$$D(f\varphi) = fD(\varphi) = 2D(f)\varphi \tag{2.1}$$

for all $f \in C^{b\varphi}(K)$. Replacing f by $f\varphi$ in (2.1) we obtain,

$$D(f\varphi^2) = f\varphi D(\varphi) = 2D(f\varphi)\varphi. \tag{2.2}$$

So,

$$\begin{aligned} f\varphi D(\varphi) &= 2D(f\varphi)\varphi \\ &= 2(2D(f)\varphi)\varphi \\ &= 4D(f)\varphi^2, f \in C^{b\varphi}(K). \end{aligned}$$

Hence,

$$f\varphi D(\varphi) = 4D(f)\varphi^2, f \in C^{b\varphi}(K). \tag{2.3}$$

Also by (2.1) we can conclude that,

$$f\varphi D(\varphi) = 2D(f)\varphi^2, f \in C^{b\varphi}(K). \tag{2.4}$$

Comparing (2.3) and (2.4) we obtain $D(f)\varphi^2 = 0$ for all $f \in C^{b\varphi}(K)$, as we wanted to show. □

Theorem 2.3. *The only derivation from $(C^{b\varphi}(K), \|\cdot\|_\infty)$ into $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is zero.*

Proof. Let $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)$ be a derivation. So by Corollary 2.2 we have $D(f)\varphi^2 = 0$ for all $f \in C^{b\varphi}(K)$. Applying [8, Proposition 2.1] we obtain $D(f) = 0$ for all $f \in C^{b\varphi}(K)$. So $D = 0$, as desired. □

The following theorem generalizes Theorem 3.1 of [9].

Theorem 2.4. *$(C^{b\varphi}(K), \|\cdot\|_\infty)$ is not $(2n-1)-$weakly amenable for all $n \in \mathbb{N}$.*

Proof. Since $(C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)} \subseteq (C^{b\varphi}(K), \|\cdot\|_\infty)^{(2n-1)}$ for all $n \in \mathbb{N}$, inspired by [Theorem 3.1, 9] the theorem can be proved.

□

Lemma 2.5. *Let $\{\Lambda_n\}_n \subseteq (C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)}$ be a sequence such that $\Lambda_n\varphi \xrightarrow{\|\cdot\|_\infty^{(1)}} \hat{0}$. Then $\{\Lambda_n\}_n$ is bounded.*

Proof. Suppose the assertion of the lemma is false. So there exists a subsequence $\{\Lambda_{n_j}\}_j$ of $\{\Lambda_n\}_n$ such that $\lim_{j\longrightarrow\infty} \|\Lambda_{n_j}\|_\infty^{(1)} = \infty$.
Define $f_j : K \longrightarrow \mathbb{C}$ by $f_j(x) = \frac{1-|\varphi(x)|}{1+(\|\Lambda_{n_j}\|_\infty^{(1)})^2|\varphi(x)|}$, $x \in K$. One can easily verify that, $\|f_j\|_\infty \leq 1$ and $\|f_j\varphi\|_\infty \leq \frac{1}{(\|\Lambda_{n_j}\|_\infty^{(1)})^2}$ for all $j \in \mathbb{N}$. It follows that $\lim_{j\longrightarrow\infty} \|f_j\varphi\|_\infty = 0$ and

$$\begin{aligned}
|\langle \Lambda_{n_j}\varphi, f_j\rangle| &= |\langle \Lambda_{n_j}, f_j\varphi\rangle| \\
&\leq \|\Lambda_{n_j}\|_\infty^{(1)} \|f_j\varphi\|_\infty \\
&\leq \|\Lambda_{n_j}\|_\infty^{(1)} \frac{1}{(\|\Lambda_{n_j}\|_\infty^{(1)})^2} \\
&= \frac{1}{\|\Lambda_{n_j}\|_\infty^{(1)}}.
\end{aligned}$$

Hence,

$$\lim_{j\longrightarrow\infty} \langle \Lambda_{n_j}\varphi, f_j\rangle = 0. \tag{2.5}$$

Also,

$$\begin{aligned}
|\langle \Lambda_{n_j}\varphi, f_j\rangle - 1| &= |\langle \Lambda_{n_j}\varphi, f_j\rangle - \langle \hat{0}, f_j\rangle| \\
&= |\langle \Lambda_{n_j}\varphi - \hat{0}, f_j\rangle| \\
&\leq \|\Lambda_{n_j}\varphi - \hat{0}\|_\infty^{(1)} \|f_j\|_\infty \\
&\leq \|\Lambda_{n_j}\varphi - \hat{0}\|_\infty^{(1)}.
\end{aligned}$$

Since $\lim_{j\longrightarrow\infty} \|\Lambda_{n_j}\varphi - \hat{0}\|_\infty^{(1)} = 0$ so,

$$\lim_{j\longrightarrow\infty} \langle \Lambda_{n_j}\varphi, f_j\rangle = 1. \tag{2.6}$$

Comparing (2.5) with (2.6) shows a contradiction. □

Theorem 2.6. *Let* $W = \overline{\left\{\Lambda\varphi \;\middle|\; \Lambda \in (C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)}\right\}}^{\|\cdot\|_\infty^{(1)}}$. *Then W is a proper closed subspace of* $(C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)}$.

Proof. Obviously W is a closed subspace of $(C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)}$. We shall show that W is proper. To this end, we will prove that $\hat{0} \notin W$. Suppose, contrary to our claim, that $\hat{0} \in W$. So there exists a sequence $\{\Lambda_n\}_n \subseteq (C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)}$ such that $\Lambda_n\varphi \xrightarrow{\|\cdot\|_\infty^{(1)}} \hat{0}$. By Lemma 2.5 $\{\Lambda_n\}_n$ is bounded. Let $\|\Lambda_n\|_\infty^{(1)} \leq M$ for all $n \in \mathbb{N}$. Define $f_n : K \longrightarrow \mathbb{C}$ by $f_n(x) = \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}$ for all $x \in K$. Clearly $\|f_n\|_\infty \leq 1$. One can easily check that $\|f_n\varphi\|_\infty \leq \frac{1}{n}$ and consequently $\|f_n\varphi\|_\infty \longrightarrow 0$. So, on the one hand,

$$\begin{aligned}
|\langle \Lambda_n\varphi, f_n\rangle - 1| &= |\langle \Lambda_n\varphi, f_n\rangle - \langle \hat{0}, f_n\rangle| \\
&= |\langle \Lambda_n\varphi - \hat{0}, f_n\rangle| \\
&\leq \|\Lambda_n\varphi - \hat{0}\|_\infty^{(1)}\|f_n\|_\infty \\
&\leq \|\Lambda_n\varphi - \hat{0}\|_\infty^{(1)} \\
&\longrightarrow 0,
\end{aligned}$$

that implies,

$$\langle \Lambda_n\varphi, f_n\rangle \longrightarrow 1. \tag{2.7}$$

On the other hand,

$$\begin{aligned}
|\langle \Lambda_n\varphi, f_n\rangle| &= |\langle \Lambda_n, f_n\varphi\rangle| \\
&\leq \|\Lambda_n\|_\infty^{(1)}\|f_n\varphi\|_\infty \\
&\leq M\|f_n\varphi\|_\infty \\
&\longrightarrow 0,
\end{aligned}$$

that implies,

$$\langle \Lambda_n\varphi, f_n\rangle \longrightarrow 0. \tag{2.8}$$

Comparing (2.7) with (2.8) yields a contradiction. So $\hat{0} \notin W$. □

Theorem 2.7. $(C^{b\varphi}(K), \|\cdot\|_\infty)$ *is not* $2n$*-weakly amenable for all* $n \in \mathbb{N}$.

Proof. By applying [Proposition 2.8.76, 2] it is sufficient to show that $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is not 2-weakly amenable.

For this purpose, let $W = \overline{\left\{\Lambda\varphi \;\middle|\; \Lambda \in (C^{b\varphi}(K), \|\cdot\|_\infty)^{(1)}\right\}}^{\|\cdot\|_\infty^{(1)}}$. By Theorem 2.6, $\hat{0} \notin W$. So

there exists an element $m \in (C^{b\varphi}(K), \|\cdot\|_\infty)^{(2)}$ such that $m\Big|_W = 0$ and $\langle m, \hat{0}\rangle \neq 0$. Define, $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^{(2)}$ by, $D(f) = f(0)m$ for all $f \in (C^{b\varphi}(K), \|\cdot\|_\infty)$. Obviously D is linear and $D \neq 0$. Indeed,
$D(1_K) = 1_K(0)m = m \neq 0$. Also, for $f \in C^{b\varphi}(K)$ we have,

$$\begin{aligned} \|D(f)\|_\infty^{(2)} &= \|f(0)m\|_\infty^{(2)} \\ &= |f(0)|\|m\|_\infty^{(2)} \\ &\leq \|m\|_\infty^{(2)}\|f\|_\infty. \end{aligned}$$

So $\|D\| \leq \|m\|_\infty^{(2)}$. This shows that D is bounded. We shall show that D is a derivation. Let $f, g \in C^{b\varphi}(K)$. So,

$$\begin{aligned} D(f \cdot g) &= D(f\varphi g) \\ &= (f\varphi g)(0)m \\ &= f(0)\varphi(0)g(0)m \\ &= 0, \end{aligned}$$

also,

$$\begin{aligned} D(f) \cdot g + f \cdot D(g) &= D(f)g\varphi + D(g)f\varphi \\ &= (f(0)m)g\varphi + (g(0)m)f\varphi \\ &= f(0)mg\varphi + g(0)mf\varphi. \end{aligned}$$

Since $m\Big|_W = 0$, clearly $mg\varphi = mf\varphi = 0$. So, we can conclude that,
$D(f) \cdot g + f \cdot D(g) = 0$. This shows that $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$ for all $f, g \in C^{b\varphi}(K)$. Since $C^{b\varphi}(K)$ is commutative and $D \neq 0$, D is not inner. Hence, $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is not 2-weakly amenable. □

Corollary 2.8. *$(C^{b\varphi}(K), \|\cdot\|_\infty)$ is not $n-$weakly amenable for all $n \in \mathbb{N}$.*

Proof. The proof is immediate by Theorems 2.4 and 2.7. □

To investigate the $n-$weak amenability of $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ we need to characterize the derivations from $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ into $(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(n)}$ for all $n \in \mathbb{N} \cup \{0\}$. To this end, we present the following lemma that is a generalization of Propositions 2.1 and 2.3 of [9].

Lemma 2.9. *Let $n \in \mathbb{N} \cup \{0\}$. Then*

1.

$$(C^{b\varphi}(K), \|\cdot\|_\infty)^{(2n)} \subseteq (C^{b\varphi}(K), \|\cdot\|_\varphi)^{(2n)}, \tag{2.9}$$
$$(C^{b\varphi}(K), \|\cdot\|_\varphi)^{(2n+1)} \subseteq (C^{b\varphi}(K), \|\cdot\|_\infty)^{(2n+1)}. \tag{2.10}$$

Moreover,

$$\|m\|_\varphi^{(2n)} \leq \|m\|_\infty^{(2n)}\|\varphi\|, m \in (C^{b\varphi}(K), \|\cdot\|_\infty)^{(2n)}, \tag{2.11}$$
$$\|\Lambda\|_\infty^{(2n+1)} \leq \|\Lambda\|_\varphi^{(2n+1)}\|\varphi\|, \Lambda \in (C^{b\varphi}(K), \|\cdot\|_\varphi)^{(2n+1)}. \tag{2.12}$$

2.

$$(C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2n)}\varphi \subseteq (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2n)}, \tag{2.13}$$

$$(C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2n+1)}\varphi \subseteq (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2n+1)}. \tag{2.14}$$

Moreover,

$$\|m\varphi\|_{\infty}^{(2n)} \leq \|m\|_{\varphi}^{(2n)}, m \in (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2n)}, \tag{2.15}$$

$$\|\Lambda\varphi\|_{\varphi}^{(2n+1)} \leq \|\Lambda\|_{\infty}^{(2n+1)}, \Lambda \in (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2n+1)}. \tag{2.16}$$

Proof. The proof is by induction on n. We first prove that all of the assertions of the lemma is valid for $n = 0$. The proofs of (2.9), (2.11), (2.13), and (2.15) are obvious and are left for the reader. Let $\Lambda \in (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(1)}, \Lambda' \in (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(1)}$ and $f, g \in C^{b\varphi}(K)$. So,

$$\begin{aligned} |\langle \Lambda, f\rangle| &\leq \|\Lambda\|_{\varphi}^{(1)}\|f\|_{\varphi} \\ &= \|\Lambda\|_{\varphi}^{(1)}\|f\varphi\|_{\infty} \\ &\leq \|\Lambda\|_{\varphi}^{(1)}\|f\|_{\infty}\|\varphi\|_{\infty} \\ &= \|\Lambda\|_{\varphi}^{(1)}\|f\|_{\infty}\|\varphi\|. \end{aligned}$$

Hence, $\|\Lambda\|_{\infty}^{(1)} \leq \|\Lambda\|_{\varphi}^{(1)}\|\varphi\|$, providing (2.10) and (2.12). Also,

$$\begin{aligned} |\langle \Lambda'\varphi, g\rangle| &= |\langle \Lambda', g\varphi\rangle| \\ &\leq \|\Lambda'\|_{\infty}^{(1)}\|g\varphi\|_{\infty} \\ &= \|\Lambda'\|_{\infty}^{(1)}\|g\|_{\varphi}. \end{aligned}$$

Therefore $\|\Lambda'\varphi\|_{\varphi}^{(1)} \leq \|\Lambda'\|_{\infty}^{(1)}$, providing (2.14) and (2.16).
Assume all of the assertions of the lemma hold for $n = p$, we will prove them for $n = p + 1$. For this purpose, let,

$$\begin{aligned} m &\in (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2p+2)}, \\ \Lambda &\in (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2p+1)}, \\ \Lambda' &\in (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2p+3)}, \\ m' &\in (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2p+2)}, \\ \Lambda'' &\in (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2p+1)}, \\ \Lambda''' &\in (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2p+3)}. \end{aligned}$$

By (2.10) and (2.12) of hypotheses we have, $\Lambda \in (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2p+1)}$ and $\|\Lambda\|_{\infty}^{(2p+1)} \leq \|\Lambda\|_{\varphi}^{(2p+1)}\|\varphi\|$. So,

$$\begin{aligned} |\langle m, \Lambda\rangle| &\leq \|m\|_{\infty}^{(2p+2)}\|\Lambda\|_{\infty}^{(2p+1)} \\ &\leq \|m\|_{\infty}^{(2p+2)}\|\Lambda\|_{\varphi}^{(2p+1)}\|\varphi\|. \end{aligned}$$

It follows that, $\|m\|_{\varphi}^{(2p+2)} \leq \|m\|_{\infty}^{(2p+2)}\|\varphi\|$, providing (2.9) and (2.11) for $n = p + 1$.
Since (2.9) and (2.11) are valid for $n = p+1$, $m \in (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2p+2)}$ and $\|m\|_{\varphi}^{(2p+2)} \leq \|m\|_{\infty}^{(2p+2)}\|\varphi\|$.
Hence,

$$\begin{aligned}|\langle \Lambda', m\rangle| &\leq \|\Lambda'\|_{\varphi}^{(2p+3)}\|m\|_{\varphi}^{(2p+2)}\\ &\leq \|\Lambda'\|_{\varphi}^{(2p+3)}\|m\|_{\infty}^{(2p+2)}\|\varphi\|,\end{aligned}$$

that implies, $\|\Lambda'\|_{\infty}^{(2p+3)} \leq \|\Lambda'\|_{\varphi}^{(2p+3)}\|\varphi\|$, providing (2.10) and (2.12) for $n = p + 1$.
Since by (2.14) and (2.16) of hypotheses, $\Lambda''\varphi \in (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2p+1)}$ and $\|\Lambda''\varphi\|_{\varphi}^{(2p+1)} \leq \|\Lambda''\|_{\infty}^{(2p+1)}$, we can define,
$m'\varphi : (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2p+1)} \longrightarrow \mathbb{C}$ by, $\langle m'\varphi, \Lambda''\rangle = \langle m', \Lambda''\varphi\rangle$. Obviously $m'\varphi$ is linear. We will prove that $m'\varphi$ is bounded on $(C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2p+1)}$.

$$\begin{aligned}|\langle m'\varphi, \Lambda''\rangle| &= |\langle m', \Lambda''\varphi\rangle|\\ &\leq \|m'\|_{\varphi}^{(2p+2)}\|\Lambda''\varphi\|_{\varphi}^{(2p+1)}\\ &\leq \|m'\|_{\varphi}^{(2p+2)}\|\Lambda''\|_{\infty}^{(2p+1)},\end{aligned}$$

that implies, $\|m'\varphi\|_{\infty}^{(2p+2)} \leq \|m'\|_{\varphi}^{(2p+2)}$, providing (2.13) and 2.15 for $n = p + 1$.
Since (2.13) and (2.15) are valid for $n = p + 1$, $m'\varphi \in (C^{b\varphi}(K), \|\cdot\|_{\infty})^{(2p+2)}$ and $\|m'\varphi\|_{\infty}^{(2p+2)} \leq \|m'\|_{\varphi}^{(2p+2)}$. Hence we can define,
$\Lambda'''\varphi : (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(2p+2)} \longrightarrow \mathbb{C}$ by, $\langle \Lambda'''\varphi, m'\rangle = \langle \Lambda''', m'\varphi\rangle$. It follows that,

$$\begin{aligned}|\langle \Lambda'''\varphi, m'\rangle| &= |\langle \Lambda''', m'\varphi\rangle|\\ &\leq \|\Lambda'''\|_{\infty}^{(2p+3)}\|m'\varphi\|_{\infty}^{(2p+2)}\\ &\leq \|\Lambda'''\|_{\infty}^{(2p+3)}\|m'\|_{\varphi}^{(2p+2)}.\end{aligned}$$

So, $\|\Lambda'''\varphi\|_{\varphi}^{(2p+3)} \leq \|\Lambda'''\|_{\infty}^{(2p+3)}$, providing (2.14) and (2.16) for $n = p + 1$. □

The following theorem generalizes Theorem 4.2 of [9].

Theorem 2.10. *Let $n \in \mathbb{N} \cup \{0\}$ and let $D : (C^{b\varphi}(K), \|\cdot\|_{\varphi}) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_{\varphi})^{(n)}$ be a bounded linear map. Then D is a derivation if and only if $D = 0$.*

Proof. One can prove the theorem by applying Lemma 2.9 and by modifying the proof of [Theorem 4.2, 9].

□

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