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N-strongly quasi-invariant measure on double coset spaces

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Abstract

Let G be a locally compact group, H and K be two closed subgroups of G, and N be the normalizer group of K in G. In this paper, the existence and properties of a rho-function for the triple (K, G, H) and an N-strongly quasi-invariant measure of double coset space $K\backslash G/H$ is investigated. In particular, it is shown that any such measure arises from a rho-function. Furthermore, the conditions under which an N-strongly quasi-invariant measure arises from a rho-function are studied.

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1. Introduction

Let G be a locally compact group and H and K be closed subgroups of G. The double coset space of G by H and K respectively, is

$$K\backslash G/H = \{KxH;\ x \in G\},$$

which introduced by Liu in [13].
When K is trivial, a double coset $K\backslash G/H$ changes to a homogeneous space G/H. The existence of quasi-invariant measures on homogeneous spaces G/H (with merely measurable rho-functions) was first proved by Mackey [15] under the assumption that G is second countable. Bruhat [3] and Loomis [14] showed how to obtain strongly quasi-invariant measures with no countability hypotheses. This work is extended in a special case in [6]. Also, the existence of a homomorphism rho-function causes the existence of a relatively invariant measure on G/H is in [16]. One may refer to [16, 8, 1, 9, 10, 11] to find more informations about homogeneous space G/H.
When $K = H$, a double coset space $K\backslash G/H$ changes to a hypergroup in which the homogeneous space G/H is a semi hypergroup [12]. It is worthwhile to note that the hypergroup plays important role in physics.

In this paper, we construct an N-strongly quasi-invariant measure on $K\backslash G/H$ when H and K are closed subgroups, not necessarily compact. Also we investigate when K is a normal closed subgroup of G then $K\backslash G/H$ possesses a G-strongly quasi-invariant measure. In addition, when H is trivial we show the existence of an N-strongly quasi-invariant measure on the right cosets of K in G.
It is worth mentioning that in [7] the conditions for the existence of N-relatively invariant measures and N-invariant measures are investigated.
Some preliminaries and notations about coset space $K\backslash G/H$ and related measures on it are stated in Section. 2.
In Section. 3, we construct a rho-function for the triple (K, G, H) and introduce an N-strongly quasi-invariant measure which arises from this rho-function.
In particular, we obtain in Section. 4., conditions under which an N-strongly quasi-invariant measure arises from a rho-function.

2. Notations and preliminaries

Let G be a locally compact Hausdorff group and let H and K be closed subgroups of G. Throughout this paper, we denote the left Haar measures on G, H and K respectively, by dx, dh, dk, and their modular functions by Δ_G, Δ_H and Δ_K, respectively. If S is a locally compact Hausdorff space, a (left) action of G on S is a continuous map $(x, s) \mapsto xs$ from $G \times S$ to S such that (i) $s \to xs$ is a homeomorphism of S for each $x \in G$, and (ii) $x(ys) = (xy)s$ for all $x, y \in G$ and $s \in S$. A space S equipped with an action of G is called a G-space. A G-space S is called transitive if for every $s, t \in S$ there exists $x \in G$ such that $xs = t$.
The standard examples of transitive G-spaces are the quotient spaces G/H (where H is a closed subgroup of G), equipped with the quotient topology on which G acts by left multiplication. We

shall use the term homogeneous space to mean a transitive space S that is isomorphic to a quotient space G/H. In homogeneous space G/H, if μ is a positive Radon measure on G/H, Borel set E is called negligible with respect to μ, if $\mu(E) = 0$. Let μ_x denote its transfer by $x \in G$, that is $\mu_x(E) = \mu(x \cdot E)$ for any Borel set $E \subseteq G/H$. μ is called strongly quasi invariant if there is a positive continuous function λ on $G \times G/H$ such that $d\mu_x(yH) = \lambda(x, yH)d\mu(yH)$, for all $x, y \in G$. A rho-function for the pair (G, H) is defined to be a positive locally integrable function ρ on G which satisfies

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x), \quad (x \in G,\ h \in H).$$

It is known that for each pair (G, H) there is a strictly positive rho-function which constructs a strongly quasi-invariant measure μ on G/H such that

$$\int_G f(x)\rho(x)dx = \int_{\frac{G}{H}} \int_H f(xh)dhd\mu(xH), \tag{2.1}$$

for all $f \in C_c(G)$, the space of all continuous functions on G with compact supports.
And conversely, each strongly quasi-invariant measure on G/H arises from a rho-function which satisfies (2.1) for a rho-function ρ, and all such measures are strongly equivalent. That is to say, all strongly quasi-invariant measures on G/H have the same negligible sets (see [8, 16]).
The notion of double coset space is a natural generalization of that of coset space arising by two subgroups, simultaneously. Recall that if $K\backslash G/H$ is a double coset space of G by H and K, then elements of $K\backslash G/H$ are given by $\{KxH;\ x \in G\}$.
The canonical mapping of which, is $q : G \to K\backslash G/H$, defined by $q(x) = KxH$, which is abbreviated by $\ddot{x}$, and which is surjective. The double coset space $K\backslash G/H$ equipped with the quotient topology, which is the largest topology, that makes q continuous. In this topology q is also an open mapping and **proper**–that is for each compact set $F \subseteq K\backslash G/H$ there is a compact set $E \subseteq G$ with $q(E) = F$. Based on the above mentioned case, $K\backslash G/H$ is a locally compact and Hausdorff space.
Let N be the normalizer of K in G, i.e.,

$$N = \{g \in G;\ gK = Kg\}.$$

Then, there is a naturally defined mapping

$$\varphi : N \times K\backslash G/H \to K\backslash G/H$$

given by

$$\varphi(n, q(x)) := KnxH.$$

It can be verified that φ is a well-defined, continuous, transitive action of N on $K\backslash G/H$. Considering $K\backslash G/H$ with this transitive action, we now denote $\varphi(n, q(x))$ by $n \cdot q(x)$.
We define the mapping Q from $C_c(G)$ to $C_c(K\backslash G/H)$ by

$$Q(f)(KxH) = \int_K \int_H f(k^{-1}xh)dhdk.$$

It is evident that Q is a well-defined continuous linear map, as well as $supp(Q(f)) \subseteq q(supp\ (f))$. In the following, the properties of this mapping is investigated. However, we first recall that the

definition of IN-group and verification of a property of it is used in the sequel.
A locally compact group G is called an IN-group if there is a compact unit neighbourhood U in G which is invariant under inner automorphism, that is, for any $x \in G$, $xUx^{-1} = U$. It is known that the IN-groups are unimodular.

Lemma 2.1. *If K is also an IN-group, then $\int_K f(k)dk = \int_K f(nkn^{-1})dk$, for all $f \in C_c(K)$ and $n \in N$.*

Proof. Let for $n \in N$, $\lambda_n : C_c(K) \to \mathbb{C}$ be given by

$$\lambda_n(f) = \int_K f(nkn^{-1})dk.$$

Then for every $t \in K$, we have

$$\lambda_n(L_{t^{-1}} f) = \int_K L_{t^{-1}} f(nkn^{-1})dk = \int_K f(tnkn^{-1})dk = \int_K f(nkn^{-1})dk.$$

This shows that λ_n is left invariant, so it induced a left Haar measure λ_n on K. Therefore, there is $c > 0$ such that

$$\int_K f(k)d\lambda_n(k) = c \int_K f(k)dk.$$

Since K is an IN-group, then there is a compact unit neighbourhood U in K such that $xUx^{-1} = U$ for all $x \in K$. Thus, $|n^{-1}Un| \leq |U|$ for each $n \in N$, where $|U|$ denotes the measure of U. Therefore, we can write

$$\begin{aligned} |c-1||U| &= \big||c|U| - |U|\big| \\ &= \big|\lambda_n(U) - |U|\big| = \big||nUn^{-1}| - |U|\big| \\ &\leq \big||U| - |U|\big| = 0. \end{aligned}$$

This implies that $c = 1$. □

Lemma 2.2. *For any compact set $F \subseteq K\backslash G/H$ there exists $f \in C_c^+(G)$ such that $Qf = 1$ on F.*

Proof. Let E be a compact neighbourhood of F in $K\backslash G/H$. Then there exists a compact set F_1 in G such that $q(F_1) = E$. Choose $g \in C_c^+(G)$ such that $g > 0$ on F_1 and using the Urysohn's Lemma, there is $\varphi \in C_c(K\backslash G/H)$ with $supp(\varphi) \subseteq E$ and $\varphi = 1$ on F. Now, set $f = \frac{\varphi \circ q}{(Qg)\circ q} g$. Since $Qg > 0$ on $supp(\varphi)$, then $f \in C_c(G)$ and $supp(f) \subseteq supp(g)$. Also $Q(f) = Q(\varphi \circ q \cdot \frac{g}{(Qg)\circ q}) = \varphi \cdot Q(\frac{g}{Qg\circ q}) = \varphi$. □

Note that for $f \in C_c(G)$ and $y \in G$, we consider $L_y f(x) = f(y^{-1}x)$ and $R_y f(x) = f(xy)$, and for each $n \in N$ and $F \in C_c(K\backslash G/H)$, we define $L_n F(\ddot{x}) = F((n^{-1}x)^{\ddot{}})$ and $R_n f(\ddot{x}) = F((xn)^{\ddot{}})$.

Lemma 2.3. *Given the notation at the beginning of the section, the map $Q : C_c(G) \to C_c(K\backslash G/H)$ has the following properties .*

(i) $Q(C_c(G)) = C_c(K\backslash G/H)$

(ii) If K is also an IN-group, then for each $n \in N$

$$Q(L_n f) = L_n Q(f), \quad f \in C_c(G).$$

Proof. For (i) suppose that $F \in C_c(K\backslash G/H)$. Since q is proper, then there is a compact subset $D \subseteq G$ such that $q(D) = supp\ (F)$. Let $f \in C_c(G)$ be such that $f(d) > 0$ for all $d \in D$. Consider the function f_1 defined on G by

$$f_1(x) = \begin{cases} \frac{f(x)\cdot F(q(x))}{Q(f)(q(x))} & \text{if} \quad Q(f)(q(x)) \neq 0 \\ 0 & \text{if} \quad Q(f)(q(x)) = 0 \end{cases}.$$

Since $Q(f)(q(x)) > 0$ for $x \in q^{-1}((supp\ (F)))$, and $F(q(x)) = 0$, for $x \in G \setminus q^{-1}(supp\ (F))$, which is an open subset in G, $f_1 \in C_c(G)$ and $Q(f_1) = F$.
Finally, for (ii) according to Lemma 2.1, we may state that

$$\begin{aligned} Q(L_n f)(KxH) &= \int_K \int_H f(nk^{-1}xh)dhdk \\ &= \int_H \int_K f(nk^{-1}n^{-1}nxh)dkdh \\ &= \int_H \int_K f(k^{-1}nxh)dkdh \\ &= L_n Q(f)(KxH). \end{aligned}$$

□

Next theorem gives a necessary and sufficient condition for the existence of a positive Radon measure on $K\backslash G/H$.

Theorem 2.4. *If* μ *is a positive Radon measure on* $K\backslash G/H$*, then positive Radon measure* $\tilde{\mu}$ *on* G *is defined by*

$$\int_G f(x)d\tilde{\mu}(x) = \int_{K\backslash G/H} Q(f)(\ddot{x})d\mu(\ddot{x}), \tag{2.2}$$

satisfying

$$\int_G f(kxh^{-1})d\tilde{\mu}(x) = \Delta_K(k)\Delta_H(h) \int_G f(x)d\tilde{\mu}(x). \tag{2.3}$$

Conversely, if a positive Radon measure $\tilde{\mu}$ *on* G *has the property* (2.3)*, then the equation* (2.2) *defines a positive Radon measure* μ *on* $K\backslash G/H$*.*

Proof. Suppose that μ is a positive Radon measure on $K\backslash G/H$, then $\tilde{\mu}$ defined by (2.2) is clearly a positive Radon measure on G. Also, for each $h_0 \in H$, $k_0 \in K$, $f \in C_c(G)$, we have

$$\begin{aligned}
\int_G f(k_0 x h_0^{-1}) d\tilde{\mu}(x) &= \int_{K\backslash G/H} \int_K \int_H L_{k_0^{-1}} \circ R_{h_0^{-1}} f(k^{-1}xh) dhdkd\tilde{\mu}(x) \\
&= \int_{K\backslash G/H} \int_K \int_H f(k_0 k^{-1} x h h_0^{-1}) dhdkd\mu(\ddot{x}) \\
&= \Delta_H(h_0)\Delta_K(k_0) \int_{K\backslash G/H} Q(f)(\ddot{x}) d\mu(\ddot{x}) \\
&= \Delta_H(h_0)\Delta_K(k_0) \int_G f(x) d\tilde{\mu}(x).
\end{aligned}$$

Conversely, suppose that the positive Radon measure $\tilde{\mu}$ on G has the property 2.3, then take

$$\mu : C_c(K\backslash G/H) \to (0, +\infty),$$

by

$$\mu(Q(f)) = \int_G f(x) d\tilde{\mu}.$$

Now we show that μ is well-defined, let $f \in C_c(G)$ such that $Q(f) = 0$. According to Lemma 2.2, there is g in $C_c(G)$ such that $Q(g) \equiv 1$ on $Q(supp\ (f))$. By using the Fubini's Theorem, we have

$$\begin{aligned}
\int_G f(x) d\tilde{\mu} &= \int_G f(x) Q(g(q(x)) d\tilde{\mu}(x) \\
&= \int_G f(x) \int_K \int_H g(k^{-1}xh) dhdkd\tilde{\mu}(x) \\
&= \int_K \int_H \int_G f(kxh^{-1}) g(x) \Delta_K(k)\Delta_H(h) d\tilde{\mu}(x) dhdk \\
&= \int_G g(x) \int_H \int_K f(k^{-1}xh) dkdhd\tilde{\mu}(x) \\
&= \int_G g(x) Q(f)(q(x)) d\tilde{\mu}(x) = 0.
\end{aligned}$$

It is easy to check that μ is a positive linear functional, therefore it induces a positive Radon measure μ on $K\backslash G/H$ such that,

$$\int_G f(x) d\tilde{\mu}(x) = \int_{K\backslash G/H} Q(f)(\ddot{x}) d\mu(\ddot{x}).$$

□

Corollary 2.5. *Considering the assumptions of Theorem 2.4, there is a correspondence between the positive Radon measure $\tilde{\mu}$ on G and μ on the double coset space $G//H$, such that*

$$\int_G f(x) d\tilde{\mu}(x) = \int_{G//H} Q(f)(\ddot{x}) d\mu(\ddot{x})$$

and

$$\int_G f(hxh^{-1})d\tilde{\mu}(x) = \int_G f(x)d\tilde{\mu}(x)$$

for all $f \in C_c(G)$.

3. The existence of N-strongly quasi invariant measure

In this section, we refine and generalize the concept of strongly quasi invariant measure on double coset spaces. Moreover, we investigate the existence of N-strongly quasi-invariant measure on these spaces. We start our work with the following definitions.

Definition 3.1. Let G be a locally compact group and H and K be closed subgroups of it.

For a positive Radon measure μ on $K\backslash G/H$, assume that μ_n is its transfer by $n \in N$, that is, $\mu_n(E) = \mu(n \cdot E)$, for any Borel set $E \subseteq K\backslash G/H$. μ is called

N-strongly quasi invariant if there is a continuous positive function λ on $N \times K\backslash G/H$ such that for all $n \in N$, $d\mu_n(\ddot{y}) = \lambda(n, \ddot{y})d\mu(\ddot{y})$ ($\ddot{y} \in K\backslash G/H$). We call such λ the modular function of μ.

Remark 3.2. Note that if K is normal in G, then the N-strongly quasi-invariant measure μ is the G-strongly quasi invariant on $K\backslash G/H$ and if $K = \{e\}$, μ is the strongly quasi invariant measure on G/H.

Definition 3.3. Suppose that G is a locally compact group and H and K are closed subgroups of G. A rho-function for the triple (K, G, H) is a non-negative locally integrable function ρ on G, which satisfies

$$\rho(kxh) = \frac{\Delta_K(k)\Delta_H(h)}{\Delta_G(h)}\rho(x).$$

In the following, it is shown that for every triple (K, G, H) there exists a rho-function and an N-strongly quasi-invariant measure on $K\backslash G/H$, which arises from this rho-function. For this, first it is shown that for each $f \in C_c(G)$ there exists a rho-function ρ_f for the triple (K, G, H).

Proposition 3.4. *Suppose that G is a locally compact group and H and K are closed subgroups of G. Then for each* $f \in C_c(G)$ *there exists a continuous rho-function* ρ_f *on G.*

Proof. For each $f \in C_c(G)$, take

$$\rho_f(x) = \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh)dhdk.$$

It is clear that ρ_f is a well-defined positive linear map and according to Fubini's formula we have

$$\begin{aligned}\int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh)dhdk &= \int_H \int_K \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh)dkdh \\ &= \int_{K\times H} \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh)d(k \times h).\end{aligned}$$

First, we show that ρ_f is uniformly continuous. Suppose that V is a compact unit neighbourhood in G. Since $f \in C_c(G)$, for given $\varepsilon > 0$ there is a symmetric neighbourhood U of e such that $U \subseteq V$ and for each $y \in Ux$, $|f(x) - f(y)| < \varepsilon$.
Take $M = V \cdot supp(f) \cdot V$. If $x \in G \backslash KMH$, then $f(k^{-1}xh) = f(k^{-1}yh) = 0$ for all $k \in K$ and $h \in H$, and if $x \in KMH$, there is $k_0 \in K$ and $h_0 \in H$ such that $k_0^{-1}xh_0 \in M$. If $y \in Ux$, then we have two cases:

(1) If $k^{-1}k_0^{-1}y \in supp(f)$, then $k^{-1}k_0^{-1}x \in M$. Therefore, $k^{-1} \in Mh_0M^{-1} \cap K$. Also, if $k^{-1}k_0^{-1}x \in supp\,(f)$, then $k^{-1} \in Mh_0M^{-1} \cap K$.

(2) If $yh_0h \in supp\,(f)$, then $h \in M^{-1}k_0^{-1}M \cap H$. Also, if $xh_0h \in supp\,(f) \subseteq M$, then $h \in M^{-1}k_0^{-1}M \cap H$.
Now put $L = L_1 \times L_2$, where $L_1 = Mh_0M^{-1} \cap K$ and $L_2 = M^{-1}k_0^{-1}M \cap H$. The set L is compact in $K \times H$ and if $(k, h) \notin L$ we have

$$f(k^{-1}k_0^{-1}xh_0h) = f(k^{-1}k_0^{-1}yh_0h) = 0.$$

Hence, according to the above mentioned, we can write

$$\begin{aligned}
|\rho_f(x) - \rho_f(y)| &\leq \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} |f(k^{-1}xh) - f(k^{-1}yh)| dhdk \\
&= \int_K \int_H \frac{\Delta_G(h'h)}{\Delta_H(h'h)\Delta_K(k^{-1})} |f(k^{-1}xh'h) - f(k^{-1}yh'h)| dhdk \\
&= \int_K \int_H \frac{\Delta_G(h'h)}{\Delta_H(h'h)\Delta_K((k'k)^{-1})} |f(k^{-1}k'^{-1}xhh) - f(k^{-1}k'^{-1}yh'h)| \\
&= \frac{\Delta_G(h')}{\Delta_H(h')\Delta_K(k')} \quad \int_{L_1} \int_{L_2} |f(k^{-1}k'^{-1}xh'h) - f(k^{-1}k'yh'h)| dhdk \\
&< d \cdot d(h \times k)(L) \cdot \varepsilon
\end{aligned}$$

where $\delta(h', k')$ denotes $\frac{\Delta_G(h')}{\Delta_H(h')\Delta_K(k')}$ and d is $\max\{\delta(h', k'), (h', k') \in L\}$.

Next, if $k_1 \in K$ and $h_1 \in H$ are arbitrary, we have

$$\begin{aligned}
\rho_f(k_1xh_1) &= \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}k_1xh_1h) dhdk \\
&= \int_K \int_H \frac{\Delta_G(h_1^{-1}h)}{\Delta_H(h_1^{-1}h)\Delta_K(k^{-1})} f(k^{-1}k_1xh) dhdk \\
&= \frac{\Delta_H(h_1)}{\Delta_G(h_1)} \int_H \int_K \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1}k_1^{-1})} f(k^{-1}xh) dkdh \\
&= \frac{\Delta_H(h_1)\Delta_K(k_1)}{\Delta_G(h_1)} \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xh) dhdk \\
&= \frac{\Delta_H(h_1)\Delta_K(k_1)}{\Delta_G(h_1)} \rho_f(x).
\end{aligned}$$

This proves ρ_f is a rho-function. □

In the next proposition, we will prove that for the triple (K, G, H) there is a positive continuous rho-function whose support is G. But first we need the following technical Lemma.

Lemma 3.5. *Let U be a symmetric unit neighbourhood of G with compact closure and N be the normalizer of K in G. If $U_N = U \cap N$ is taken, there exists a subset A of G with the following properties:*

(i) For every $x \in G$, we have $KxH \bigcap U_N a \neq \emptyset$, for some $a \in A$.

(ii) If M is a compact subset of G, then $\{a \in A; KMH \bigcap U_N a \neq \emptyset\}$ is finite.

Proof. Let $\mathcal{A} = \{A \subseteq G;$ for all $a \neq b$ in $A, a \notin KU_N bH\}$. According to Zorn's Lemma, $\mathcal{A}$ has a maximal element, say A. We claim that A satisfies (i) and (ii).

(i) If $x \in A$ the claim is clear. If $x \in G \setminus A$ where such that $KxH \bigcap U_N a = \emptyset$, for all $a \in A$, then we could add x to A and make A strictly larger. So (i) holds for A.

(ii) Let M be a compact subset of G and $A_M = \{a \in A; KMH \cap U_N a \neq \emptyset\}$.
For every $a \in A_M$, $KMH \cap U_N a \neq \emptyset$ implies $KaH \cap U_N M \neq \emptyset$ and conversely.
Pick $x_a \in KaH \cap U_N M$. If A_M is infinite, then $\{x_a; a \in A_M\}$ would have a cluster point x, say, in the compact set $\bar{U}_N M$.
Let V be a unit neighbourhood such that $VV^{-1} \subseteq U$. Then, by choosing $V_N = V \cap N$, we have $V_N V_N^{-1} \subseteq U_N$. Since the x_a is a cluster at x, there exist distinct $a, b \in A_M$ such that $x_a, x_b \in V_N x$. This implies that $x_a x_b^{-1} \in V_N V_N^{-1} \subseteq U_N$. But $x_a \in KaH$ and $x_b \in KbH$, so $x_a \in KU_N bH$ which forces $a \in KU_N bH$, in contradiction to $A \in \mathcal{A}$. So, A_M is finite and (ii) is met.

□

Next we use Lemma 3.5 and Proposition 3.4 to give a rho-function for each triple (K, G, H) mentioned above, which is strictly positive on G.

Proposition 3.6. *With the above notation, there exists a rho-function ρ for the triple (K, G, H), which is continuous and everywhere strictly positive on G.*

Proof. Choose $f \in C_c^+(G)$ such that $f(e) > 0$ and $f(x) = f(x^{-1})$ for all $x \in G$. Put $U = \{x \in G; f(x) > 0\}$, by choosing $U_N = U \cap N$ and according to Lemma 3.5 there is subset A of G with properties (i) and (ii) which are mentioned in this Lemma.
Let for every $y \in A$, $f^y(x) = f(xy^{-1})$ for $x \in G$. By using Proposition 3.4, we can define a continuous rho-function ρ_{f^y} by

$$\rho_{f^y}(x) = \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)\Delta_K(k^{-1})} f(k^{-1}xhy^{-1})dhdk.$$

Now, by using the fact that $\rho_{f^y}(x) = 0$ if $x \notin KU_N yH$ and applying the Proposition 3.4, for any compact subset M of G, we have ρ_{f^y} as being zero on M for all but finitely many $y \in A$. Thus $\rho = \sum_{y \in A} \rho_{f^y}$ is a continuous function on G. Also, it is evident that ρ is a rho-function.
According to Lemma 3.5 (i), for each $x \in G$, there is $y \in A$ such that $f^y(kxh) > 0$ for some $k \in K$ and $h \in H$. Therefore, $\rho_{f^y}(x) > 0$ and hence $\rho(x) > 0$. □

Next, we use Proposition 3.4 to construct a positive measure on $K\backslash G/H$.

Theorem 3.7. *Let ρ be a rho-function for the triple (K, G, H). Then there exists a positive Radon measure μ_ρ on $K\backslash G/H$ such that*

$$\int_{K\backslash G/H} Q(f)(\ddot{x})d\mu_\rho(\ddot{x}) = \int_G f(x)\rho(x)dx$$

for all $f \in C_c(G)$.

Proof. By applying Proposition 3.6, for each triple (K, G, H) we can get a rho-function ρ. Take the linear functional I_ρ on $C_c(K\backslash G/H)$ by

$$I_\rho(Q(f)) = \int_G f(x)\rho(x)dx.$$

By using Lemma 2.2, there exists $g \in C_c(G)$ such that $Q(g)(\ddot{x}) = 1$ on $supp(Q(f))$. That is, $\int_K \int_H g(k^{-1}xh))dhdk = 1$ for all $x \in Supp\, f$ therefore we can write

$$\begin{aligned}
\int_G f(x)\rho(x)dx &= \int_G f(x)\rho(x)Q(g)(\ddot{x})dx \\
&= \int_K \int_H \int_G f(x)\rho(x)g(k^{-1}xh)dxdhdk \\
&= \int_G \int_K \int_H f(kxh^{-1})\Delta_H(h^{-1})\Delta_K(k)\rho(x)g(x)dhdkdx \\
&= \int_G g(x)\rho(x)\Big(\int_K \int_H f(k^{-1}xh)dhdk\Big)dx.
\end{aligned}$$

Now if $Q(f) = 0$, then $\int_G f(x)\rho(x)dx = 0$. Therefore, I_ρ is a well-defined positive linear functional on $C_c(K\backslash G/H)$. We conclude that there exists a positive Radon measure μ_ρ on $K\backslash G/H$ such that

$$\int Q(f)(\ddot{x})d\mu_\rho(\ddot{x}) = \int_G f(x)\rho(x)dx.$$

□

We add the IN-group condition for closed subgroup K of G in Theorem 3.7 to achive our result.

Theorem 3.8. *Suppose also that K is an IN-group. Given any rho-function ρ for the triple (K, G, H), there is an N-strongly quasi-invariant measure μ_ρ on $K\backslash G/H$ such that*

$$\begin{aligned}
\int_G f(y)\rho(y)dy &= \int_{K\backslash G/H} Q(f)(\ddot{y})d\mu_\rho(\ddot{y}) \\
&= \int_{K\backslash G/H} \int_K \int_H f(k^{-1}yh)dhdkd\mu_\rho(\ddot{y}).
\end{aligned}$$

Proof. By applying Theorem 3.7, we can get a unique measure μ_ρ on $K\backslash G/H$, which satisfies the following:

$$\int_G f(y)\rho(y)dy = \int_{K\backslash G/H}\int_K\int_H f(k^{-1}yh)dhdkd\mu_\rho(\ddot{y}).$$

μ_ρ is an N-strongly quasi invariant. Indeed, let

$$\lambda : N\times K\backslash G/H\times \longrightarrow (0,+\infty)$$

by

$$\lambda(n,\ddot{y}) = \frac{\rho(ny)}{\rho(y)}. \tag{3.1}$$

By using the fact that K is an IN-group, one can prove that λ is well-defined. The continuity of rho-function ρ results in the fact that λ is also continuous. Moreover, for each $n\in N$, we have

$$\begin{aligned}\int_{K\backslash G/H} Q(f)(\ddot{y})d\mu_n(\ddot{y}) &= \int_{K\backslash G/H} Q(L_n f)(\ddot{y})d\mu(y)\\ &= \int_G L_n f(y).\rho(y)dy\\ &= \int_{K\backslash G/H} Q(f.\lambda(n,.))(\ddot{y})d\mu_\rho(\ddot{y}).\end{aligned}$$

Therefore,

$$\frac{d\mu_n(\ddot{y})}{d\mu(\ddot{y})} = \lambda(n,\ddot{y}).$$

□

Remark 3.9. According to Theorem 3.8, if K is also normal in G, then $K\backslash G/H$ has a G- strongly quasi-invariant measure.

In the following proposition we list some properties of μ_n.

Proposition 3.10. *Let ρ be a rho-function for the triple (K,G,H).*

(i). If A is a closed subset of $K\backslash G/H$ such that $\rho(n)=0$, for all $n\in N\setminus q^{-1}(A)$, then $Supp(\mu_\rho)\subseteq A$.

(ii). For each $n\in N$, $L_n\rho$ is also a rho-function for the triple (K,G,H) and $\mu_{L_n\rho} = (\mu_\rho)_{n^{-1}}$

(iii). Suppose that K is an IN-group, then if $f\in C_c^+(G)$ and take $\rho=\rho_f$, therefore for any $\alpha\in C_c(K\backslash G/H)$

$$\int_{K\backslash G/H}\alpha(\ddot{x})d\mu_\rho(\ddot{x}) = \int_G \alpha(q(x))f(x)dx.$$

Proof. The proof of (i) and (ii) are straightforward. For each $\alpha \in C_c(K\backslash G/H)$, there is $\varphi \in C_c(G)$ such that $Q(\varphi) = \alpha$. Therefore, we have

$$\begin{aligned}\int_{K\backslash G/H} \alpha(\ddot{x})d\mu_\rho(\ddot{x}) &= \int_{K\backslash G/H} Q(\varphi)(\ddot{x})d\mu_\rho(\ddot{x}) \\ &= \int_G \varphi(x)\rho(x)dx \\ &= \int_G \varphi(x) \int_K \int_H \frac{\Delta_G(h)}{\Delta_H(h)} f(k^{-1}xh)dhdkdx \\ &= \int_G \int_{H\times K} \varphi(k^{-1}xh)f(x)d(h\times k)dx \\ &= \int_G f(x)\Big(\int_{H\times K} \varphi(k^{-1}xh)d(h\times k)\Big)dx \\ &= \int_G f(x)Q(\varphi)(q(x))dx.\end{aligned}$$

This proves (iii). □

4. rho-function and N-strongly quasi-invariant measure

Suppose that G is a locally compact group, H and K are closed subgroups of G and N is the normalizer group of K in G. Also, suppose that ω is a left Haar measure on N with the modular function Δ_N. In this section, we want to consider under which conditions an N-strongly quasi-invariant measure on $K\backslash G/H$ arises from a rho-function.
First, we recall that if X is a locally compact Hausdorf space and μ is a positive Radon measure on X, then subset B is called locally negligible, if for each compact subset M of X, $\mu(B \cap M) = 0$.

Remark 4.1. In [2] has been shown that N is not locally negligible if and only if N is open subgroup of G.

Lemma 4.2. *If N is an open subgroup of G then each $f \in C_c(N)$ may be regarded as a function in $C_c(G)$ and $Q : C_c(G)\big|_{C_c(N)}$ is surjective on $B = \{F \in C_c(K\backslash G/H),\ supp\ (F) \subseteq q(N)\}$.*

Proof. Suppose N is open and $f \in C_c(N)$. So $f : N \to \mathbb{C}$ is continuous and $supp\ (f) \subseteq N$. Now we define,

$$f_1(x) = \begin{cases} f(x) & x \in N \\ 0 & x \notin N \end{cases}$$

Then $f_1\big|_N = f$ is continuous. Also $f_1\big|_{N^c} = 0$. So $f_1^{-1}(\{0\}) = G\backslash N$ and since N is open then $G\backslash N$ is closed. Therefore, $f_1\big|_{N^c}$ is continuous and $supp\ (f_1) = supp\ (f) \subseteq N \subseteq G$ is compact. Thus, $f_1 \in C_c(G)$ and $f_1\big|_N = f$.
Now suppose $F \in B$. Since $supp\ (F) \subseteq q(N)$, then we may suppose that f has support contained in N. It follows that there exists f_1 in $C_c(G)$ such that $F = Q(f_1)$. □

Our main result in this section is as follows:

Theorem 4.3. *Suppose also that K is an IN-group, N is not locally negligible, and $H \subseteq N$. Then every N-strongly quasi-invariant measure μ on $K\backslash G/H$ arises from a rho-function. That is, there is a rho-function $\rho : G \to (0, +\infty)$, such that*

$$\int_{K\backslash G/H}\int_K\int_H f(k^{-1}xh)dhdkd\mu(\ddot{x}) = \int_G f(x)\rho(x)dx \text{ for all } f \in C_c(N), \tag{4.1}$$

and all such measures are N-strongly equivalent. That is to say that they have the same negligible sets on $q(N)$.

Proof. Suppose that μ is an N-strongly quasi-invariant measure on $K\backslash G/H$, then there is a positive continuous function λ on $N \times (K\backslash G/H)$, such that $(d\mu_x/d\mu)(\ddot{y}) = \lambda(x, \ddot{y})$. It is easy to check that

$$\lambda(n_1n_2, p) = \lambda(n_1, n_2p)\lambda(n_2, p).$$

According to Remark 4.1, N is an open subgroup of G. Therefore, by applying Lemma 4.2, each function in $C_c(N)$ may be regarded as a function in $C_c(G)$. Also, Range $Q\big|_{C_c(N)}$ is $\{F \in C_c(K\backslash G/H);\ supp\ (F) \subseteq q(N)\}$. The mapping $f \mapsto \int_{K\backslash G/H} Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH)$ is a left invariant positive linear functional on $C_c(N)$. Indeed;

$$\begin{aligned}
&\int_{K\backslash G/H} Q(L_m f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH)\\
&\quad= \int_{K\backslash G/H} L_m Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH)\\
&\quad= \int_{K\backslash G/H} Q(f)(Km^{-1}nH)\lambda(n, KH)^{-1}d\mu(KnH)\\
&\quad= \int_{K\backslash G/H} Q(f)(KnH)\lambda(mn, KH)^{-1}d\mu_m(KnH)\\
&\quad= \int_{K\backslash G/H} Q(f)(KnH)\lambda(m, KnH)^{-1}\lambda(n, KH)^{-1}\lambda(m, KnH)d\mu(KnH)\\
&\quad= \int_{K\backslash G/H} Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH).
\end{aligned}$$

By uniqueness of Haar measure on N, there is $c > 0$ such that

$$\int_{K\backslash G/H} Q(f)(KnH)\lambda(n, KH)^{-1}d\mu(KnH) = c\int_N f(x)d\omega(x). \tag{4.2}$$

Let $\rho_1 : N \to (0, +\infty)$ be given by $\rho_1(n) = c\lambda(n, KH)$. By replacing f by $f \cdot \lambda(n, KH)$ in (4.2), we see that

$$\int_{K\backslash G/H}\int_K\int_H f(k^{-1}nh)dhdk = \int_N f(n)\rho_1(n)dn \tag{4.3}$$

Now ρ_1 can be extended on G by the following definition:

$$\rho : G \to (0, +\infty)$$
$$\rho(x) = \begin{cases} \rho_1(x) & x \in N \\ 0 & x \notin N \end{cases},$$

then ρ is a positive continuous function on G. Moreover, if $h_0 \in H$ and $k_0 \in K$, then by using $H \subseteq N$, we can write

$$\begin{aligned}
\int_G f(x)\rho(k_0xh_0)dx &= \int_N f(x)\rho(k_0xh_0)d\omega(x) \\
&= \int_G f(k_0^{-1}xh_0^{-1})\rho(x)\Delta_N(h_0^{-1})d\omega(x) \\
&= \int_{K\backslash G/H}\int_H\int_K f(k_0^{-1}k^{-1}xhh_0^{-1})dkdhd\mu(\ddot{x}) \\
&= \Delta_G(h_0^{-1})\int_{K\backslash G/H}\int_K\int_H f(k^{-1}xh)\Delta_H(h_0)\Delta_K(k_0)dhdkd\mu(\ddot{x}) \\
&= \Delta_G(h_0^{-1})\Delta_H(h_0)\Delta_K(k_0)\int_G f(x)\rho(x)d\omega(x) \\
&= \frac{\Delta_K(k_0)\Delta_H(h_0)}{\Delta_G(h_0)}\int_N f(x)\rho(x)d\omega(x).
\end{aligned}$$

This being for all $f \in C_c(N)$, $\rho(k_0nh_0) = \frac{\Delta_H(h_0)\Delta_K(k_0)}{\Delta_G(h_0)}\rho(n)$. When $x \notin N$, the equality $\rho(k_0xh_0) = \frac{\Delta_H(h_0)\Delta_K(k_0)}{\Delta_G(h_0)}\rho(x)$ is trivial. This proves that ρ is a rho-function.
Suppose that μ_1 and μ_2 are N-strongly quasi-invariant measures on $K\backslash G/H$ associated with rho-function ρ_1 and ρ_2 on G, respectively. Then, we have

$$\frac{\rho_1(knh)}{\rho_2(knh)} = \frac{\frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\cdot\rho_1(n)}{\frac{\Delta_H(h)\Delta_K(k)}{\Delta_G(h)}\cdot\rho_2(n)} = \frac{\rho_1(n)}{\rho_2(n)}$$

for all $n \in N$.
Take $\varphi : K\backslash G/H \to [0, +\infty)$ by

$$\varphi(KxH) = \begin{cases} \frac{\rho_1(x)}{\rho_2(x)} & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases},$$

clearly φ is well-defined and continuous.
Let $f \in C_c(G)$. Then we can write

$$\begin{aligned}
Q(f\cdot\frac{\rho_1}{\rho_2})(KnH) &= \int_K\int_H f(k^{-1}nh)\frac{\rho_1(k^{-1}nh)}{\rho_2(k^{-1}nh)}dhdk \\
&= \frac{\rho_1(n)}{\rho_2(n)}\int_K\int_H f(k^{-1}nh)dhdk,
\end{aligned}$$

for all $n \in N$.
Therefore, we have

$$\begin{aligned}\int_{K\backslash G/H} Q(f)(\ddot{n})d\mu_1(\ddot{n}) &= \int_N f(n)\rho_1(n)dn \\ &= \int_N f(n)(\frac{\rho_1(n)}{\rho_2(n)})\rho_2(n)dn \\ &= \int_{K\backslash G/H} Q(f)(\ddot{n})\varphi(\ddot{n})d\mu_2(\ddot{n}).\end{aligned}$$

Hence, $\frac{d\mu_1}{d\mu_2}(\ddot{n}) = \varphi(\ddot{n})$ for all $\ddot{n} \in K\backslash G/H$.
Now, if $A \subseteq q(N)$ is a negligible set with respect to μ_1, then we have

$$0 = \int_{K\backslash G/H} 1_A(\ddot{x})d\mu_1(\ddot{x}) = \int_{K\backslash G/H} 1_A(\ddot{x})\varphi(\ddot{x})d\mu_2(\ddot{x}).$$

Therefore, $\int_{K\backslash G/H} 1_A(\ddot{x})\varphi(\ddot{x})d\mu_2(\ddot{x}) = 0$. By using the fact that for each $n \in N$ we can get $\varphi(\ddot{n}) > 0$ we have $\int_{K\backslash G/H} 1_A(\ddot{x})d\mu_2(\ddot{x}) = 0$, so $\mu_2(A) = 0$.
Therefore, the negligible sets of $q(N)$ with respect to μ_1 are the same as the negligible sets, with respect to μ_2, and we are done. □

Corollary 4.4. *Let G be a semidirect product of K and H, respectively. Double coset space $K\backslash G/H$ possesses a strongly quasi-invariant measure.*

Corollary 4.5. *If $K \lhd G$, then each strongly quasi-invariant measure on $K\backslash G/H$ arises from a rho-function. In other words, there exists a rho-function ρ on G such that*

$$\int_{K\backslash G/H}\int_K\int_H f(k^{-1}xh)dhdkd\mu(\ddot{x}) = \int_G f(x)\rho(x)dx \quad \textit{for all } f \in C_c(G)$$

Proof. It is sufficient to apply Theorem 4.3 and to note the fact that $N = G$ □

Proposition 4.6. *If μ is an N-strongly quasi-invariant measure on $K\backslash G/H$ which arises from a rho-function, then supp (μ) = $K\backslash G/H$.*

Proof. Suppose that μ is an N-strongly quasi-invariant measure on $K\backslash G/H$ which arises from a rho-function ρ. Therefore, we can write

$$\int_{K\backslash G/H}\int_K\int_H f(k^{-1}xh)dhdkd\mu(\ddot{x}) = \int_G f(x)\rho(x)dx \quad \text{for all } f \in C_c(G).$$

Now if *supp* $(\mu) \neq K\backslash G/H$, then there is a non-empty open subset U of $K\backslash G/H$ such that $\mu(U) = 0$. By applying Urysohn's Lemma, there is a non-zero $F \in C_c(K\backslash G/H)$ such that *supp* $(F) \subseteq U$. Also, there is a non-zero $f \in C_c(G)$ such that $Q(f) = F$. So,

$$\begin{aligned}0 = \int_{K\backslash G/H} F(\ddot{x})d\mu(\ddot{x}) &= \int_{K\backslash G/H} Q(f)(\ddot{x})d\mu(\ddot{x}) \\ &= \int_{K\backslash G/H}\int_K\int_H f(k^{-1}xh)dhdkd\mu(\ddot{x}) \\ &= \int_G f(x)\rho(x)dx > 0\end{aligned}$$

which is a contradiction. Therefore, $supp\ (\mu) = K\backslash G/H$. □

Proposition 4.7. *Let N be open in G, and μ be any N-strongly quasi-invariant measure on $K\backslash G/H$ which arises from a rho-function. Then, for a Borel subset $A \subseteq K\backslash G/H$, $A \cap q(N)$ is locally negligible if and only if $q^{-1}(A) \cap N$ is locally negligible in G.*

Proof. Let $A \subseteq K\backslash G/H$ be a Borel set such that $A \cap q(N)$ be locally negligible. By intersecting $A \cap q(N)$ with an arbitrary compact subset of $K\backslash G/H$, we may assume, without loss of generality, that $A \cap q(N)$ is relatively compact, that is, $\overline{A \cap q(N)}$ is compact.
Let $f \in C^+(G)$ be such that $f \neq 0$. By applying Fubini's Theorem, we can write

$$\int_{K\backslash G/H} \int_N f(x) \cdot 1_{A\cap q(N)}(x \cdot \ddot{y}) d\omega(x) d\mu(\ddot{y}) = \int_N \int_{K\backslash G/H} f(x) \cdot 1_{A\cap q(N)}(x \cdot \ddot{y}) d\mu(\ddot{y}) d\omega(x). \tag{4.4}$$

Suppose that $\mu(A \cap q(N)) = 0$, then $\mu(x^{-1} \cdot A \cap q(N)) = 0$ for all $x \in N$. Thus, the right hand side of (4.4) is zero, and so the left hand side. Therefore, we may state that

$$\int_N f(x) 1_{A\cap q(N)}(x \cdot \ddot{y}) d\omega(x) = 0$$

for almost all $\ddot{y} \in K\backslash G/H$.
Let C be any compact subset of G such that $C \cap N \neq \emptyset$ and U a compact unit neighbourhood in G. Select $f \in C_c^+(G)$ so that $f(x) > 1$ for all $x \in CU^{-1} \cap N$. Since $\mu(q(U \cap N)) > 0$, there exists $y \in U \cap N$ such that

$$\int_N f(x) \cdot 1_{A\cap q(N)}(x \cdot \ddot{y}) d\omega(x) = 0.$$

So,

$$\begin{aligned} 0 &= \Delta_N(y) \int_N f(x) \cdot 1_{A\cap q(N)}(q(xy)) d\omega(x) \\ &= \int_N f(xy^{-1}) 1_{q^{-1}(A)\cap N}(x) d\omega(x). \end{aligned}$$

Now, for each $x \in CU^{-1} \cap N$, we have $f(xy^{-1}) \geq 1$ which implies that

$$\int_N 1_{q^{-1}(A)\cap N\cap C}(x) d\omega(x) = 0.$$

Thus, $q^{-1}(A) \cap N \cap C$ is negligible set for any compact set $C \subseteq G$, that is, $q^{-1}(A) \cap N$ is locally negligible.
Conversely, suppose that $q^{-1}(A) \cap N$ is locally negligible. Again, let $f \in C_c^+(N)$, and $\ddot{y} \in q(N)$ be arbitrary and from now fixed, since q is onto, choose $y \in N$ such that $q(y) = \ddot{y}$. Then, $x \mapsto f(xy^{-1})$ is continuous with compact support. So,

$$\begin{aligned} 0 &= \int_N f(xy^{-1}) \Delta_N(y^{-1}) 1_{q^{-1}(A)\cap N}(x) d\omega(x) \\ &= \int_N f(x) 1_{q^{-1}(A)\cap N}(xy) d\omega(x) \\ &= \int_N f(x) 1_{A\cap q(N)}(x \cdot q(y)) d\omega(x). \end{aligned}$$

Then, the left hand side of (4.4) is zero, therefore the right hand side is zero as well. Hence, for almost all $x \in N$

$$\begin{aligned} 0 &= \int_{K\backslash G/H} f(x) \cdot 1_{A\cap q(N)}(x \cdot \ddot{y}) d\mu(\ddot{y}) \\ &= f(x) \cdot \mu(x^{-1} \cdot A \cap q(N)). \end{aligned}$$

Since $f \neq 0$, there is $x \in N$ so that $\mu(x^{-1} \cdot A \cap q(N)) = 0$ which implies that $\mu(A \cap q(N)) = 0$. □

Theorem 4.8. *If K is also an IN-group and μ is an N-strongly quasi-invariant measure on $K\backslash G/H$, then $\tilde{\mu}$ defined by $\tilde{\mu}(f) = \int_{K\backslash G/H} Q(f)(\ddot{x} d\mu(\ddot{x})$ has the following property:*

$$\int_G f(nxh^{-1}) d\tilde{\mu}(x) = \Delta_H(h) \int_G f(x) \cdot \lambda(n, q(x)) d\tilde{\mu}(x). \tag{4.5}$$

Proof. Suppose that μ is an N-strongly quasi-invariant measure. Therefore, there is the continuous positive function λ on $N \times K\backslash G/H$ such that $d\mu_n(\ddot{x}) = \lambda(n, \ddot{x}) d\mu(\ddot{x})$ for all $n \in N$. Hence, by applying Theorem 2.4, we have

$$\begin{aligned} \int_G f(nxh^{-1}) d\tilde{\mu}(x) &= \int_G L_{n^{-1}} \circ R_{h^{-1}} f(x) d\tilde{\mu}(x) \\ &= \Delta_H(h) \int_G L_{n^{-1}} f(x) d\tilde{\mu}(x) \\ &= \Delta_H(h) \int_{K\backslash G/H} L_{n^{-1}} Q(f)(\ddot{x}) d\mu(\ddot{x}) \\ &= \Delta_H(h) \int_{K\backslash G/H} Q(f)(\ddot{x}) \lambda(n, \ddot{x}) d\mu(\ddot{x}) \\ &= \Delta_H(h) \int_G \big(f \cdot \lambda(n, q(\cdot))\big)(x) d\tilde{\mu}(x) \end{aligned}$$

□

Remark 4.9. Note that if $K = \{e\}$, then we conclude that each strongly quasi-invariant measure on G/H arises from a rho-function and if $H = \{e\}$, then $K\backslash G$ (the right cosets of K in G) has N-strongly quasi-invariant measure by the left action and if N is not locally negligible, this measure arises from a rho-function.

Remark 4.10. Take $K = H$. Now if N is not locally negligible, each N-strongly quasi-invariant measure on $G//H$ arises from a rho-function.

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