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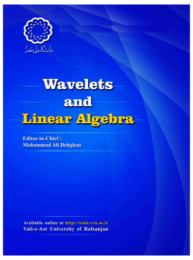


# Additive maps preserving the fixed points of Jordan products of operators

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## Abstract

Let $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$. In this paper, we determine the form of a surjective additive map $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ preserving the fixed points of Jordan products of operators, i.e., $F(A \circ B) \subseteq F(\phi(A) \circ \phi(B))$, for every $A, B \in \mathcal{B}(\mathcal{X})$, where $A \circ B = AB + BA$, and $F(A)$ denotes the set of all fixed points of operator $A$.



## 1. Introduction

Preserving problems on operator algebras have attracted attention of many mathematicians in the last decades. These problems concern the question of characterizing the form of all maps

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on operator algebras that leave invariant a certain property, and many results exposing the algebraic structure of such maps are obtained. Recently, some preserver problems concern the certain properties of different types of products of operators (cf. [2-12]).

Let $\mathcal{B}(\mathcal{X})$ denote the algebra of all bounded linear operators on a complex Banach space $\mathcal{X}$. Let $A \in \mathcal{B}(\mathcal{X})$. Recall that $x \in \mathcal{X}$ is a fixed point of $A$, whenever we have $Ax = x$. It is clear that the set of all fixed points of $A$ is a subspace of $\mathcal{X}$. Denote by $F(A)$ and $\dim F(A)$ the set of all fixed points of $A$ and the dimension of $F(A)$, respectively.

We say that a map $\phi$ on $\mathcal{B}(\mathcal{X})$ is preserving the fixed points of the operation '$*$' of operators if $F(A * B) \subseteq F(\phi(A) * \phi(B))$, for every $A, B \in \mathcal{B}(\mathcal{X})$.

Authors in [9] characterized the forms of surjective maps on $\mathcal{B}(\mathcal{X})$ which preserve the dimension of fixed points of products of operators, in both directions. More precisely, it was shown that if $\phi : \mathcal{B}(\mathcal{X}) \longrightarrow \mathcal{B}(\mathcal{X})$ is a surjective map which satisfies $\dim F(AB) = \dim F(\phi(A)\phi(B))$, for every $A, B \in \mathcal{B}(\mathcal{X})$, then there exists an invertible operator $S \in \mathcal{B}(\mathcal{X})$ such that $\phi(A) = SAS^{-1}$ or $\phi(A) = -SAS^{-1}$ for all $A \in \mathcal{B}(\mathcal{X})$. Authors in [10], considered the maps $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ and $\phi : \mathcal{M}_n \to \mathcal{M}_n$ satisfying $F(A + B) = F(\phi(A) + \phi(B))$ and $\dim F(A + B) = \dim F(\phi(A) + \phi(B))$, respectively. Moreover, authors in [11], considered the forms of surjective maps on $\mathcal{B}(\mathcal{X})$ which preserve the fixed points of triple Jordan products of operators, in both directions, i.e., $F(ABA) = F(\phi(A)\phi(B)\phi(A))$.

The Jordan product of $A, B \in \mathcal{B}(\mathcal{X})$ is defined as $A \circ B = AB + BA$. The aim of this paper is to continue these works by studying surjective additive maps on $\mathcal{B}(\mathcal{X})$ which preserve the fixed points of Jordan products of operators. The complete form of our main result is as following:

**Main Theorem.** Let $\mathcal{X}$ be a complex Banach space with $\dim\ \mathcal{X} \geq 2$ and let $\phi : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ be a surjective additive map which satisfies

$$F(A \circ B) \subseteq F(\phi(A) \circ \phi(B)),$$

for every $A, B \in \mathcal{B}(\mathcal{X})$. Then $\phi(A) = A$, for every $A \in \mathcal{B}(\mathcal{X})$ or, $\phi(A) = -A$, for every $A \in \mathcal{B}(\mathcal{X})$.

## 2. Proofs

Denote by $\mathcal{X}^*$ the dual space of $\mathcal{X}$. For every nonzero $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$, the symbol $x \otimes f$ stands for the rank-one linear operator on $\mathcal{X}$ defined by $(x \otimes f)y = f(y)x$ for any $y \in \mathcal{X}$. Note that every rank-one operator in $\mathcal{B}(\mathcal{X})$ can be written in this way. The rank-one operator $x \otimes f$ is idempotent (resp. nilpotent) if and only if $f(x) = 1$ (resp. $f(x) = 0$). Moreover, it is easy to check that $F(x \otimes f) =< x >$ if and only if $f(x) = 1$ and $F(x \otimes f) = \{0\}$ if and only if $f(x) \neq 1$. In order to prove the main theorem, we need some auxiliary propositions and lemmas.

**Proposition 2.1.** *Let $A \in \mathcal{B}(\mathcal{X})$. If $F(A \circ B) = \{0\}$, for every $B \in \mathcal{B}(\mathcal{X})$, then $A = 0$.*

*Proof.* Let $x \in \mathcal{X}$ be arbitrary. If $x$ and $Ax$ are linearly independent, then we can find a functional $f$ such that $f(x) = 0$ and $f(Ax) = 1$. Hence by setting $B = x \otimes f$ we get a contradiction, because

$$(A \circ B)x = (Ax \otimes f + x \otimes fA)x = x$$

which implies that $F(A \circ B) \neq \{0\}$. Therefore, $x$ and $Ax$ are linearly dependent, for every $x \in \mathcal{X}$ which by Theorem 2.3 in [1] $A = \lambda I$, for some scalar $\lambda$. If $\lambda \neq 0$, setting $B = \frac{\lambda^{-1}}{2} I$ implies

$$\mathcal{X} = F(I) = F(\lambda I \circ \frac{\lambda^{-1}}{2} I),$$

which is a contradiction and this completes the proof. □

**Proposition 2.2.** *Let $A, C \in \mathcal{B}(\mathcal{X})$ and* rank$A \geq 2$*. If $F(A \circ B) \subseteq F(C \circ B)$, for every rank-at-most-two operator $B$, then $A = C$.*

*Proof.* If $A$ is scalar, then $A = \alpha I$ for some scalar $\alpha \neq 0$. For any nonzero $x$ and functional $f$ with $f(x) = \frac{1}{2\alpha}$ we have $x \in F(A \circ x \otimes f)$, so $(Cx \otimes f + x \otimes fC)x = \frac{1}{2\alpha} Cx + f(Cx)x = x$, giving that $x$ and $Cx$ are linearly dependent and thus $C = \beta I$. It is then easy to see that $\beta = \alpha$ forcing that $C = A$.

Let now $A$ be non-scalar and additionally assume that there exists an $x$, such that $Ax \notin < x, Cx >$. Then we can fix a functional $f$ satisfying $f(x) = f(Cx) = 0$ and $f(Ax) = 1$. Observe that $(A \circ x \otimes f)x = x$ so, $x \in F(C \circ x \otimes f)$, a contradiction with $(C \circ x \otimes f)x = 0$. It follows that $Ax \in < x, Cx >$ for every $x \in \mathcal{X}$. If $\dim \mathcal{X} \geq 3$, then by Lemma 2.4 in [6], it follows that $A = \alpha I + \beta C$ for some scalars $\alpha, \beta$. Since $A$ is not scalar, $\beta \neq 0$ and so, $C = aI + bA$ for some scalars $a, b$. For every rank-one nilpotent $x \otimes f$, such that $x$ and $y := Ax$ are linearly independent and $f(x) = 0$, $f(y) = 1$ we have $x \in F(A \circ x \otimes f)$. Therefore, $(Cx \otimes f + x \otimes fC)x = f(Cx)x = x$, so $f(Cx) = 1$, giving further that $b = 1$. We next show that $a = 0$. Let us choose functionals $g, h$ such that $g(x) = g(y) = 1$, $h(x) = 0$ and $h(y) = -1$. Setting $B = x \otimes g + y \otimes h$ gives that $x \in F(A \circ B)$. Then, $x = C \circ Bx = (2a + 1)x$, giving $a = 0$ as desired.

It remains to verify the two-dimensional case. Then we can assume that $A$ and $C$ are $2 \times 2$ complex matrices with $A$ invertible. With no loss of generality we may assume that $A$ is upper triangular with nonzero (possibly equal) diagonal entries $\lambda_1, \lambda_2$. Let $E_{11}$ and $E_{22}$ be standard matrix units. By choosing $B_j = \frac{1}{2\lambda_j} E_{jj}$, $j = 1, 2$, and further computing and comparing the fixed points of $A \circ B_j$ and $C \circ B_j$, $j = 1, 2$, we easily obtain that $C = A$. □

**Lemma 2.3.** *$\phi$ is injective.*

*Proof.* Let $\phi(A) = 0$. Thus $F(\phi(A) \circ T) = \{0\}$, for every $T \in \mathcal{B}(\mathcal{X})$. Since $\phi$ is surjective, from assumption we obtain $F(A \circ B) = \{0\}$, for every $B \in \mathcal{B}(\mathcal{X})$. By Proposition 2.1, $A = 0$. □

**Lemma 2.4.** *Let $0 \neq N = x \otimes f$, for some $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$ such that $f(x) = 0$. Then $x \notin F(\phi(N)$.*

*Proof.* Let $P = x \otimes g$, for some $g \in \mathcal{X}^*$ such that $g(x) = 1$. Hence $Q = P + nN$ is an idempotent operator, for every $n \in \mathbb{N} \cup \{0\}$ and so

$$F(\frac{1}{2} Q \circ Q) = F(Q) = < x > .$$

On the other hand

$$F(\frac{1}{2} Q \circ Q) \subseteq F(\frac{1}{2} \phi(Q) \circ \phi(Q)) = F(\phi(Q)^2) = F([\phi(P) + n\phi(N)]^2),$$

for every $n \in \mathbb{N} \cup \{0\}$. Therefore, we obtain $[\phi(P) + n\phi(N)]^2 x = x$ and then

$$[\phi(P)^2 + n^2\phi(N)^2 + n(\phi(P)\phi(N) + \phi(N)\phi(P))]x = x$$

for more than two values of $n$. The coefficient at $n^2$ must be zero and so $\phi(N)^2 x = 0$. From this, we infer that $x \notin F(N)$, because, otherwise, $\phi(N)x = x$, then $\phi(N)^2 x = \phi(N)x = x$ and so $x = 0$, which is a contradiction. □

**Lemma 2.5.** *$\phi(I) = I$ or $\phi(I) = -I$.*

*Proof.* Let $\phi(A) = I$ and $x \in \mathcal{X}$. Assume that $x$ and $Ax$ are linearly independent. Thus there exists a functional $f$ such that $f(x) = 0$ and $f(Ax) = 2$. We have

$$[\frac{1}{2}A \circ (x \otimes f)]x = [\frac{1}{2}Ax \otimes f + \frac{1}{2}x \otimes fA]x = x$$

and then

$$x \in F(\phi(\frac{1}{2}A) \circ \phi(x \otimes f)) = F(\frac{I}{2} \circ \phi(x \otimes f)) = F(\phi(x \otimes f))$$

which by Lemma 2.4 is a contradiction. Therefore, $x$ and $Ax$ are linearly dependent for every $x \in \mathcal{X}$ and then $A = \lambda I$, for some scalar $\lambda$. We have

$$\mathcal{X} = F(\lambda^{-1} I \circ \frac{\lambda}{2} I) \subseteq F(\phi(\lambda^{-1} I) \circ \phi(\frac{\lambda}{2} I))$$

$$= F(\phi(\lambda^{-1} I) \circ \frac{I}{2}) = F(\phi(\lambda^{-1} I))$$

and then $\phi(\lambda^{-1} I) = I$. This together with $\phi(\lambda I) = I$ and Lemma 2.3 implies $\lambda^{-1} I = \lambda I$ and then $\lambda = 1$ or $-1$. This completes the proof. □

*Remark* 2.6. Without losing any generality (replacing $\phi$ by $-\phi$ if needed) we assume that $\phi(I) = I$. Thus we have

$$F(A) \subseteq F(\phi(A)), \tag{1}$$

for every $A \in \mathcal{B}(\mathcal{X})$.

**Lemma 2.7.** *$\phi(A) = A$, for every rank-one operator $A$.*

*Proof.* Let $A = x \otimes f$, for some $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. Unitality of $\phi$ together with (1) implies

$$\ker A = F(I - A) \subseteq F(I - \phi(A)) = \ker \phi(A)$$

and then $\ker f \subseteq \ker \phi(A)$. Injectivity of $\phi$ yields that $\ker f = \ker \phi(A)$, because if $\ker f$ is a proper subset of $\ker \phi(A)$, then since $\ker f$ is a maximal subspace of $\mathcal{X}$, $\ker \phi(A) = \mathcal{X}$ and so $\phi(A) = 0$, which is in contrast to injectivity of $\phi$. So $\ker f = \ker \phi(A)$ implies that $\phi(A) = y \otimes f$, for some $y \in \mathcal{X}$. The rest of the proof is divided into three cases.

**Case 1.** If $f(x) = 1$, then $< x >= F(A) \subseteq F(\phi(A))$ which implies that $\phi(A)x = yf(x) = y = x$. Hence $\phi(A) = A$.

**Case 2.** If $f(x) = 0$, then there exits a functional $g$ such that $g(x) = 1$ and then by Case 1

$$\phi(x \otimes (f + g)) = x \otimes (f + g).$$

On the other hand

$$\phi(x \otimes (f + g)) = \phi(x \otimes f) + \phi(x \otimes g) = \phi(x \otimes f) + x \otimes g.$$

Thus $\phi(x \otimes f) = x \otimes f$.

**Case 3.** Let $f(x) = a \neq 0, 1$. First we show that $x$ and $y$ are linearly dependent. Otherwise there exists a functional $g$ such that $g(x) = 1$ and $g(y) = 0$. Let $\phi((1 - a)x \otimes g) = w \otimes g$, for some $w \in \mathcal{X}$. Since $(f + (1 - a)g)x = 1$, we have

$$< x >= F(x \otimes (f + (1 - a)g))$$

$$= F(x \otimes f + (1 - a)x \otimes g) \subseteq F(y \otimes f + w \otimes g)$$

which implies that $(y \otimes f + w \otimes g)x = x$ and then $ay + w = x$. Hence $g(w) = 1$ which is a contradiction, because if $g(w) = 1$, then from Case 1, $w \otimes g = \phi(w \otimes g)$ and injectivity of $\phi$ follows that $(1 - a)x \otimes g = w \otimes g$ and so $(1 - a)x = w$. Thus $(1 - a)g(x) = g(w)$ and then $a = 0$, which is not correct.

Therefore, $x$ and $y$ are linearly dependent and so $\phi(x \otimes f) = a' x \otimes f$, for some scalar $a'$. Let $u$ be a vector such that $f(u) = 0$. By Case 2 and the first part of Case 3, there exists a scalar $b'$ such that

$$\phi(u \otimes f) = \phi((u + x) \otimes f - x \otimes f)$$

and so

$$u \otimes f = b'(u + x) \otimes f - a' x \otimes f.$$

Thus

$$(u \otimes f)x = (b'(u + x) \otimes f - a' x \otimes f)x$$

which implies that $au = ab'(u + x) - aa' x$ and then $0 = (b' - 1)u + (b' - a')x$. It is clear that $x$ and $u$ are linearly independent, because otherwise, from $f(u) = 0$ we obtain $f(x) = 0$ which is a contradiction. This together with the last relation implies $b' = 1$ and $b' = a'$ and so $a' = 1$. This completes the proof. □

***Proof of Main Theorem.*** The assertion immediately follows by Lemma 2.7, additivity of $\phi$ and Proposition 2.2.

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