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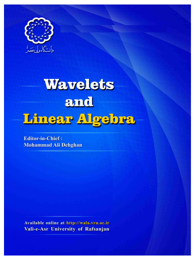

Weak and cyclic amenability of certain function algebras

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Abstract

We consider $C^{b\varphi}(K)$ to be the space $C^b(K)$ equipped with the product $f \cdot g = f\varphi g$ for all $f, g \in C^b(K)$ where, $K = \overline{B_1^{(0)}}$ is the closed unit ball of a non-zero normed vector space A and φ is a non-zero element of A^* such that $\|\varphi\| \leq 1$. We define $\|f\|_\varphi = \|f\varphi\|_\infty$ for all $f \in C^{b\varphi}(K)$. Some relations between the dual spaces of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ are investigated. Also we characterize the derivations from $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ into $(C^{b\varphi}(K), \|\cdot\|_\infty)^*$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)^*$ respectively. Finally we investigate the weak and cyclic amenability of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^{b\varphi}(K), \|\cdot\|_\varphi)$.

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1. Introduction and Preliminaries

Let A be a Banach algebra and X be a Banach $A-$bimodule. A derivation from A into X is a bounded linear mapping $D : A \longrightarrow X$ such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. Given $x \in X$, the mapping $\delta_x : A \longrightarrow X$ defined by $\delta_x(a) = ax - xa, a \in A$, is a derivation that is called an inner derivation. A derivation $D : A \longrightarrow X$ is called inner if, there exists $x \in X$ such that $D = \delta_x$.

A Banach algebra A is amenable if, for each Banach $A-$bimodule X, every derivation $D : A \longrightarrow X^*$ is inner and it is weakly amenable if, every derivation $D : A \longrightarrow A^*$ is inner.

For a Banach algebra A, a derivation $D : A \longrightarrow A^*$ is called cyclic if, $\langle D(a), b\rangle + \langle D(b), a\rangle = 0$ for all $a, b \in A$. A is called cyclic amenable if, every cyclic derivation from A into A^* is inner [2].

Let us denote by $K = \overline{B_1^{(0)}}$ the closed unit ball of a non-zero normed vector space A and let φ be a non-zero element of A^* such that $\|\varphi\| \leq 1$. We denote by $C^b(K)$ the space of all complex-valued, bounded and continuous functions on K. It is easily seen that $C^b(K)$ is a unital algebra with respect to the pointwise algebraic operations. The function 1_K is the identity of $C^b(K)$. We consider the uniform norm on K by

$$\|f\|_\infty = \sup\Big\{|f(x)| \quad \Big| \quad x \in K\Big\},$$

for all $f \in C^b(K)$. Immediately $(C^b(K), \|\cdot\|_\infty)$ is a commutative, unital Banach algebra such that $\|\varphi\|_\infty = \|\varphi\|$.

Recall that every metric space is completely regular.
So by [1, Examples 3.2.2 (i)], $(C^b(K), \|\cdot\|_\infty)$ is a commutative C^*-algebra. It is well-known that every commutative C^*-algebra is amenable [7, Example 2.3.4]. Hence $(C^b(K), \|\cdot\|_\infty)$ is an amenable Banach algebra.

On the space $C^b(K)$ we define the product $(f \cdot g)(x) = f(x)\varphi(x)g(x)$, $x \in K$, for all $f, g \in C^b(K)$. Obviously $(C^b(K), \cdot)$ is an algebra that we denote it by $C^{b\varphi}(K)$. In [5] we show that $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is a non-unital, commutative Banach algebra. Idempotent, nilpotent, zero divisor elements and also bounded approximate identities of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ are characterized in [5]. Also some relations between character spaces of $(C^{b\varphi}(K), \|\cdot\|_\infty)$ and $(C^b(K), \|\cdot\|_\infty)$ are investigated in [5].

In [3] for a Banach algebra A, R. A. Kamyabi-Gol and M. Janfada defined a new product " $\cdot$ " on A by, $a \cdot c = a\varepsilon c$ for all $a, c \in A$, where ε is a fixed element of the closed unit ball $\overline{B_1^{(0)}}$ of A. $(A, \cdot)$ is an associative Banach algebra which is denoted by A_ε. It is understood that the Banach algebra $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is a special case of the algebra A_ε. Some properties such as, Arens regularity, amenability and derivations on A_ε are investigated in [3]. Also biflatness, biprojectivity, $\varphi-$amenability and $\varphi-$contractibility of A_ε are investigated in [4].
Let A^* be the dual space of a Banach algebra A. It is well-known that with the operations

$$\langle \Lambda a, b\rangle = \langle \Lambda, ab\rangle$$
$$\langle a\Lambda, b\rangle = \langle \Lambda, ba\rangle, a, b \in A, \Lambda \in A^*,$$

A^* is a Banach $A-$bimodule.

2. Dual spaces of $(C^{b\varphi}(K), ||\cdot||_\infty)$ and $(C^{b\varphi}(K), ||\cdot||_\varphi)$

In this section let A be a non-zero normed vector space and let φ be a non-zero linear functional on A with $||\varphi|| \leq 1$. Also let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. We set $||f||_\varphi = ||f\varphi||_\infty$ for all $f \in C^b(K)$.

By [6, Theorem 2.7] and [6, Proposition 2.2], $||\cdot||\varphi$ is not an algebraic and complete norm on $C^b(K)$. Obviously $||f||_\varphi \leq ||\varphi||_\infty ||f||_\infty = ||\varphi|| ||f||_\infty$. We set $(C^{b\varphi}(K), ||\cdot||_\infty)^*$ and $(C^{b\varphi}(K), ||\cdot||_\varphi)^*$ as the dual spaces of $(C^{b\varphi}(K), ||\cdot||_\infty)$ and $(C^{b\varphi}(K), ||\cdot||_\varphi)$ with the norms $||\cdot||_\infty^*$ and $||\cdot||_\varphi^*$ respectively. Given $x \in K$, the mapping $\hat{x} : C^b(K) \longrightarrow \mathbb{C}$ defined by $\langle \hat{x}, f\rangle = f(x)$ is a linear functional.

Proposition 2.1. *The inclusion* $(C^{b\varphi}(K), ||\cdot||_\varphi)^* \subseteq (C^{b\varphi}(K), ||\cdot||_\infty)^*$ *is hold and* $||\Lambda||_\infty^* \leq ||\Lambda||_\varphi^* ||\varphi||$ *for all* $\Lambda \in (C^{b\varphi}(K), ||\cdot||_\varphi)^*$. *Whereas,* $(C^{b\varphi}(K), ||\cdot||_\infty)^* \nsubseteq (C^{b\varphi}(K), ||\cdot||_\varphi)^*$.

Proof. Let $\Lambda \in (C^{b\varphi}(K), ||\cdot||_\varphi)^*$. Then

$$\begin{aligned}|\langle \Lambda, f\rangle| &\leq ||\Lambda||_\varphi^* ||f||_\varphi \\ &= ||\Lambda||_\varphi^* ||f\varphi||_\infty \\ &\leq ||\Lambda||_\varphi^* ||\varphi|| ||f||_\infty,\end{aligned}$$

for all $f \in C^{b\varphi}(K)$. It follows that $||\Lambda||_\infty^* \leq ||\Lambda||_\varphi^* ||\varphi||$. So, $\Lambda \in (C^{b\varphi}(K), ||\cdot||_\infty)^*$.

Define $\Lambda : (C^{b\varphi}(K), ||\cdot||_\infty) \longrightarrow \mathbb{C}$ by $\langle \Lambda, f\rangle = \langle \hat{0}, f\rangle = f(0)$. Clearly Λ is linear and $|\langle \Lambda, f\rangle| = |f(0)| \leq ||f||_\infty$ for all $f \in C^{b\varphi}(K)$. So $\Lambda \in (C^{b\varphi}(K), ||\cdot||_\infty)^*$. Let $f_n(x) = \frac{1-|\varphi(x)|}{1+n|\varphi(x)|}$ for all $x \in K$ and $n \in \mathbb{N}$. One can easily verify that $||f_n||_\varphi \longrightarrow 0$ whereas, $\langle \Lambda, f_n\rangle = f_n(0) = 1 \nrightarrow 0$. Hence $\Lambda \notin (C^{b\varphi}(K), ||\cdot||_\varphi)^*$. □

Corollary 2.2. *The inclusion* $(C^{b\varphi}(K), ||\cdot||_\varphi)^* \subseteq (C^b(K), ||\cdot||_\infty)^*$ *is hold. Whereas,* $(C^b(K), ||\cdot||_\infty)^* \nsubseteq (C^{b\varphi}(K), ||\cdot||_\varphi)^*$.

Proof. Obviously $(C^{b\varphi}(K), ||\cdot||_\infty)^* = (C^b(K), ||\cdot||_\infty)^*$. So by Proposition 2.1 the inclusion $(C^{b\varphi}(K), ||\cdot||_\varphi)^* \subseteq (C^b(K), ||\cdot||_\infty)^*$ is hold and
$(C^b(K), ||\cdot||_\infty)^* \nsubseteq (C^{b\varphi}(K), ||\cdot||_\varphi)^*$. □

Proposition 2.3. $(C^{b\varphi}(K), ||\cdot||_\infty)^*\varphi \subseteq (C^{b\varphi}(K), ||\cdot||_\varphi)^*$.

Proof. Let $\Lambda \in (C^{b\varphi}(K), ||\cdot||_\infty)^*$. Define $\Lambda\varphi : C^{b\varphi}(K) \longrightarrow \mathbb{C}$ by $\langle \Lambda\varphi, f\rangle = \langle \Lambda, f\varphi\rangle, f \in C^{b\varphi}(K)$. Clearly $\Lambda\varphi$ is linear. For each $f \in C^{b\varphi}(K)$

$$\begin{aligned}|\langle \Lambda\varphi, f\rangle| &= |\langle \Lambda, f\varphi\rangle| \\ &\leq ||\Lambda||_\infty^* ||f\varphi||_\infty \\ &= ||\Lambda||_\infty^* ||f||_\varphi.\end{aligned}$$

It follows that $||\Lambda\varphi||_\varphi^* \leq ||\Lambda||_\infty^*$. So $\Lambda\varphi \in (C^{b\varphi}(K), ||\cdot||_\varphi)^*$. □

Proposition 2.4. *1.* $\hat{x} \in (C^{b\varphi}(K), ||\cdot||_\infty)^*$ *and* $||\hat{x}||_\infty^* \leq 1$ *for all* $x \in K$.

2. $\hat{x} \in (C^{b\varphi}(K), \|\cdot\|_\varphi)^*$ *and* $\|\hat{x}\|_\varphi^* \leq \frac{1}{|\varphi(x)|}$ *for all* $x \in K \setminus \ker\varphi$.

3. $\hat{x} \notin (C^{b\varphi}(K), \|\cdot\|_\varphi)^*$ *for all* $x \in K \cap \ker\varphi$.

Proof. The part *(1)* is obvious. Let $x \in K \setminus \ker\varphi$ and $f \in C^{b\varphi}(K)$. Then

$$\begin{aligned}|\langle \hat{x}, f\rangle| &= |f(x)| \\ &= \frac{|f(x)\varphi(x)|}{|\varphi(x)|} \\ &= \frac{|(f\varphi)(x)|}{|\varphi(x)|} \\ &\leq \frac{\|f\varphi\|_\infty}{|\varphi(x)|} \\ &= \frac{\|f\|_\varphi}{|\varphi(x)|}.\end{aligned}$$

This shows that $\|\hat{x}\|_\varphi^* \leq \frac{1}{|\varphi(x)|}$, providing part *(2)*. Let $x \in K \cap \ker\varphi$. Similar to the proof of Proposition 2.1 define $f_n(t) = \frac{1-|\varphi(t)|}{1+n|\varphi(t)|}$ for all $t \in K$ and $n \in \mathbb{N}$. As we claimed, $\|f_n\|_\varphi \longrightarrow 0$ whereas, $\langle \hat{x}, f_n\rangle = f_n(x) = 1 \nrightarrow 0$. Hence $\hat{x} \notin (C^{b\varphi}(K), \|\cdot\|_\varphi)^*$, providing *(3)*. □

Theorem 2.5. *Let* $M = \{f\varphi + \alpha 1_K \quad | \quad f \in C^b(K), \alpha \in \mathbb{C}\}$. *Then*

1. *If* $\ker\varphi = \{0\}$ *or equivalently* $\dim A = 1$ *then* $\overline{M}^{\|\cdot\|_\infty} = C^b(K)$.

2. *If* $\ker\varphi \neq \{0\}$ *or equivalently* $\dim A > 1$ *then* $\overline{M}^{\|\cdot\|_\infty} \subsetneqq C^b(K)$.

Proof. One can easily check that M is a subalgebra of $C^b(K)$.
(1) Let $\ker\varphi = \{0\}$. So $\dim A = 1$ and K is compact. Let $x, y \in K$ and $x \neq y$. Since $\ker\varphi = \{0\}$ so, $\varphi(x) \neq \varphi(y)$. This shows that M separates the points of K. Also for each $x \in K$ we have $1_K(x) = 1 \neq 0$. Therefore M vanishes nowhere. We shall show that $\bar{\varphi} \in \overline{M}^{\|\cdot\|_\infty}$, where $\bar{\varphi}$ is the complex conjugate of φ. To do this define $f_n : K \longrightarrow \mathbb{C}$ by,

$$f_n(x) = \begin{cases} \frac{\bar{\varphi}(x)}{\varphi(x)} e^{\frac{-1}{n\|x\|}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

for all $n \in \mathbb{N}$. Clearly f_n is continuous and bounded for all $n \in \mathbb{N}$. Indeed, $\|f_n\|_\infty = e^{\frac{-1}{n}}$ for all $n \in \mathbb{N}$. For each $x \in K\setminus\{0\}$ we have,

$$\begin{aligned}|f_n(x)\varphi(x) - \bar{\varphi}(x)| &= |\frac{\bar{\varphi}(x)}{\varphi(x)} e^{\frac{-1}{n\|x\|}} \varphi(x) - \bar{\varphi}(x)| \\ &= |\bar{\varphi}(x) e^{\frac{-1}{n\|x\|}} - \bar{\varphi}(x)| \\ &= |\bar{\varphi}(x)||e^{\frac{-1}{n\|x\|}} - 1| \\ &\leq \|\varphi\|\|x\|(1 - e^{\frac{-1}{n\|x\|}}) \\ &\leq \|x\|(1 - e^{\frac{-1}{n\|x\|}}).\end{aligned}$$

So

$$|f_n(x)\varphi(x) - \bar{\varphi}(x)| \leq \|x\|(1 - e^{\frac{-1}{n\|x\|}}) \tag{2.1}$$

for all $x \in K\backslash\{0\}$. Define $h(t) = t(1 - e^{\frac{-1}{nt}})$ for all $t \in (0,1]$. So $h'(t) = 1 - e^{\frac{-1}{nt}} - \frac{1}{nt}e^{\frac{-1}{nt}}, t \in (0,1]$. We claim that $h'(t) > 0$ for all $t \in (0,1]$. Define $P(y) = 1 - e^y + ye^y, y \in (-\infty, 0]$. So $P'(y) = ye^y \leq 0, y \in (-\infty, 0]$. Hence P is strictly decreasing on $(-\infty, 0)$. As $\frac{-1}{nt} < 0$ so $P(\frac{-1}{nt}) > P(0) = 0$. It follows that $1 - e^{\frac{-1}{nt}} - \frac{1}{nt}e^{\frac{-1}{nt}} > 0, t \in (0,1]$. This shows that h is strictly increasing on $(0,1]$ and so $\|h\|_\infty = h(1) = 1 - e^{\frac{-1}{n}}$. So by inequality (2.1) we can conclude that $\|f_n\varphi - \bar{\varphi}\|_\infty \leq 1 - e^{\frac{-1}{n}}$. Hence $\|f_n\varphi - \bar{\varphi}\|_\infty \longrightarrow 0$ and so $\bar{\varphi} \in \overline{M}^{\|\cdot\|_\infty}$. We will proceed to show that $\overline{M}^{\|\cdot\|_\infty}$ is self-adjoint. For this purpose, we first prove that $\bar{g} \in \overline{M}^{\|\cdot\|_\infty}$ for all $g = f\varphi + \alpha 1_K \in M$. Since $f_n\varphi \xrightarrow{\|\cdot\|_\infty} \bar{\varphi}$ so

$$\begin{aligned}
(\bar{f}f_n)\varphi + \bar{\alpha}1_K &= \bar{f}(f_n\varphi) + \bar{\alpha}1_K \\
&\xrightarrow{\|\cdot\|_\infty} \bar{f}\bar{\varphi} + \bar{\alpha}1_K \\
&= \bar{g}.
\end{aligned}$$

This shows that $\bar{g} \in \overline{M}^{\|\cdot\|_\infty}$. Now let $v \in \overline{M}^{\|\cdot\|_\infty}$. Then there exists a sequence $\{g_n\}_n \subseteq M$ such that $g_n \xrightarrow{\|\cdot\|_\infty} v$. Since $\bar{g_n} \xrightarrow{\|\cdot\|_\infty} \bar{v}$ and $\bar{g_n} \in \overline{M}^{\|\cdot\|_\infty}$ so $\bar{v} \in \overline{M}^{\|\cdot\|_\infty}$. Therefore $\overline{M}^{\|\cdot\|_\infty}$ is a self-adjoint subalgebra of $C^b(K)$ that separates the points of K and vanishes nowhere. By applying Stone-Weierstrass theorem we can conclude that $\overline{M}^{\|\cdot\|_\infty} = C^b(K)$.

(2) Let $\ker\varphi \neq 0$. Choose $0 \neq x_0 \in \ker\varphi$. The function $w : K \longrightarrow \mathbb{C}$ defined by $w(x) = \|x\|, x \in K$, is bounded and continuous but $w \notin \overline{M}^{\|\cdot\|_\infty}$. Indeed, suppose contrary to our claim that $w \in \overline{M}^{\|\cdot\|_\infty}$. So there exists a sequence $\{f_n\varphi + \alpha_n 1_K\}_n \subseteq M$ such that $f_n\varphi + \alpha_n 1_K \xrightarrow{\|\cdot\|_\infty} w$. As uniform convergence implies pointwise convergence, it follows that $(f_n\varphi + \alpha_n 1_K)(x) \longrightarrow w(x)$ for all $x \in K$. So,

$$\begin{aligned}
\alpha_n &= f_n(0)\varphi(0) + \alpha_n 1_K(0) \\
&\longrightarrow \|0\| \\
&= 0,
\end{aligned}$$

and also

$$\begin{aligned}
\alpha_n &= f_n(\frac{x_0}{\|x_0\|})\varphi(\frac{x_0}{\|x_0\|}) + \alpha_n 1_K(\frac{x_0}{\|x_0\|}) \\
&\longrightarrow \|\frac{x_0}{\|x_0\|}\| \\
&= 1,
\end{aligned}$$

which is a contradiction. □

3. Weak and cyclic amenability of $(C^{b\varphi}(K), \|\cdot\|_\infty)$

Clearly each amenable Banach algebra is also weakly amenable. Hence $(C^b(K), \|\cdot\|_\infty)$ is weakly amenable. The following theorem shows that $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is not weakly amenable.

Theorem 3.1. *$(C^{b\varphi}(K), \|\cdot\|_\infty)$ is not weakly amenable.*

Proof. Define $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^*$ by $D(f) = f(0)\hat{0}$ for all $f \in C^{b\varphi}(K)$. A straightforward calculation reveals that D is well-defined and linear. Obviously, D is non-zero. Indeed,

$$\begin{aligned}\langle D(1_K), 1_K\rangle &= 1_K(0)1_K(0)\\ &= 1\\ &\neq 0.\end{aligned}$$

Also,

$$\begin{aligned}|\langle D(f), g\rangle| &= |f(0)g(0)|\\ &\leq \|f\|_\infty \|g\|_\infty.\end{aligned}$$

So $\|D\| \leq 1$. We shall show that D is a derivation. For $f, g, h \in C^{b\varphi}(K)$,

$$\begin{aligned}\langle D(f\cdot g), h\rangle &= \langle D(f\varphi g), h\rangle\\ &= (f\varphi g)(0)h(0)\\ &= f(0)\varphi(0)g(0)h(0)\\ &= 0.\end{aligned}$$

So $D(f \cdot g) = 0$ for all $f, g \in C^{b\varphi}(K)$.
On the other hand,

$$\begin{aligned}\langle D(f)\cdot g + f\cdot D(g), h\rangle &= \langle D(f)g\varphi, h\rangle + \langle f\varphi D(g), h\rangle\\ &= \langle D(f), g\varphi h\rangle + \langle D(g), hf\varphi\rangle\\ &= f(0)(g\varphi h)(0) + g(0)(hf\varphi)(0)\\ &= 0.\end{aligned}$$

Hence $D(f)\cdot g + f \cdot D(g) = 0$ for all $f, g \in C^{b\varphi}(K)$. Therefore

$$D(f\cdot g) = D(f)\cdot g + f\cdot D(g), \quad f, g \in C^{b\varphi}(K).$$

Since $D \neq 0$ and $C^{b\varphi}(K)$ is commutative, so D is not inner. This shows that $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is not weakly amenable. □

Theorem 3.2. *Let $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^*$ be a bounded linear map. Then D is a derivation if and only if $D(f\varphi) = fD(\varphi) = 2D(f)\varphi$ for all $f \in C^{b\varphi}(K)$.*

Proof. Let D be a derivation. Define $\tilde{D} : (C^b(K), \|\cdot\|_\infty) \longrightarrow (C^b(K), \|\cdot\|_\infty)^*$ by $\tilde{D}(f) = D(f\varphi) - fD(\varphi)$, $f \in C^b(K)$. Clearly $\tilde{D}$ is well-defined and linear. Also $\tilde{D}$ is bounded. Indeed for $f \in C^b(K)$

we have,

$$\begin{aligned}\|\tilde{D}(f)\|_\infty^* &= \|D(f\varphi) - fD(\varphi)\|_\infty^* \\ &\leq \|D(f\varphi)\|_\infty^* + \|fD(\varphi)\|_\infty^* \\ &\leq \|D\|\|f\varphi\|_\infty + \|f\|_\infty\|D(\varphi)\|_\infty^* \\ &\leq \|D\|\|f\|_\infty\|\varphi\|_\infty + \|f\|_\infty\|D\|\|\varphi\|_\infty \\ &= (2\|D\|\|\varphi\|)\|f\|_\infty.\end{aligned}$$

So $\|\tilde{D}\| \leq 2\|D\|\|\varphi\|$. Hence $\tilde{D}$ is bounded. Since

$$\begin{aligned}D(\varphi) &= D(1_K \cdot 1_K) \\ &= 2D(1_K)\varphi,\end{aligned}$$

and $f\varphi = f \cdot 1_K$ so,

$$\tilde{D}(f) = D(f)\varphi - \frac{f}{2}D(\varphi), \quad f \in C^b(K). \tag{3.1}$$

We shall show that $\tilde{D}$ is a derivation.

$$\begin{aligned}\tilde{D}(fg) &= D(fg\varphi) - fgD(\varphi) \\ &= D(f \cdot g) - fgD(\varphi) \\ &= D(f) \cdot g + f \cdot D(g) - fgD(\varphi) \\ &= D(f)g\varphi + f\varphi D(g) - fgD(\varphi), \quad f, g \in C^b(K).\end{aligned}$$

On the other hand,

$$\begin{aligned}\tilde{D}(f)g + f\tilde{D}(g) &= (D(f)\varphi - \frac{f}{2}D(\varphi))g + f(D(g)\varphi - \frac{g}{2}D(\varphi)) \\ &= D(f)g\varphi - \frac{fg}{2}D(\varphi) + f\varphi D(g) - \frac{fg}{2}D(\varphi) \\ &= D(f)g\varphi + f\varphi D(g) - fgD(\varphi), \quad f, g \in C^b(K).\end{aligned}$$

Hence $\tilde{D}(fg) = \tilde{D}(f)g + f\tilde{D}(g), f, g \in C^b(K)$. By amenability of $(C^b(K), \|\cdot\|_\infty)$, there exists $\Lambda \in (C^b(K), \|\cdot\|_\infty)^*$ such that $\tilde{D}(f) = \delta_\Lambda(f) = f\Lambda - \Lambda f = 0$ for all $f \in C^b(K)$. It follows that $D(f\varphi) = fD(\varphi)$ for all $f \in C^b(K)$. Also by (3.1) we have $D(f)\varphi - \frac{f}{2}D(\varphi) = 0$. So $fD(\varphi) = 2D(f)\varphi$ for all $f \in C^b(K)$. Therefore,

$$D(f\varphi) = fD(\varphi) = 2D(f)\varphi, \quad f \in C^{b\varphi}(K).$$

For the converse, let the equalities $D(f\varphi) = fD(\varphi) = 2D(f)\varphi$ hold for all $f \in C^{b\varphi}(K)$. So

$$\begin{aligned}D(f \cdot g) &= D(f\varphi g) \\ &= D(fg\varphi) \\ &= fgD(\varphi), \quad f, g \in C^{b\varphi}(K).\end{aligned}$$

Also,

$$\begin{aligned} D(f)\cdot g + f\cdot D(g) &= D(f)\varphi g + f\varphi D(g)\\ &= D(f)\varphi g + fD(g)\varphi\\ &= (\frac{f}{2}D(\varphi))g + f(\frac{g}{2}D(\varphi))\\ &= \frac{fg}{2}D(\varphi) + \frac{fg}{2}D(\varphi)\\ &= fgD(\varphi), \quad f,g\in C^{b\varphi}(K). \end{aligned}$$

This shows that D is a derivation. □

Corollary 3.3. *Let $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^*$ be a derivation and let $\ker\varphi = \{0\}$. Then $D = 0$ if and only if $D(1_K) = 0$.*

Proof. Let $D(1_K) = 0$. So $D(\varphi) = 2D(1_K)\varphi = 0$. Hence by Theorem 3.2, $D(f\varphi) = 0$ for all $f \in C^b(K)$. It follows that $D\Big|_M = 0$, where M is our notation in Theorem 2.5. Since D is continuous and $\overline{M}^{\|\cdot\|_\infty} = C^b(K)$ so $D = 0$. The converse is obvious. □

Theorem 3.4. *Let $\ker\varphi = \{0\}$. Then $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is cyclic amenable. Moreover, the only cyclic derivation on $(C^{b\varphi}(K), \|\cdot\|_\infty)$ is zero.*

Proof. Let $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^*$ be a cyclic derivation. So by Theorem 3.2, $D(f\varphi) = fD(\varphi)$ for all $f \in C^{b\varphi}(K)$. Since D is cyclic and $D(\varphi) = 2D(1_K)\varphi$ so,

$$\begin{aligned} 0 &= \langle D(f\varphi), 1_K\rangle + \langle D(1_K), f\varphi\rangle\\ &= \langle fD(\varphi), 1_K\rangle + \langle D(1_K), f\varphi\rangle\\ &= \langle D(\varphi), f\rangle + \langle D(1_K)\varphi, f\rangle\\ &= \langle D(\varphi), f\rangle + \langle \frac{D(\varphi)}{2}, f\rangle\\ &= \frac{3}{2}\langle D(\varphi), f\rangle, \quad f\in C^{b\varphi}(K). \end{aligned}$$

This shows that $D(\varphi) = 0$. Hence

$$\begin{aligned} \langle D(1_K), f\varphi\rangle &= \langle D(1_K)\varphi, f\rangle\\ &= \langle \frac{D(\varphi)}{2}, f\rangle\\ &= 0, \quad f\in C^{b\varphi}(K). \end{aligned}$$

Clearly $\langle D(1_K), 1_K\rangle = 0$. So we can conclude that $D(1_K)\Big|_M = 0$. The continuity of $D(1_K)$ and the equality $\overline{M}^{\|\cdot\|_\infty} = C^b(K)$ imply that $D(1_K) = 0$. Applying Corollary 3.3 shows that $D = 0 = \delta_0$. □

Inspired by the proof of Theorem 3.4 we can conclude the following result.

Corollary 3.5. *Let* $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^*$ *be a cyclic derivation. Then* $D(\varphi) = 0$ *and* $D(1_K)\Big|_{\overline{M}^{\|\cdot\|_\infty}} = 0$.

Proposition 3.6. *Let* $D : (C^{b\varphi}(K), \|\cdot\|_\infty) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\infty)^*$ *be a cyclic derivation. Then* $\overline{ImD} \subsetneqq (C^{b\varphi}(K), \|\cdot\|_\infty)^*$, *where ImD is the image of D.*

Proof. By Corollary 3.5 and Theorem 3.2, $D(f)\varphi = 0$ for all $f \in C^{b\varphi}(K)$. Since $\varphi \neq 0$ so there exists $\Lambda \in (C^{b\varphi}(K), \|\cdot\|_\infty)^*$ such that $\langle \Lambda, \varphi \rangle \neq 0$. We claim that $\Lambda \notin \overline{ImD}$. Assume by absurd that $\Lambda \in \overline{ImD}$. Then there exists a sequence $\{f_n\}_n \subseteq C^{b\varphi}(K)$ such that $D(f_n) \xrightarrow{\|\cdot\|_\infty^*} \Lambda$. So $0 = D(f_n)\varphi \xrightarrow{\|\cdot\|_\infty^*} \Lambda\varphi$. This shows that $\Lambda\varphi = 0$ and so $\langle \Lambda, \varphi \rangle = \langle \Lambda\varphi, 1_K \rangle = 0$ that is a contradiction. □

4. Weak and cyclic amenability of $(C^{b\varphi}(K), \|\cdot\|_\varphi)$

In this section we characterize the derivations from $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ into $(C^{b\varphi}(K), \|\cdot\|_\varphi)^*$. Also we investigate weak and cyclic amenability of $(C^{b\varphi}(K), \|\cdot\|_\varphi)$.

Theorem 4.1. *Let* $N = \{f\varphi \mid f \in C^b(K)\}$. *Then* $\overline{N}^{\|\cdot\|_\varphi} = C^b(K)$.

Proof. Let $g \in C^b(K)$. Define $f_n : K \longrightarrow \mathbb{C}$ by,

$$f_n(x) = \begin{cases} 0 & x \in \ker\varphi \cap K \\ g(x)\frac{e^{\frac{-1}{n|\varphi(x)|}}}{\varphi(x)} & x \in K \setminus \ker\varphi \end{cases}$$

for all $n \in \mathbb{N}$. One can easily verify that each f_n is continuous and $\|f_n\|_\infty \leq \frac{n}{e}\|g\|_\infty$. Because the maximum value of the function $s(t) = \frac{e^{\frac{-1}{nt}}}{t}, t \in (0, \|\varphi\|]$, occurs in $t = \frac{1}{n}$. We shall show that $\|f_n\varphi - g\|_\varphi = \|f_n\varphi^2 - g\varphi\|_\infty \longrightarrow 0$. To this end let $x \in K \setminus \ker\varphi$. So

$$\begin{aligned} |f_n(x)\varphi^2(x) - g(x)\varphi(x)| &= |g(x)\varphi(x)e^{\frac{-1}{n|\varphi(x)|}} - g(x)\varphi(x)| \\ &= |g(x)||\varphi(x)||e^{\frac{-1}{n|\varphi(x)|}} - 1| \\ &= |g(x)||\varphi(x)|(1 - e^{\frac{-1}{n|\varphi(x)|}}) \\ &\leq \|g\|_\infty|\varphi(x)|(1 - e^{\frac{-1}{n|\varphi(x)|}}). \end{aligned}$$

Let $t = |\varphi(x)|$ for all $x \in K \setminus \ker\varphi$. Clearly $0 < t \leq \|\varphi\|$. Define $h(t) = t(1 - e^{\frac{-1}{nt}})$ for all $t \in (0, \|\varphi\|]$. An argument similar to the proof of Theorem 2.5 reveals that $\|h\|_\infty = \|\varphi\|(1 - e^{\frac{-1}{n\|\varphi\|}})$. Hence $|f_n(x)\varphi^2(x) - g(x)\varphi(x)| \leq \|g\|_\infty\|\varphi\|(1 - e^{\frac{-1}{n\|\varphi\|}})$ for all $x \in K \setminus \ker\varphi$. Therefore $|f_n(x)\varphi^2(x) - g(x)\varphi(x)| \leq \|g\|_\infty\|\varphi\|(1 - e^{\frac{-1}{n\|\varphi\|}})$ for all $x \in K$. This shows that $\|f_n\varphi^2 - g\varphi\|_\infty \leq \|g\|_\infty\|\varphi\|(1 - e^{\frac{-1}{n\|\varphi\|}})$ for all $n \in \mathbb{N}$. So $\|f_n\varphi - g\|_\varphi = \|f_n\varphi^2 - g\varphi\|_\infty \longrightarrow 0$. It follows that $\overline{N}^{\|\cdot\|_\varphi} = C^b(K)$. □

Theorem 4.2. *Let* $D : (C^{b\varphi}(K), \|\cdot\|_\varphi) \longrightarrow (C^{b\varphi}(K), \|\cdot\|_\varphi)^*$ *be a bounded linear map. Then D is a derivation if and only if* $D = 0$.

Proof. Let D be a derivation. By Corollary 2.2, $(C^{b\varphi}(K), \|\cdot\|_\varphi)^* \subseteq (C^b(K), \|\cdot\|_\infty)^*$. So we can define $\tilde{D} : (C^b(K), \|\cdot\|_\infty) \longrightarrow (C^b(K), \|\cdot\|_\infty)^*$ by $\tilde{D}(f) = D(f\varphi) - fD(\varphi)$ for all $f \in C^b(K)$. Clearly $\tilde{D}$ is linear. We claim that $\tilde{D}$ is bounded. Let $f \in C^b(K)$. So by Proposition 2.1 we can conclude that,

$$\begin{aligned}
\|\tilde{D}(f)\|_\infty^* &= \|D(f\varphi) - fD(\varphi)\|_\infty^* \\
&\leq \|D(f\varphi)\|_\infty^* + \|fD(\varphi)\|_\infty^* \\
&\leq \|D(f\varphi)\|_\infty^* + \|f\|_\infty \|D(\varphi)\|_\infty^* \\
&\leq \|D(f\varphi)\|_\varphi^* \|\varphi\| + \|f\|_\infty \|D(\varphi)\|_\varphi^* \|\varphi\| \\
&\leq \|D\| \|f\varphi\|_\varphi \|\varphi\| + \|f\|_\infty \|D\| \|\varphi\|_\varphi \|\varphi\| \\
&= \|D\| \|f\varphi^2\|_\infty \|\varphi\| + \|f\|_\infty \|D\| \|\varphi^2\|_\infty \|\varphi\| \\
&\leq \|D\| \|f\|_\infty \|\varphi^2\|_\infty \|\varphi\| + \|f\|_\infty \|D\| \|\varphi^2\|_\infty \|\varphi\| \\
&= \|D\| \|f\|_\infty \|\varphi\|^3 + \|f\|_\infty \|D\| \|\varphi\|^3 \\
&= (2\|D\| \|\varphi\|^3) \|f\|_\infty.
\end{aligned}$$

So $\tilde{D}$ is bounded. An argument similar to the proof of Theorem 3.2 can be applied to show that $\tilde{D}$ is a derivation and consequently,

$$D(f\varphi) = fD(\varphi) = 2D(f)\varphi \tag{4.1}$$

for all $f \in C^b(K)$. Replacing f by $f\varphi$ in (4.1) we can conclude that,

$$D(f\varphi^2) = f\varphi D(\varphi) = 2D(f\varphi)\varphi. \tag{4.2}$$

So we have,

$$\begin{aligned}
f\varphi D(\varphi) &= 2D(f\varphi)\varphi \\
&= 2(2D(f)\varphi)\varphi \\
&= 4D(f)\varphi^2, \quad f \in C^b(K).
\end{aligned}$$

Hence $(fD(\varphi) - 4D(f)\varphi)\varphi = 0$ for all $f \in C^b(K)$. It follows that

$$\begin{aligned}
0 &= \langle (fD(\varphi) - 4D(f)\varphi)\varphi, g \rangle \\
&= \langle fD(\varphi) - 4D(f)\varphi, g\varphi \rangle, \quad g \in C^b(K).
\end{aligned}$$

So $fD(\varphi) - 4D(f)\varphi\Big|_N = 0$, where N is our notation in Theorem 4.1. Since $fD(\varphi) - 4D(f)\varphi \in (C^{b\varphi}(K), \|\cdot\|_\varphi)^*$ so $fD(\varphi) - 4D(f)\varphi\Big|_{\overline{N}^{\|\cdot\|_\varphi}} = 0$. Applying Theorem 4.1 we can conclude that,

$$fD(\varphi) = 4D(f)\varphi \tag{4.3}$$

for all $f \in C^b(K)$. Combining (4.1) with (4.3) we obtain $D(f)\varphi = 0$ for all $f \in C^b(K)$. Hence,

$$\begin{aligned} 0 &= \langle D(f)\varphi, g\rangle \\ &= \langle D(f), g\varphi\rangle \end{aligned}$$

for all $g \in C^b(K)$. It follows that $D(f)\Big|_N = 0$ for all $f \in C^b(K)$. So $D(f)\Big|_{\overline{N}^{\|\cdot\|_\varphi}} = 0$ for all $f \in C^b(K)$. This shows that $D(f) = 0$ for all $f \in C^b(K)$ and so $D = 0$. The Converse is obvious. □

Corollary 4.3. $(C^{b\varphi}(K), \|\cdot\|_\varphi)$ *is weakly and cyclicly amenable.*

Proof. The proof immediately yields from Theorem 4.2. □

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