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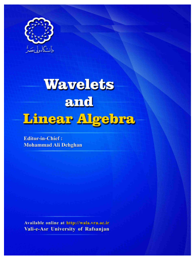

ON CONTINUOUS WEAVING G-FRAMES IN HILBERT SPACES

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Abstract

In this note, we intend to introduce the concept of weaving continuous g-frames in Hilbert spaces. In addition, we present some new result for these frames and also we show that it is enough to check that on smaller measurable space than the given measurable space. We investigate the relationship between these frames and c-woven also, the sufficient and condition will be given. Finally, we verify the perturbation of weaving c-g-frames.

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1. Introduction and Preliminaries

Frames were introduced by Duffin and Scheaffer [11] in the context of non-harmonic Fourier series. They are very useful in characterization of function spaces and fields of applications such as filter bank theory, signal and image processing, coding and wireless communications[17].

Nowadays, frames have a significant role in both pure and applied mathematics, so that these are a fundamental research area in mathematics, computer science and engineering, but technical advances and measure amounts of data which cannot be handled with a signal processing system have been increased using the various frames as g-frame [18], fusion frame [7], K-frame [14] and weaving frames [4] and etc.

Continuous frames (or briefly c-frames) were been proposed by Kaiser [15] and also independently by Ali et al. [2] to a family indexed by some locally compact space endowed with a Radon measure. C-frames are the first generalization frames to measure spaces. For more studies about these frames, we refer to [12, 16]. By combining the above mentioned extensions of frames, the new and more general notion called *continuous g-frame* has been introduced in [1, 10].

Recently, Bemrose et al. [4] introduced a new concept of weaving frames which is motivated by a question in distributed signal processing. Also, the continuous version of weaving frames and their properteis is introduced by Vashisht and Deepshikha in [19, 20, 21]. In this paper, we introduce the concept of weaving continuous g-frames in Hilbert spaces and we generalize some results in [4, 19] to c-g-frames.

Throughout this paper, (Ω, μ) is a measure space with positive measure μ, H and $\{H_\omega\}_{\omega\in\Omega}$ are Hilbert spaces and a family of Hilbert spaces, respectively, and $\mathcal{B}(H, K)$ is the set of all bounded and linear operators from H to K. If $H = K$, then $\mathcal{B}(H, H)$ will be denoted by $\mathcal{B}(H)$. For each $m > 1$ where $m \in \mathbb{N}$, we define $[m] := \{1, 2, \cdots, m\}$ and $[m]^c = \{m + 1, m + 2, \cdots\}$.

First, we will need to the pseudo-inverse operator. If an operator U has closed range, then there exists a right-inverse operator $U^\dagger$ (pseudo-inverse of U) in the following sense (see [8])

Lemma 1.1. *Let $U \in \mathcal{B}(K, H)$ be a bounded operator with closed range $\mathcal{R}(U)$. Then there exists a bounded operator $U^\dagger \in \mathcal{B}(H, K)$ for which*

$$UU^\dagger x = x, \quad x \in \mathcal{R}(U).$$

We first recall the definition of continuous frame from [3, 16].

Definition 1.2. Suppose that (Ω, μ) is a measure space with positive measure μ. A mapping $F : \Omega \to H$ is called a continuous frame (or briefly c-frame) for H with respect to (Ω, μ), if

(i) For all $h \in H$, $\omega \mapsto \langle h, F(\omega)\rangle$ is a measurable function on Ω,

(ii) there exist positive constants A and B such that for each $f \in H$,

$$A\|h\|^2 \le \int_\Omega |\langle h, F(\omega)\rangle|^2 \, d\mu(x) \le B\|h\|^2. \tag{1.1}$$

Now, we summarize some facts about c-g-frames from [1]. Define

$$\Pi_{\omega\in\Omega} H_\omega = \{F : \Omega \longrightarrow \cup_{\omega\in\Omega} H_\omega : \ F(\omega) \in H_\omega\}.$$

We say that $F \in \Pi_{\omega\in\Omega} H_\omega$ is strongly measurable if F as a mapping of Ω to $\oplus_{\omega\in\Omega} H_\omega$ is measurable. Let

$$\mathfrak{L}^2(H_\omega, \mu) = \left\{ F \in \Pi_{\omega\in\Omega} H_\omega : F \text{ is strongly measurable}, \int_\Omega \|F(\omega)\|^2 d\mu(\omega) < \infty \right\}.$$

With inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(\omega), G(\omega)\rangle d\mu(\omega).$$

It can be proved that $\mathfrak{L}^2(H_\omega, \mu)$ is a Hilbert space ([1]). We will denote the norm of $F \in \mathfrak{L}^2(H_\omega, \mu)$ by $\|F\|_2$.

Definition 1.3. A family $\{\Lambda_\omega \in \mathcal{B}(H, H_\omega)\}_{\omega\in\Omega}$ is called a continuous g-frame (or briefly c-g-frame) for H with respect to $\{H_\omega\}_{\omega\in\Omega}$, if

(i) the mapping

$$\Omega \longmapsto \mathbb{C},$$
$$\omega \longmapsto \|\Lambda_\omega f\|,$$

is measurable for any $f \in H$.

(ii) there exist constants $0 < A \le B < \infty$ such that for each $f \in H$,

$$A\|f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \le B\|f\|^2. \tag{1.2}$$

If A, B can be chosen such that $A = B$, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a tight c-g-frame and if $A = B = 1$, it is called Parseval c-g-frame. A family $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a c-g-Bessel family if the right hand inequality (1.2) holds and the number B is called the Bessel constant.

Theorem 1.4. ([1]). Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel family for H with respect to $\{H_\omega\}_{\omega\in\Omega}$ with the bound B. Then the mapping T_Λ of $\mathfrak{L}^2(H_\omega, \mu)$ to H defined by

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle d\mu(\omega), \quad F \in \mathfrak{L}^2(H_\omega, \mu), \; g \in H,$$

is linear and bounded with $\|T_\Lambda\| \le \sqrt{B}$. Furthermore, for each $g \in H$ and $\omega \in \Omega$,

$$T_\Lambda^*(g)(\omega) = \Lambda_\omega g.$$

In the continuous g-frame, frame operator $S_\Lambda = T_\Lambda T_\Lambda^*$ is defined by

$$S_\Lambda : H \longrightarrow H,$$
$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle f, \Lambda_\omega \Lambda_\omega^* g\rangle \, d\mu.$$

Therefore,

$$A Id_H \le S_\Lambda \le B Id_H$$

and we obtain, if $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a c-g-frame, then S_Λ is a positive, self-adjoint and invertible operator.

Definition 1.5. *([19]). A family of c-frames* $\{F_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ *for* H *with respect to* μ *is said to be c-woven if there exist universal same positive constants* $0 < A \leq B < \infty$ *such that for each partition* $\{\sigma_i\}_{i\in[m]}$ *of* Ω*, the family* $\{F_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ *is a c-frame for* H *with bounds* A *and* B*. Each family* $\{F_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ *is called a weaving.*

2. Continuous Weaving g-Frames

In this section, we introduce the notation of continuous g-woven in Hilbert spaces and discuss some of their properties.

Definition 2.1. A family of c-g-frames $\{\Lambda_{\omega i} \in \mathcal{B}(H, H_\omega)\}_{\omega\in\Omega, i\in[m]}$ for H is said to be continuous g-woven (or c-g-woven) if there exist universal constants $0 < A \leq B$ such that for each partition $\{\sigma_i\}_{i\in[m]}$ of Ω, the family $\{\Lambda_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ is a c-g-frame for H with bounds A and B.

In the above definition, A and B is called universal c-g-frame bounds. It is easy to show that every c-g-woven has an universal upper c-g-frame bound. Indeed, let $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ be a c-g-Bessel family for H with bound B_i for each $i \in [m]$. Then, for any partition $\{\sigma_i\}_{i\in[m]}$ of Ω and $f \in H$ we have

$$\sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu \leq \sum_{i\in[m]} \int_{\Omega} \|\Lambda_{\omega i} f\|^2 \, d\mu \leq \Big(\sum_{i\in[m]} B_i \Big) \|f\|^2.$$

In the next results, we construct a c-g-woven by using a bounded linear operator.

Theorem 2.2. *Let* $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ *be a c-g-woven for* H *with universal bounds* A, B*. If* $U \in \mathcal{B}(H)$ *has closed range, then* $\{\Lambda_{\omega i} U^*\}_{\omega\in\Omega, i\in[m]}$ *is a c-g-woven for* $\mathcal{R}(U)$ *with frame bounds* $A\|U^\dagger\|^{-2}$ *and* $B\|U\|^2$.

Proof. First, since $U^* f \in H$ and $\omega \mapsto \|\Lambda_{\omega i} f\|$ is a measurable function for each $f \in H$ and $i \in [m]$, then $\omega \mapsto \|\Lambda_{\omega i} U^* f\|$ is measurable for any $f \in H$ and $i \in [m]$. On the other hand, for each $f \in \mathcal{R}(U)$, we have

$$\begin{aligned} A\|f\|^2 &= A\|(U^\dagger)^* U^* f\|^2 \\ &\leq A\|U^\dagger\|^2 \|U^* f\|^2 \\ &\leq \|U^\dagger\|^2 \sum_{i\in[m]} \int_{\Omega} \|\Lambda_{\omega i} U^* f\|^2 \, d\mu. \end{aligned}$$

The upper bound condition is given similarly. □

Corollary 2.3. *Let* $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ *be a c-g-woven for* H *with universal bounds* A, B*. If* $U \in \mathcal{B}(H)$ *is invertible, then* $\{\Lambda_{\omega i} U\}_{\omega\in\Omega, i\in[m]}$ *is a c-g-woven for* H *with frame bounds* $A\|U^{-1}\|^{-2}$ *and* $B\|U\|^2$.

The next proposition shows that it is enough to check c-g-weaving on smaller measurable space than the original which this is an extension of Proposition 3.10 in [19].

Theorem 2.4. *For each $i \in [m]$, let $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ be a c-g-frame for H with frame bounds A_i and B_i. If there exists a measurable subset $\Sigma \subset \Omega$ such that the family of c-g-frame $\{\Lambda_{\omega i}\}_{\omega\in\Sigma, i\in[m]}$ is a c-g-woven for H with universal frame bounds A and B, then $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ is a c-g-woven for H with universal bounds A and $\sum_{i\in[m]} B_i$.*

Proof. Suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of Ω and $f \in H$. The upper bound is clear. For the lower bound, it is clear that $\{\sigma_i \cap \Sigma\}_{i\in[m]}$ is a partition of Σ. Thus, $\{\Lambda_{\omega i}\}_{\omega\in\sigma_i\cap\Sigma, i\in[m]}$ is a c-g-frame for H with the lower frame bound A. Hence, for each $f \in H$

$$\begin{aligned}\sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) &\geq \sum_{i\in[m]} \int_{\sigma_i\cap\Sigma} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\ &\geq A\|f\|^2.\end{aligned}$$

□

Casazza and Lynch in [6] showed that It is possible to remove vectors from woven frames and still be left with woven frames. After, this topic was been presented in [19]. Now, we study it for c-g-woven in the following Theorem.

Theorem 2.5. *Let $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ be a c-g-woven for H with universal bounds A and B. If there exists $0 < D < A$ and a measurable subset $\Sigma \subset \Omega$ and $n \in [m]$ such that for each $f \in H$,*

$$\sum_{i\in[m]\setminus\{n\}} \int_{\Omega\setminus\Sigma} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \leq D\|f\|^2, \qquad \forall f \in H,$$

Then the family $\{\Lambda_{\omega i}\}_{\omega\in\Sigma, i\in[m]}$ is a c-g-woven for H with frame bounds $A - D$ and B.

Proof. Suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of Σ and $\{\tau_i\}_{i\in[m]}$ is a partition of $\Omega \setminus \Sigma$. For a given $f \in H$, we define

$$\begin{aligned}\varphi &: \Sigma \longrightarrow \mathbb{C}, \\ \varphi(\omega) &= \sum_{i\in[m]} \chi_{\sigma_i}(\omega)\|\Lambda_{\omega i} f\|,\end{aligned}$$

and

$$\begin{aligned}\phi &: \Lambda \longrightarrow \mathbb{C}, \\ \phi(\omega) &= \sum_{i\in[m]} \chi_{\sigma_i\cup\tau_i}(\omega)\|\Lambda_{\omega i} f\|,\end{aligned}$$

where, χ_{σ_i} is the characteristic function of σ_i. Since $\{\Lambda_{\omega i}\}_{\omega\in\sigma_i\cup\tau_i, i\in[m]}$ is a c-g-frame for H and $\varphi = \phi|_\Sigma$, then φ and ϕ are measurable. So, for each $f \in H$, we have

$$\begin{aligned}\sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) &\leq \sum_{i\in[m]} \int_{\sigma_i\cup\tau_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\ &\leq B\|f\|^2.\end{aligned}$$

Now, for the lower frame bound, assume that $\{\varsigma_i\}_{i\in[m]}$ is a partition of $\Omega \setminus \Sigma$ such that $\varsigma_n = \emptyset$. Then $\{\varsigma_i \cup \sigma_i\}_{i\in[m]}$ is a partition of Ω and so, for any $f \in H$ we have,

$$
\begin{aligned}
&\sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&= \sum_{i\in[m]\setminus\{n\}} \Big(\int_{\varsigma_i \cup \sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) - \int_{\varsigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \Big) + \int_{\sigma_n} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&\geq \sum_{i\in[m]\setminus\{n\}} \Big(\int_{\varsigma_i \cup \sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) - \int_{\Omega\setminus\Sigma} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \Big) + \int_{\sigma_n} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&= \sum_{i\in[m]} \int_{\varsigma_i \cup \sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) - \sum_{i\in[m]\setminus\{n\}} \int_{\Omega\setminus\Sigma} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&\geq (A - D)\|f\|^2.
\end{aligned}
$$

□

The following presents a relationship between the norms of the c-g-frame operator of original c-g-frame and the weaving.

Theorem 2.6. *Let $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ be a c-g-woven for H with universal bounds A and B. If $S^{(i)}_{\Lambda}$ is the c-g-frame operator of $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ for each $i \in [m]$, S_{Λ,σ_i} represents the c-g-frame operator of $\{\Lambda_{\omega i}\}_{\omega\in\sigma_i, i\in[m]}$ for each partition $\{\sigma_i\}_{i\in[m]}$ of Ω and $S^{(i)}_{\Lambda,\sigma_i}$ denotes the c-g-frame operator $S^{(i)}_{\Lambda}$ with integral restricted to σ_i, then for each $f \in H$,*

$$\sum_{i\in[m]} \|S^{(i)}_{\Lambda,\sigma_i} f\|^2 \leq B \|S_{\Lambda,\sigma_i}\| \|f\|^2.$$

Proof. Suppose that $f \in H$. We can write

$$
\begin{aligned}
\sum_{i\in[m]} \|S^{(i)}_{\Lambda,\sigma_i} f\|^2 &= \sum_{i\in[m]} \big(\sup_{\|g\|=1} |\langle S^{(i)}_{\Lambda,\sigma_i} f, g\rangle| \big)^2 \\
&= \sum_{i\in[m]} \big(\sup_{\|g\|=1} |\langle T^{(i)}_{\Lambda,\sigma_i} (T^{(i)}_{\Lambda,\sigma_i})^* f, g\rangle| \big)^2 \\
&\leq \sum_{i\in[m]} B \|(T^{(i)}_{\Lambda,\sigma_i})^* f\|^2 \\
&= B \sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&= B \langle S_{\Lambda,\sigma_i} f, f\rangle \\
&\leq B \|S_{\Lambda,\sigma_i}\| \|f\|^2.
\end{aligned}
$$

□

The next result provides a necessary and sufficient condition for c-g-woven which connects to c-woven.

Theorem 2.7. *Let $\Omega_i \subseteq \Omega$ be measurable subsets for all $i \in [m]$, and let F_i and G_i be c-frame mappings on Ω_i for H_ω with the pair frame bounds (A_{F_i}, B_{F_i}) and (A_{G_i}, B_{G_i}), respectively, for each $\omega \in \Omega$. Assume that $\Lambda_{\omega i}, \Theta_{\omega i} \in \mathcal{B}(H, \Omega_i)$ for any $i \in [m]$ such that $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ are strongly measurable. Then the following assertions are equivalent.*

(I) $\{\Lambda^*_{\omega i} F_i\}_{\omega\in\Omega, i\in[m]}$ *and* $\{\Theta^*_{\omega i} G_i\}_{\omega\in\Omega, i\in[m]}$ *are c-woven for H.*

(II) $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ *and* $\{\Theta_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ *are c-g-woven for H.*

Proof. $(I) \Rightarrow (II)$. Suppose that $\sigma \subset \Omega$ is a measurable subset and $f \in H$. Let $\{\Lambda^*_{\omega i} F_i\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta^*_{\omega i} G_i\}_{\omega\in\Omega, i\in[m]}$ are c-woven for H with universal frame bounds C, D and $A = \inf\{A_{F_{\omega i}}, A_{G_{\omega i}}\}$. We have for each $i \in [m]$,

$$\begin{aligned}
&A \int_\sigma \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) + A \int_{\sigma^c} \|\Theta_{\omega i} f\|^2 \, d\mu(\omega) \\
&\leq \int_\sigma A_{F_{\omega i}} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) + \int_{\sigma^c} A_{G_{\omega i}} \|\Theta_{\omega i} f\|^2 \, d\mu(\omega) \\
&\leq \int_\sigma \int_{\Omega_i} |\langle \Lambda_{\omega i} f, F_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle \Theta_{\omega i} f, G_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) \\
&= \int_\sigma \int_{\Omega_i} |\langle f, \Lambda^*_{\omega i} F_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle f, \Theta^*_{\omega i} G_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) \\
&\leq D\|f\|^2.
\end{aligned}$$

With the same way, we conclude that

$$\begin{aligned}
&B \int_\sigma \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) + B \int_{\sigma^c} \|\Theta_{\omega i} f\|^2 \, d\mu(\omega) \\
&\geq \int_\sigma \int_{\Omega_i} |\langle f, \Lambda^*_{\omega i} F_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle f, \Theta^*_{\omega i} G_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) \\
&\geq C\|f\|^2,
\end{aligned}$$

where $B = \sup\{B_{F_{\omega i}}, B_{G_{\omega i}}\}$. Thus, we obtain $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ are c-g-woven for H with universal frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

$(II) \Rightarrow (I)$. Suppose that $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ are c-g-woven for H with universal frame bounds C and D. Now, we can write for each $f \in H$,

$$\begin{aligned}
&\int_\sigma \int_{\Omega_i} |\langle f, \Lambda^*_{\omega i} F_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle f, \Theta^*_{\omega i} G_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) \\
&= \int_\sigma \int_{\Omega_i} |\langle \Lambda_{\omega i} f, F_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) + \int_{\sigma^c} \int_{\Omega_i} |\langle \Theta_{\omega i} f, G_i(x)\rangle|^2 \, d\mu(x)\, d\mu(\omega) \\
&\geq \int_\sigma A_{F_{\omega i}} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) + \int_{\sigma^c} A_{G_{\omega i}} \|\Theta_{\omega i} f\|^2 \, d\mu(\omega) \\
&\geq A\Big(\int_\sigma \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) + \int_{\sigma^c} \|\Theta_{\omega i} f\|^2 \, d\mu(\omega) \\
&\geq AC\|f\|^2.
\end{aligned}$$

Also, we can get

$$\int_{\sigma}\int_{\Omega_i} |\langle f, \Lambda^*_{\omega i}F_i(x)\rangle|^2\, d\mu(x)\, d\mu(\omega) + \int_{\sigma^c}\int_{\Omega_i} |\langle f, \Theta^*_{\omega i}G_i(x)\rangle|^2\, d\mu(x)\, d\mu(\omega) \le BD\|f\|^2.$$

So, $\{\Lambda^*_{\omega i}F_i\}_{\omega\in\Omega, i\in[m]}$ and $\{\Theta^*_{\omega i}G_i\}_{\omega\in\Omega, i\in[m]}$ are c-woven for H. with universal bounds AC and BD. $\square$

The following theorem is extension of Lemma 4.3 of [4] for c-g-frames.

Theorem 2.8. *Let $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ be a family of c-g-frames for H with respect to a σ-finite measure μ. Suppose that for a partition collection of disjoint finite sets $\{\tau_i\}_{i\in[m]}$ of Ω and for any $\varepsilon > 0$ there exists a partition $\{\sigma_i\}_{i\in[m]}$ of the set $\Omega \setminus \bigcup_{i\in[m]} \tau_i$ such that $\{\Lambda_{\omega i}\}_{\omega\in(\sigma_i\cup\tau_i), i\in[m]}$ has a lower c-g-frame bound less than ε. Then $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ is not a c-g-woven.*

Proof. Since (Ω, μ) is a σ-finite measure space, then $\Omega = \cup_{i\in\mathbb{N}}\Omega_i$, where Ω_i are disjoint measurable sets and $\mu(\Omega_i) < \infty$ for all $i \in \mathbb{N}$. Assume that $\tau_{i1} = \emptyset$ for all $i \in [m]$ and $\varepsilon = 1$. Then, by the assumption, there exists a partition $\{\sigma_{i1}\}_{i\in[m]}$ of Ω such that $\{\Lambda_{\omega i}\}_{\omega\in(\sigma_{i1}\cup\tau_{i1}), i\in[m]}$ has a lower bound (also, optimal lower bound) less that 1. Thus, there is a vector $f_1 \in H$ with $\|f_1\| = 1$ such that

$$\sum_{i\in[m]}\int_{\sigma_{i1}\cup\tau_{i1}} \|\Lambda_{\omega i}f_1\|^2\, d\mu(\omega) < 1.$$

Since

$$\sum_{i\in[m]}\int_{\Omega} \|\Lambda_{\omega i}f_1\|^2\, d\mu(\omega) < \infty,$$

so, there is a $k_1 \in \mathbb{N}$ such that

$$\sum_{i\in[m]}\int_{\mathbb{K}_1} \|\Lambda_{\omega i}f_1\|^2\, d\mu(\omega) < 1,$$

where, $\mathbb{K}_1 = \cup_{i\ge k_1+1}\Omega_i$.

Continuing this way, for $\varepsilon = \frac{1}{n}$ and a partition $\{\tau_{in}\}_{i\in[m]}$ of $\Omega_1 \cup \cdots \cup \Omega_{k_n-1}$ such that

$$\tau_{in} = \tau_{i(n-1)} \cup \left(\sigma_{i(n-1)} \cap (\Omega_1 \cup \cdots \cup \Omega_{k_n-1})\right)$$

for all $i \in [m]$, there exists a partition $\{\sigma_{in}\}_{i\in[m]}$ of $\Omega \setminus (\Omega_1 \cup \cdots \cup \Omega_{k_n-1})$ such that $\{\Lambda_{\omega i}\}_{\omega\in(\sigma_{in}\cup\tau_{in}), i\in[m]}$ has a lower bound less than $\dfrac{1}{n}$. Therefore, there is a $f_n \in H$ and $k_n \in \mathbb{N}$ such that $\|f_n\| = 1$, $k_n > k_{n-1}$ and

$$\sum_{i\in[m]}\int_{\mathbb{K}_n} \|\Lambda_{\omega i}f_n\|^2\, d\mu(\omega) < \frac{1}{n},$$

where, $\mathbb{K}_n = \cup_{i\ge k_n+1}\Omega_i$. Choose a partition $\{\varsigma_i\}_{i\in[m]}$ of Ω, where $\varsigma_i := \cup_{j\in\mathbb{N}}\{\tau_{ij}\} = \tau_{i(n+1)} \cup (\varsigma_i \cap \Omega \setminus (\Omega_1 \cup \cdots \cup \Omega_n))$. Assume that $\{\Lambda_{\omega i}\}_{\omega\in\varsigma_i, i\in[m]}$ is a c-g-frame for H with the optimal lower frame bound A. Then, by the Archimedean Property, there exists a $n \in \mathbb{N}$ such that $r > \frac{2}{A}$. Now, there exists a $f_r \in H$ with $\|f_r\| = 1$ such that

$$\begin{aligned}
&\sum_{i\in[m]} \int_{\varsigma_i} \|\Lambda_{\omega i} f_r\|^2 \, d\mu(\omega) \\
&= \sum_{i\in[m]} \int_{\tau_{i(r+1)}} \|\Lambda_{\omega i} f_r\|^2 \, d\mu(\omega) + \sum_{i\in[m]} \int_{\varsigma_i \cap \Omega \setminus (\Omega_1 \cup \cdots \cup \Omega_r)} \|\Lambda_{\omega i} f_r\|^2 \, d\mu(\omega) \\
&\leq \sum_{i\in[m]} \int_{\tau_{ir} \cup \sigma_{ir}} \|\Lambda_{\omega i} f_r\|^2 \, d\mu(\omega) + \sum_{i\in[m]} \int_{\cup_{k\geq r+1} \Omega_k} \|\Lambda_{\omega i} f_r\|^2 \, d\mu(\omega) \\
&< \frac{1}{r} + \frac{1}{r} \\
&< A\|f_r\|^2,
\end{aligned}$$

and this is a contradiction with the lower bound of A. □

Corollary 2.9. *Let $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ be a c-g-woven for H with respect to a σ-finite measure μ. Then there exists a collection of disjoint measurable subsets $\{\tau_i\}_{i\in[m]}$ of Ω and $A > 0$ such that for any partition $\{\sigma_i\}_{i\in[m]}$ of the set $\Omega \setminus \cup_{i\in[m]} \tau_i$, the family $\{\Lambda_{\omega i}\}_{\omega\in(\tau_i\cup\sigma_i), i\in[m]}$ is a c-g-frame for H with the lower frame bound A.*

Corollary 2.10. *Suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Theta_\omega\}_{\omega\in\Omega}$ are c-g-frames for H with optimal upper frame bounds B_1 and B_2, respectively, and they are c-g-woven for H. Then, $B_1 + B_2$ is not an optimal upper frame bound for the c-g-woven.*

Proof. Let $\varepsilon > 0$. Assume that $B_1 + B_2$ is an optimal upper frame bound for the c-g-woven. So, there exists $\sigma \subset \Omega$ such that

$$\sup_{\|f\|=1} \Big(\int_\sigma \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) + \int_{\sigma^c} \|\Theta_{\omega i} f\|^2 \, d\mu(\omega) \Big) = B_1 + B_2.$$

There exist a $f_1 \in H$ such that $\|f_1\| = 1$ and

$$\int_\sigma \|\Lambda_{\omega i} f_1\|^2 \, d\mu(\omega) + \int_{\sigma^c} \|\Theta_{\omega i} f_1\|^2 \, d\mu(\omega) \geq B_1 + B_2 - \varepsilon.$$

Thus, by the assumption,

$$\int_{\Omega\setminus\sigma} \|\Lambda_{\omega i} f_1\|^2 \, d\mu(\omega) + \int_{\Omega\setminus\sigma^c} \|\Theta_{\omega i} f_1\|^2 \, d\mu(\omega) \leq \varepsilon.$$

Now, by Theorem 2.8, we conclude that $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Theta_\omega\}_{\omega\in\Omega}$ are not c-g-woven and this is a contradiction. □

A sufficient condition for c-g-woven is given in next theorem.

Theorem 2.11. *Let* $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ *be a c-g-frame for* H *with frame bounds* A_i *and* B_i *for each* $i \in [m]$. *Suppose that there exists* $M > 0$ *such that for all* $f \in H$, $i \neq k \in [m]$ *and all measurable subset* $\Delta \subset \Omega$,

$$\int_{\Delta} \|(\Lambda_{\omega i} - \Lambda_{\omega k})f\|^2 \, d\mu(\omega) \leq M \min\Big\{ \int_{\Delta} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega), \int_{\Delta} \|\Lambda_{\omega k} f\|^2 \, d\mu(\omega) \Big\}.$$

Then, the family $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ *is a c-g-woven with universal bounds*

$$\frac{A}{(m-1)(M+1)+1} \quad \textit{and} \quad B,$$

where, $A := \sum_{i\in[m]} A_i$ *and* $B = \sum_{i\in[m]} B_i$.

Proof. The upper bound is evident. For the lower bound, suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of Ω and $f \in H$. Therefore,

$$\begin{aligned}
\sum_{i\in[m]} A_i \|f\|^2 &\leq \sum_{i\in[m]} \int_{\Omega} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&= \sum_{i\in[m]} \sum_{k\in[m]} \int_{\sigma_k} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&\leq \sum_{i\in[m]} \Big(\int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) \\
&\qquad + \sum_{\substack{k\in[m] \\ k\neq i}} \int_{\sigma_k} \Big\{ \|\Lambda_{\omega i} f - \Lambda_{\omega k} f\|^2 \, d\mu(\omega) + \|\Lambda_{\omega k} f\|^2 \, d\mu(\omega) \Big\} \Big) \\
&\leq \sum_{i\in[m]} \Big(\int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega) + \sum_{\substack{k\in[m] \\ k\neq i}} \int_{\sigma_k} (M+1) \|\Lambda_{\omega k} f\|^2 \, d\mu(\omega) \Big) \\
&= \{(m-1)(M+1)+1\} \sum_{i\in[m]} \int_{\sigma_i} \|\Lambda_{\omega i} f\|^2 \, d\mu(\omega).
\end{aligned}$$

□

3. Perturbation For C-G-Woven

Perturbation of frames has been discussed by Cazassa and Christensen in [5]. For weaving frames, Bemrose and et.al. have studied in [4], also Vashisht and Deepshikha presented for continuous case in [19]. we aim to present it for c-g-woven.

Theorem 3.1. *Suppose for each* $i \in [m]$, *the family* $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ *be a c-g- frame for* H *with frame bounds* A_i *and* B_i. *Assume that there exist constants* λ_i, η_i *and* $\gamma_i (i \in [m])$ *such that for some fixed* $n \in [m]$,

$$A := A_n - \sum_{i\in[m]\setminus\{n\}} (\lambda_i + \eta_i \sqrt{B_n} + \gamma_i \sqrt{B_i})(\sqrt{B_n} + \sqrt{B_i}) > 0$$

and

$$\Big|\int_\Omega \langle(\Lambda^*_{\omega n}-\Lambda^*_{\omega i})F(w),g\rangle\,d\mu\Big| \le \eta_i\Big|\int_\Omega \langle\Lambda^*_{\omega n}F(w),g\rangle\,d\mu\Big| + \gamma_i\Big|\int_\Omega \langle\Lambda^*_{\omega i}F(w),g\rangle\,d\mu\Big| + \lambda_i\|F\|_2$$

for every $F \in \mathfrak{L}^2(H_\omega,\mu)$ and $g \in H$. Then for any partition $\{\sigma_j\}_{j\in[m]}$ of Ω, $\{\Lambda_{\omega i}\}_{\omega\in\sigma_j, j\in[m]}$ is a c-g-frame for H with universal frame bounds A and $\sum_{i\in[m]} B_i$. Hence the family of c-g-frame $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ is woven for H.

Proof. It is clear that $\{\Lambda_{\omega i}f\}_{\omega\in\sigma_j, j\in[m]}$ is strongly measurable for each $f \in H$ and any partition $\{\sigma_j\}_{j\in[m]}$ of Ω, also the family $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ is a c-g-Bessel family with Bessel bound $\sum_{i\in[m]} B_i$. Now, we show that $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ has the lower frame condition. Assume that T_{Λ_i} is the synthesis operator of $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$. Then for any $F \in \mathfrak{L}^2(H_\omega,\mu)$, we have

$$\begin{aligned}\|T_{\Lambda_i}F\| = \sup_{\|g\|=1}\Big|\langle T_{\Lambda_i}F,g\rangle\Big| &\le \sup_{\|g\|=1}\Big|\int_\Omega \langle\Lambda^*_{\omega i}F(w),g\rangle\,d\mu\Big| \\ &\le \sup_{\|g\|=1}\Big(\int_\Omega \|F\|^2\,d\mu\Big)^{\frac{1}{2}}\Big(\int_\Omega \|\Lambda_{\omega i}g\|^2\,d\mu\Big)^{\frac{1}{2}} \\ &\le \sqrt{B_i}\|F\|_2.\end{aligned}$$

Now, for any $F \in \mathfrak{L}^2(H_\omega,\mu)$, $i \in [m]\setminus\{n\}$, we have

$$\begin{aligned}&\|(T_{\Lambda_n}-T_{\Lambda_i})F\| \\ &= \sup_{\|g\|=1}\Big|\langle(T_{\Lambda_n}-T_{\Lambda_i})F,g\rangle\Big| \\ &= \sup_{\|g\|=1}\Big|\int_\Omega \langle(\Lambda^*_{\omega n}-\Lambda^*_{\omega i})F(w),g\rangle\,d\mu\Big| \\ &\le \eta_i \sup_{\|g\|=1}\Big|\int_\Omega \langle\Lambda^*_{\omega n}F(w),g\rangle\,d\mu\Big| + \gamma_i \sup_{\|g\|=1}\Big|\int_\Omega \langle\Lambda^*_{\omega i}F(w),g\rangle\,d\mu\Big| + \lambda_i\|F\|_2 \\ &= \eta_i\|T_{\Lambda_n}F\| + \gamma_i\|T_{\Lambda_i}F\| + \lambda_i\|F\|_2 \\ &\le \eta_i\sqrt{B_n}\|F\|_2 + \gamma_i\sqrt{B_i}\|F\|_2 + \lambda_i\|F\|_2 \\ &= (\eta_i\sqrt{B_n} + \gamma_i\sqrt{B_i} + \lambda_i)\|F\|_2.\end{aligned}$$

Thus,

$$\|T_{\Lambda_n}-T_{\Lambda_i}\| \le \eta_i\sqrt{B_n} + \gamma_i\sqrt{B_i} + \lambda_i. \tag{3.1}$$

For each $i \in [m]$ and $\sigma \subset X$, we define

$$T_i^{(\sigma)} : \Big(\oplus_{\omega\in\sigma} H_\omega,\mu\Big)_{L^2} \to H,$$
$$\langle T_i^{(\sigma)}G,h\rangle = \int_\sigma \langle\Lambda^*_{\omega i}G(w),h\rangle\,d\mu,$$

for all $G \in \mathfrak{L}^2(H_\omega, \mu)$, we have
$\|T_i^{(\sigma)} G\| = \|T_{\Lambda_i}(G.\chi_\sigma)\| \leq \|T_{\Lambda_i}\| \|G.\chi_\sigma\| \leq \|T_{\Lambda_i}\| \|F\|_2 \leq \sqrt{B_i} \|F\|_2$.
Thus, $\|T_i^{(\sigma)}\| \leq \sqrt{B_i}$, for each $i \in [m]$. Similarly with (2), we get for each $i \in [m] \setminus \{n\}$,

$$\|T_n^{(\sigma)} - T_i^{(\sigma)}\| \leq \eta_i \sqrt{B_n} + \gamma_i \sqrt{B_i} + \lambda_i. \tag{3.2}$$

For every $f \in H$ and $i \in [m] \setminus \{n\}$, We compute

$$\begin{aligned}
&\left\|\left(T_n^{(\sigma)}(T_n^{(\sigma)})^* - T_i^{(\sigma)}(T_i^{(\sigma)})^*\right)f\right\| \\
&\leq \left\|\left(T_n^{(\sigma)}(T_n^{(\sigma)})^* - T_n^{(\sigma)}(T_i^{(\sigma)})^*\right)f\right\| \\
&\quad + \left\|\left(T_n^{(\sigma)}(T_n^{(\sigma)})^* - T_n^{(\sigma)}(T_i^{(\sigma)})^*\right)f\right\| \\
&\leq \left\|T_n^{(\sigma)}\right\| \left\|((T_n^{(\sigma)})^* - (T_i^{(\sigma)})^*)f\right\| \\
&\quad + \left\|T_i^{(\sigma)}\right\| \left\|(T_n^{(\sigma)} - T_i^{(\sigma)})f\right\| \\
&\leq (\eta_i \sqrt{B_n} + \gamma_i \sqrt{B_i} + \lambda_i)(\sqrt{B_n} + \sqrt{B_i})\|f\|.
\end{aligned}$$

Now, suppose that $\{\sigma_i\}_{i\in[m]}$ is a partition of Ω and T_Λ be the synthesis operator associated with the c-g-Bessel family $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$, by using (4), we have,

$$\begin{aligned}
\|T_\Lambda^* f\|^2 &= |\langle f, T_\Lambda T_\Lambda^* f\rangle| \\
&= \Big| \sum_{i\in[m]} \int_{\sigma_i} \langle f, \Lambda_{\omega i}^* \Lambda_{\omega i} f\rangle \, d\mu(\omega) \Big| \\
&= \Big| \int_{\sigma_1} \langle f, \Lambda_{\omega 1}^* \Lambda_{\omega 1} f\rangle \, d\mu(\omega) + ... + \int_{\sigma_n} \langle f, \Lambda_{\omega n}^* \Lambda_{\omega n} f\rangle \, d\mu(\omega) \\
&\quad + ... \int_{\sigma_m} \langle f, \Lambda_{\omega m}^* \Lambda_{\omega m} f\rangle \, d\mu(\omega) \Big| \\
&= \Big| \int_{\sigma_1} \langle f, \Lambda_{\omega 1}^* \Lambda_{\omega 1} f\rangle \, d\mu(\omega) + ... + \sum_{i\in[m]} \int_{\sigma_i} \langle f, \Lambda_{\omega i}^* \Lambda_{\omega i} f\rangle \, d\mu(\omega) \\
&\quad - \sum_{i\in[m]\setminus\{n\}} \int_{\sigma_i} \langle f, \Lambda_{\omega i}^* \Lambda_{\omega i} f\rangle \, d\mu(\omega) + \int_{\sigma_m} \langle f, \Lambda_{\omega m}^* \Lambda_{\omega m} f\rangle \, d\mu(\omega) \Big| \\
&= \Big| \int_\Omega \langle f, \Lambda_{\omega n}^* \Lambda_{\omega n} f\rangle \, d\mu(\omega) - \sum_{i\in[m]\setminus\{n\}} \Big(\int_{\sigma_i} \langle f, \Lambda_{\omega n}^* \Lambda_{\omega n} f\rangle \, d\mu(\omega) \\
&\quad - \int_{\sigma_i} \langle f, \Lambda_{\omega i}^* \Lambda_{\omega i} f\rangle \, d\mu(\omega) \Big) \Big| \\
&\geq \Big| \int_\Omega \langle f, \Lambda_{\omega n}^* \Lambda_{\omega n} f\rangle \, d\mu(\omega) \Big| - \sum_{i\in[m]\setminus\{n\}} \Big| \int_{\sigma_i} \langle f, \Lambda_{\omega n}^* \Lambda_{\omega n} f\rangle \, d\mu(\omega) \\
&\quad - \int_{\sigma_i} \langle f, \Lambda_{\omega i}^* \Lambda_{\omega i} f\rangle \, d\mu(\omega) \Big) \Big|
\end{aligned}$$

$$\begin{aligned}
&\geq A_n\|f\|^2 - \sum_{i\in[m]\setminus\{n\}} \left|\langle T_n^{(\sigma)}(T_n^{(\sigma)})^* f, f\rangle - \langle T_i^{(\sigma)}(T_i^{(\sigma)})^* f, f\rangle\right| \\
&\geq A_n\|f\|^2 - \sum_{i\in[m]\setminus\{n\}} \|f\|\|(T_n^{(\sigma)}(T_n^{(\sigma)})^* - T_i^{(\sigma)}(T_i^{(\sigma)})^*)f\| \\
&\geq A_n\|f\|^2 - \sum_{i\in[m]\setminus\{n\}} \|f\|^2(\eta_i\sqrt{B_n} + \gamma_i\sqrt{B_i} + \lambda_i)(\sqrt{B_n} + \sqrt{B_i}) \\
&= A\|f\|^2.
\end{aligned}$$

This completes the proof. □

With similar proof of Theorem 3.1, we can show the following result when the index n is not fixed.

Corollary 3.2. *For each $i \in [m]$, let the family $\{\Lambda_{\omega i}\}_{\omega\in\Omega}$ be a c-g- frame for H with frame bounds A_i and B_i. Assume that there exist constants λ_i, η_i, $\gamma_i (i \in [m-1])$ and $n \in [m]$ so that*

$$A := A_1 - \sum_{i\in[m-1]\setminus\{n\}} (\lambda_i + \eta_i\sqrt{B_i} + \gamma_i\sqrt{B_{i+1}})(\sqrt{B_i} + \sqrt{B_{i+1}}) > 0$$

and

$$\begin{aligned}
&\left|\int_\Omega \langle(\Lambda^*_{\omega i} - \Lambda^*_{\omega(i+1)})F(w), g\rangle\, d\mu\right| \\
&\qquad \leq \eta_i\left|\int_\Omega \langle\Lambda^*_{\omega i}F(w), g\rangle\, d\mu\right| + \gamma_i\left|\int_\Omega \langle\Lambda^*_{\omega(i+1)}F(w), g\rangle\, d\mu\right| + \lambda_i\|F\|_2,
\end{aligned}$$

for every $F \in \mathfrak{L}^2(H_\omega,\mu)$ and $g \in H$. Then for any partition $\{\sigma_j\}_{j\in[m]}$ of Ω, $\{\Lambda_{\omega i}\}_{\omega\in\sigma_j, j\in[m]}$ is a c-g-frame for H with universal frame bounds A and $\sum_{i\in[m]} B_i$. Hence the family of c-g-frame $\{\Lambda_{\omega i}\}_{\omega\in\Omega, i\in[m]}$ is woven for H.

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