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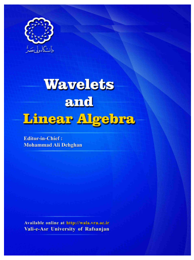

Characterizing Global Minimizers of the Difference of Two Positive Valued Affine Increasing and Co-radiant Functions

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Abstract

Many optimization problems can be reduced to a problem with an increasing and co-radiant objective function by a suitable transformation of variables. Functions, which are increasing and co-radiant, have found many applications in microeconomic analysis. In this paper, the abstract convexity of positive valued affine increasing and co-radiant (ICR) functions are discussed. Moreover, the basic properties of this class of functions such as support set, subdifferential and maximal elements of support set are characterized. Finally, as an application, necessary and sufficient conditions for the global minimum of the difference of two strictly positive valued affine ICR functions are presented.

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1. Introduction

In the study of mathematical analysis, functions which can be represented as the upper envelopes of a subset of a certain class H of sufficiently simple (elementary) functions are called abstract convex with respect to H [7, 9].

Some classes of increasing functions are abstract convex. For example, the class of increasing and positively homogeneous (IPH) functions [4], increasing and convex-along-rays (ICAR) functions [8], and increasing and co-radiant (ICR) functions [2, 5] are of this type. Also, the class of non-positive valued affine ICR functions [1] are another class of increasing functions which are abstract convex [1]. In this paper, the abstract convexity of positive valued affine ICR functions is investigated, where a positive valued affine ICR function is the affine transformation of a positive valued ICR function on a constant. Moreover, some properties of positive valued affine ICR functions in the framework of abstract convexity are obtained. Finally, the maximal elements of the support set of this class of functions are characterized.

One of the most important global optimization problems is minimizing a DC function (difference of two convex functions) [10], i.e.,

$$\text{minimize } f(x) \text{ subject to } x \in X,$$

where $f(x) = q(x) - p(x)$ and p, q are convex functions. In a general case, DC function can be replaced by DAC function (difference of two abstract convex functions). For example, one can observe minimizing of the difference of two increasing and convex along rays functions [8, 6], minimizing of the difference of two ICR functions [3] and minimizing of the difference of two non-positive valued affine ICR functions [1]. In this paper, the functions p and q are replaced by positive valued affine ICR functions and then necessary and sufficient conditions for the global minimum of f are presented.

The paper is organized as follows. Necessary definitions and results about the positive valued ICR functions are provided in Section 2. Section 3 is devoted to studying the abstract convexity of positive valued affine ICR functions. In Section 4, the maximal elements of the support set for strictly positive valued affine ICR functions are investigated. Finally, in Section 5, necessary and sufficient optimality conditions for the difference of two strictly positive valued affine ICR functions are given.

2. Preliminaries

In this section, some preliminaries and notations are stated. Let X be a real topological vector space which is equipped with a closed convex pointed cone $S \subseteq X$ (the latter means that $S \cap (-S) = \{0\}$). In this case, $x \leq y$ if $y - x \in S$, and $x < y$ if $y - x \in S \setminus \{0\}$. Let $f : X \to [0, +\infty]$ be a function. A vector $x^* \in X$ is called a global minimum of the function f over X if $f(x^*) = \inf_{x \in X} f(x)$.

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Definition 2.1. A function $f : X \to [0, +\infty]$ is called co-radiant if $f(\lambda x) \geq \lambda f(x)$ for all $x \in X$ and all $\lambda \in (0, 1]$. It is clear that f is co-radiant if $f(\lambda x) \leq \lambda f(x)$ for all $x \in X$ and all $\lambda \geq 1$.

Definition 2.2. A function $f : X \to [0, +\infty]$ is called increasing if $x \geq y$ implies $f(x) \geq f(y)$. A function $f : X \to [0, +\infty]$ is called strictly increasing on $A \subseteq X$, if for each $x, y \in A$ such that $x < y$ implies $f(x) < f(y)$.

Remark 2.3. A function $f : X \to [0, +\infty]$ is called an ICR function if f is an increasing and co-radiant function. In particular, we say that an ICR function $f : X \to [0, +\infty]$ is strictly ICR, if f is strictly increasing on $X \setminus (-S)$ and f is co-radiant.

Definition 2.4. [7]. Let X be a non-empty set and $f : X \longrightarrow [0, +\infty]$ be a function. Consider $\mathcal{H} : = \{h : X \to [0, +\infty] : h \text{ is a function}\}$.

(1) The support set of f with respect to $\mathcal{H}$ is defined by

$$supp(f, \mathcal{H}) : = \{h \in \mathcal{H} : h(x) \leq f(x), \ \forall \ x \in X\}.$$

(2) The function f is called abstract convex with respect to $\mathcal{H}$ or $\mathcal{H}$-convex if there exists a subset $\triangle$ of $\mathcal{H}$ such that

$$f(x) = \sup_{h \in \triangle} h(x), \quad \forall \ x \in X.$$

(3) The subdifferential of the function f at a point $x_0 \in dom f := \{x \in X : f(x) < +\infty\}$ with respect to $\mathcal{H}$ or $\mathcal{H}$-subdifferential of f is defined by

$$\partial_{\mathcal{H}} f(x_0) : = \{h \in \mathcal{H} : h(x_0) \in \mathbb{R}, f(x) - f(x_0) \geq h(x) - h(x_0), \forall \ x \in X\}.$$

Note that the set $\mathcal{H}$ in the Definition 2.4 is called the set of elementary functions.

The function $h_c : X \longrightarrow [0, +\infty]$ is defined by

$$h_c(x) := h(x) + c, \quad (x \in X)$$

is called an $\mathcal{H}$-affine function, where $h \in \mathcal{H}$ and $c \in \mathbb{R}$. In addition, the set of all $\mathcal{H}$-affine functions is denoted by $H_{\mathcal{H}} := \{h_c : h \in \mathcal{H}, \ c \in \mathbb{R}\}$.

The function $l : X \times X \times \mathbb{R}_{++} \longrightarrow [0, +\infty]$ is defined by

$$l(x, y, \alpha) := \max\{0 \leq \lambda \leq \alpha : \lambda y \leq x\}, \ \forall \ x, y \in X, \ \forall \ \alpha > 0 \tag{2.1}$$

was introduced in [2] with the convention $\max \emptyset = 0$. We define

$$\mathbb{R}_{++} := \{\alpha \in \mathbb{R} : \alpha > 0\}.$$

In the following proposition, some properties of the function l are stated.

Proposition 2.5. *[2]. For every $x,\ y,\ x',\ y' \in X$; $\gamma \in (0,1]$; $\mu,\ \alpha,\ \alpha' \in \mathbb{R}_{++}$, one has*

$$l(\mu x, y, \alpha) = \mu l(x, y, \frac{\alpha}{\mu}),$$

$$l(x, \mu y, \alpha) = \frac{1}{\mu} l(x, y, \mu\alpha), \tag{2.2}$$

$$x \le x' \Longrightarrow l(x, y, \alpha) \le l(x', y, \alpha),$$

$$y \le y' \Longrightarrow l(x, y', \alpha) \le l(x, y, \alpha),$$

$$\alpha \le \alpha' \Longrightarrow l(x, y, \alpha) \le l(x, y, \alpha'),$$

$$l(\gamma x, y, \alpha) \ge \gamma l(x, y, \alpha),$$

$$l(x, \gamma y, \alpha) \le \frac{1}{\gamma} l(x, y, \alpha),$$

$$l(x, y, \alpha) = \alpha \Leftrightarrow \alpha y \le x. \tag{2.3}$$

The following theorem for the class of positive ICR functions will be used later.

Theorem 2.6. *[2]. Let $f : X \longrightarrow [0, +\infty]$ be a function. Then the following assertions are equivalent:*

(i) f is ICR.

(ii) $\lambda f(y) \le f(x)$ for all $x, y \in X$ and all $\lambda \in (0,1]$ such that $\lambda y \le x$.

(iii) $l(x, y, \alpha) f(\alpha y) \le \alpha f(x)$ for all $x, y \in X$ and all $\alpha \in \mathbb{R}_{++}$, with the convention $0 \times (+\infty) = 0$.

We now consider the set of elementary positive valued ICR functions

$$L := \left\{ l_{(y,\alpha)} :\ y \in X,\ \alpha \in \mathbb{R}_{++} \right\},$$

where, for each $(y, \alpha) \in X \times \mathbb{R}_{++}$, the function $l_{(y,\alpha)} : X \longrightarrow [0, +\infty]$ is defined by $l_{(y,\alpha)}(x) = l(x, y, \alpha)$ for all $x \in X$. Then the following results are valid.

Theorem 2.7. *[2]. Let $f : X \to [0, +\infty]$ be a function. Then f is an ICR function if and only if there exists a set $A \subseteq L$ such that*

$$f(x) = \sup_{l_{(y,\alpha)} \in A} l_{(y,\alpha)}(x),\ (x \in X).$$

In this case, one can take $A := \left\{ l_{(y,\alpha)} \in L : f(\alpha y) \ge \alpha \right\}$. Hence, f is an ICR function if and only if it is L-convex.

Theorem 2.8. *[2]. Let $f : X \to [0, +\infty]$ be an ICR function. Then*

$$supp(f, L) = \left\{ l_{(y,\alpha)} \in L : f(\alpha y) \ge \alpha \right\}.$$

Definition 2.9. Let $H_0 := \{f : X \longrightarrow [0, +\infty] : f \text{ is a function}\}$. Consider H_0 with the natural pointwise order relation of functions. A function $f \in H_0$ is called a maximal element of the set H_0 if

$$\bar{f} \in H_0,\ \bar{f}(x) \geq f(x),\ \ \forall\ x \in X \implies \bar{f}(x) = f(x),\ \forall\ x \in X.$$

The following theorem characterizes the maximal elements of the support set of $f : X \longrightarrow [0, +\infty]$ with respect to L, where f is a positive valued ICR function.

Theorem 2.10. *[3]. Let $f : X \longrightarrow [0, +\infty]$ be a strictly ICR function. Let $y \in X$ be such that $\varepsilon := \max\{\alpha : f(\alpha y) \geq \alpha\} < +\infty$. Then, $l_{(y,\varepsilon)}$ is a maximal element of the support set of f if and only if $f(\varepsilon y) = \varepsilon$.*

Remark 2.1. In this paper, the class of positive valued ICR functions $f : X \longrightarrow [0, +\infty]$ with $\liminf_{x \longrightarrow 0^+} f(x) = 0$ are considered.

3. Abstract Convexity of Positive Valued Affine ICR Functions

In this section, first the support set of positive valued affine ICR functions is characterized. Then the abstract convexity of this class of functions in terms of H_L(the set of all L-affine functions) is investigated, where

$$H_L := \left\{ l_{(y,\alpha),c} : l_{(y,\alpha)} \in L,\ c \in \mathbb{R} \right\},$$

and

$$l_{(y,\alpha),c}(x) := l_{(y,\alpha)}(x) + c,\ \forall\ c \in \mathbb{R}, \forall\ x \in X.$$

Finally, the subdifferential of this class of functions is obtained.

Definition 3.1. A function f_a of the form $f_a := f + a$ is called (strictly) positive valued affine ICR function if $a \in \mathbb{R}$ and $f : X \longrightarrow [0, +\infty]$ is a (strictly) positive valued ICR function.

Now the support set of positive valued ICR functions in terms of H_L is characterized.

Proposition 3.2. *Let $f : X \longrightarrow [0, +\infty]$ be an ICR function. Then*

$$supp(f, H_L) = \left\{ l_{(y,\alpha),c} \in H_L\ :\ c \leq 0,\ f(\alpha y) \geq \alpha + c \right\}.$$

Proof. Let $l_{(y,\alpha),c} \in supp(f, H_L)$. Then

$$l_{(y,\alpha),c}(x) \leq f(x), \quad \forall\ x \in X. \tag{3.1}$$

By definition l in (2.1) and in view of Remark 2.2, $\liminf_{x \longrightarrow 0^+} f(x) = 0$, then $c \leq 0$. Furthermore, by setting $x = \alpha y$ in (3.1) and using (2.3), the inequality $\alpha + c \leq f(\alpha y)$ is obtained. Therefore, $l_{(y,\alpha),c} \in \left\{ l_{(y,\alpha),c} \in H_L\ :\ c \leq 0,\ f(\alpha y) \geq \alpha + c \right\}$.

Now, suppose that $l_{(y,\alpha),c} \in \left\{ l_{(y,\alpha),c} \in H_L\ :\ c \leq 0,\ f(\alpha y) \geq \alpha + c \right\}$ be arbitrary. By using the part (*iii*) of Theorem 2.6, we have

$$l(x, y, \alpha) f(\alpha y) \leq \alpha f(x), \quad \forall x\ \in X.$$

Since $l_{(y,\alpha),c} \in \left\{ l_{(y,\alpha),c} \in H_L \ : \ c \leq 0, \ f(\alpha y) \geq \alpha + c \right\}$, hence

$$(\alpha + c) l(x, y, \alpha) \leq l(x, y, \alpha) f(\alpha y) \leq \alpha f(x). \tag{3.2}$$

So, the definition of l and (3.2) imply that

$$\alpha l(x, y, \alpha) + c\alpha \leq \alpha f(x).$$

Thus, $l_{(y,\alpha),c}(x) \leq f(x)$. Hence $l_{(y,\alpha),c} \in supp(f, H_L)$. □

Now, it is easy to characterize the support set of f_a in terms of H_L by using $supp(f, H_L)$.

Corollary 3.3. *The support set of the positive valued affine ICR function f_a is the set*

$$supp(f_a, H_L) = \left\{ l_{(y,\alpha),c} \in H_L \ : \ c \leq a, \ f_a(\alpha y) \geq \alpha + c \right\}.$$

Proof. By Proposition 3.2 and since $supp(f_a, H_L) = supp(f, H_L) + a$, the result follows. □

Theorem 3.4. *Let $f_a = f + a$ be a positive valued affine ICR function. Then there exists a set $\triangle \subseteq H_L$ such that*

$$f_a(x) = \sup_{l_{(y,\alpha),c} \in \triangle} l_{(y,\alpha),c}(x), \quad \forall \ x \in X,$$

where $\triangle = \left\{ l_{(y,\alpha),c} \in H_L : c \leq a, \ \ f(\alpha y) \geq \alpha + a \right\}$.

Proof. By using Proposition 3.2, one can take $\triangle := supp(f_a, H_L)$. So

$$\sup_{l_{(y,\alpha),c} \in \triangle} l_{(y,\alpha),c}(x) \leq f_a(x). \tag{3.3}$$

Also, by Theorem 2.7,

$$f(x) = \sup\nolimits_{l_{(y,\alpha)} \in A} l_{(y,\alpha)}(x),$$

where $A = supp(f, L)$. Therefore

$$f_a(x) = \sup_{l_{(y,\alpha)} \in A} l_{(y,\alpha)}(x) + a = \sup_{l_{(y,\alpha),a} \in A_a} l_{(y,\alpha),a}(x),$$

where $A_a = A + a$. Since $A_a \subseteq \triangle$, then

$$f_a(x) \leq \sup_{l_{(y,\alpha),c} \in \triangle} l_{(y,\alpha),c}(x). \tag{3.4}$$

So, the equations (3.3) and (3.4) imply that

$$f_a(x) = \sup_{l_{(y,\alpha),c} \in \triangle} l_{(y,\alpha),c}(x), \ \forall \ x \in X.$$

□

Corollary 3.5. *Any positive valued affine ICR function is H_L-convex.*

In the following, the H_L-subdifferential of the class of positive valued affine ICR function is characterized.

Proposition 3.6. *Let $x_0 \in domf$ and $f_a = f + a$, where $a \in \mathbb{R}$ and f is a positive valued ICR function on X. Then*

$$\partial_{H_L} f_a(x_0) = \partial_L f(x_0) + \mathbb{R}.$$

Proof. For any $c \in \mathbb{R}$, one has

$$\begin{aligned} & l_{(y,\alpha),c} \in \partial_{H_L} f_a(x_0) \\ \Longleftrightarrow\ & l_{(y,\alpha),c}(x) - l_{(y,\alpha),c}(x_0) \le f_a(x) - f_a(x_0), \quad \forall\, x \in X \\ \Longleftrightarrow\ & l_{(y,\alpha)}(x) - l_{(y,\alpha)}(x_0) \le f(x) - f(x_0), \quad \forall\, x \in X \\ \Longleftrightarrow\ & l_{(y,\alpha)} \in \partial_L f(x_0), \end{aligned}$$

which completes the proof. □

4. Maximal Elements of the Support Set of Strictly Positive Valued Affine ICR Functions

This section is devoted to finding the maximal elements of the support set of strictly positive valued affine ICR functions. In the following, by concentrating on the support set of strictly positive valued affine ICR functions some results are obtained. Note that $S \subseteq X$ is a closed convex pointed cone.

Lemma 4.1. *Let $f : X \longrightarrow [0, +\infty]$ be a strictly ICR function and $S \neq \{0\}$. Then, $f(x) > 0$ for all $x \in X \setminus (-S)$.*

Proof. Suppose that there exists $x' \in X \setminus (-S)$ in a way $f(x') = 0$. Consider $x_0 \in (-S) \setminus \{0\}$ and, for each $\gamma > 0$, put $x_\gamma := x' + \gamma x_0$. If $x_\gamma \in -S$ for all $\gamma > 0$, therefore, since $-S$ is a closed cone and $\lim_{\gamma \longrightarrow 0^+} x_\gamma = x'$, then one can conclude that $x' \in -S$, which is a contradiction. So, there exists $\gamma > 0$ such that $x_\gamma \in X \setminus (-S)$. Because of $\gamma x_0 \le 0$, then we obtain $x_\gamma \le x'$. Since $x', x_\gamma \in X \setminus (-S)$ and f is strictly increasing on $X \setminus (-S)$, it follows that $0 = f(x') > f(x_\gamma)$. This is a contradiction. Thus $f(x) > 0$ for all $x \in X \setminus (-S)$. □

Theorem 4.2. *Let $f : X \longrightarrow [0, +\infty]$ be a strictly positive valued ICR function, and let $l_{(y,\alpha),c} \in H_L$ be a maximal element of $supp(f, H_L)$. Then $f(\alpha y) = \alpha$ and $c = 0$.*

Proof. Let $y' = \dfrac{\alpha y}{f(\alpha y)}$ (by Lemma 4.1 $f(\alpha y) > 0$) and $\alpha' = f(\alpha y)$. Since $f(\alpha' y') = \alpha'$, then $l_{(y',\alpha'),0} \in supp(f, H_L)$. Therefore, by relations (2.2) and (2.3), the following equations are valid,

$$l_{(y',\alpha'),0}(\alpha y) = l(\alpha y, \frac{\alpha y}{f(\alpha y)}, f(\alpha y)) = \frac{f(\alpha y)}{\alpha} l(\alpha y, y, \alpha) = f(\alpha y). \tag{4.1}$$

Since $l_{(y,\alpha),c} \in supp(f, H_L)$, so $f(\alpha y) \ge \alpha + c$. Using (4.1) implies that $l_{(y',\alpha'),0}(\alpha y) \ge \alpha + c$. Also, Proposition 3.2 and $c \le 0$ imply that $l_{(y,\alpha),c} \in supp(l_{(y',\alpha'),0}, H_L)$, that is, $l_{(y,\alpha),c} \le l_{(y',\alpha'),0}$ on X. On the other hand, $l_{(y,\alpha),c}$ is a maximal element of $supp(f, H_L)$. Thus $l_{(y,\alpha),c} = l_{(y',\alpha'),0}$. Hence, by putting $x = 0$, one has $c = 0$. So, the equality $l_{(y,\alpha),0} = l_{(y,\alpha)}$ and the facts that $f(\alpha y) \ge \alpha$ and $supp(f, L) \subseteq supp(f, H_L)$ imply that $l_{(y,\alpha)}$ is a maximal element of $supp(f, L)$. Hence, Theorem 2.10 implies that $f(\alpha y) = \alpha$. □

The next lemma states relation between maximal elements of $supp(f, H_L)$ and $supp(f_a, H_L)$.

Lemma 4.3. *Let $f_a = f + a$ be a strictly positive valued affine ICR function. Then $l_{(y,\alpha),c} \in H_L$ is a maximal element of $supp(f_a, H_L)$ if and only if $l_{(y,\alpha),c-a}$ is a maximal element of $supp(f, H_L)$.*

Proof. It is easy to observe that $l_{(y,\alpha),c}$ is a maximal element of $supp(f, H_L)$ if and only if $l_{(y,\alpha),c} + a$ is a maximal element of $supp(f, H_L) + a$. Also, clearly, $supp(f_a, H_L) = supp(f, H_L) + a$ and $l_{(y,\alpha),c+a} = l_{(y,\alpha),c} + a$. Therefore the proof is complete. □

Proposition 4.4. *Let $f_a = f + a$ be a strictly positive valued affine ICR function, and $l_{(y,\alpha),c} \in H_L$ be a maximal element of $supp(f_a, H_L)$. Then $f(\alpha y) = \alpha$ and $c = a$.*

Proof. By Lemma 4.3, a maximal element of $supp(f, H_L)$ is of the form $l_{(y,\alpha),c-a}$. Now, Theorem 4.2 implies that $f(\alpha y) = \alpha$ and $c = a$. □

Now, necessary and sufficient conditions for characterizing the maximal elements of $supp(f_a, H_L)$ are presented.

Theorem 4.5. *Let $f_a = f + a$ be a strictly positive valued affine ICR function, and let $l_{(y,\varepsilon),c} \in H_L$, $y \in X \setminus (-S)$, and $\varepsilon := \max\{\alpha > 0 : f(\alpha y) \geq \alpha\} < +\infty$. Then $l_{(y,\varepsilon),c}$ is a maximal element of $supp(f_a, H_L)$ if and only if $f(\varepsilon y) = \varepsilon$ and $c = a$.*

Proof. By Proposition 4.4, if $l_{(y,\varepsilon),c}$ is a maximal element of $supp(f_a, H_L)$, then $f(\varepsilon y) = \varepsilon$ and $c = a$. Conversely, let $\varepsilon = \max\{\alpha > 0 : f(\alpha y) \geq \alpha\}$, $f(\varepsilon y) = \varepsilon$ and $c = a$. So, Theorem 2.10 implies that $l_{(y,\varepsilon)}$ is a maximal element of $supp(f, L)$. Now, assume that $l_{(y',\alpha'),c'} \in supp(f_a, H_L)$ is such that

$$l_{(y,\varepsilon),c}(x) \leq l_{(y',\alpha'),c'}(x), \quad \forall\, x \in X. \tag{4.2}$$

Set $x = 0$ in (4.2), then $c \leq c'$. Because of $l_{(y',\alpha'),c'} \in supp(f_a, H_L)$, therefore we achieve $c' \leq a$. Now, by the assumption $c = a$, one can conclude that $c = c'$ and $c' = a$. Hence, the relation (4.2) implies that $l_{(y,\varepsilon)} \leq l_{(y',\alpha')}$ on X and $l_{(y',\alpha')} \in supp(f, L)$. On the other hand, $l_{(y,\varepsilon)}$ is a maximal element of $supp(f, L)$, and then $l_{(y,\varepsilon)} = l_{(y',\alpha')}$ on X. Therefore $l_{(y,\varepsilon),c}$ is a maximal element of $supp(f_a, H_L)$. □

5. Characterizing Global Minimizers of the Difference of Two Strictly Positive Valued Affine ICR Functions

In this section, necessary and sufficient conditions for the global minimum of the difference of two strictly positive valued affine ICR functions are presented. To this end, the support set is described by using maximal elements.

Lemma 5.1. *Let $f : X \longrightarrow [0, +\infty)$ be a strictly positive valued ICR function and $S \subseteq X$ be a closed convex pointed cone. Then, for any $l_{(y,\alpha),c} \in supp(f_a, H_L)$ with $y \in X \setminus (-S)$, there exists a maximal element $l_{(y',\alpha'),c'}$ of $supp(f_a, H_L)$ such that*

$$l_{(y,\alpha),c} \leq l_{(y',\alpha'),c'} \text{ on } X,$$

where $\alpha' = f(\varepsilon_y y)$, $c' = a$, $y' = \frac{\varepsilon_y y}{f(\varepsilon_y y)}$ and $\varepsilon_y := \max\{\alpha > 0 : f(\alpha y) \geq \alpha\}$ for all $y \in X \setminus (-S)$.

Proof. It is clear that the equation $f(\alpha' y') = \alpha'$ and [3, Corollary 4.1] leads to

$$\alpha' = \max\{\alpha > 0 : f(\alpha y') \geq \alpha\}.$$

So, Theorem 4.5 concludes that $l_{(y',\alpha'),c'}$ is a maximal element of $supp(f_a, H_L)$. Since $f(x) < +\infty$ for all $x \in X$, then

$$f_a(x) = \max_{l_{(y,\alpha),c} \in supp(f_a, H_L)} l_{(y,\alpha),c}(x), \quad \forall\, x \in X.$$

So, every maximal element of $supp(f_a, H_L)$ is a maximum element, and so $l_{(y',\alpha'),c'} \geq l_{(y,\alpha),c}$ on X. □

Proposition 5.2. *Let $f, f' : X \longrightarrow [0, +\infty)$ be strictly positive valued ICR functions so that $f_a := f + a$ and $f'_b := f' + b$. Let $\eta_y := \max\{\alpha > 0 : f(\alpha y) \geq \alpha\}$, $\varepsilon_y := \max\{\beta > 0 : f'(\beta y) \geq \beta\}$, $f(\eta_y y) \geq \eta_y$ and $f'(\varepsilon_y y) = \varepsilon_y$ for all $y \in X \setminus (-S)$. Then the following assertions are equivalent:*

(i) $supp(f_a, H_L) \subseteq supp(f'_b, H_L)$.

(ii) For any maximal element $l_{(y_1,\alpha_1),c_1} \in supp(f_a, H_L)$, there exists a maximal element $l_{(y_2,\alpha_2),c_2} \in supp(f'_b, H_L)$ such that

$$l_{(y_1,\alpha_1),c_1}(x) \leq l_{(y_2,\alpha_2),c_2}(x), \ \forall\, x \in X.$$

(iii) $a \leq b$ and $f'_b(\eta_y y) \geq \eta_y + a$ for each $y \in X \setminus (-S)$ with $\eta_y \geq 0$.

Proof. $(i) \Longrightarrow (ii)$. Assume that $l_{(y_1,\alpha_1),c_1}$ is a maximal element of $supp(f_a, H_L)$. So, $l_{(y_1,\alpha_1),c_1} \in supp(f'_b, H_L)$. By using Lemma 5.1 there exists a maximal element $l_{(y_2,\alpha_2),c_2} \in supp(f'_b, H_L)$ such that

$$l_{(y_1,\alpha_1),c_1}(x) \leq l_{(y_2,\alpha_2),c_2}(x), \ \forall\, x \in X.$$

$(ii) \Longrightarrow (i)$. Consider $l_{(y_1,\alpha_1),c_1} \in supp(f_a, H_L)$. By using Lemma 5.1 there exists a maximal element $l_{(y_2,\alpha_2),c_2} \in supp(f_a, H_L)$ such that $l_{(y_1,\alpha_1),c_1}(x) \leq l_{(y_2,\alpha_2),c_2}(x)$, $\forall\, x \in X$. Also, there exists a maximal element $l_{(y_3,\alpha_3),c_3} \in supp(f'_b, H_L)$ such that

$$l_{(y_2,\alpha_2),c_2}(x) \leq l_{(y_3,\alpha_3),c_3}(x), \ \forall\, x \in X.$$

Hence

$$l_{(y_1,\alpha_1),c_1}(x) \leq l_{(y_2,\alpha_2),c_2}(x) \leq l_{(y_3,\alpha_3),c_3}(x) \leq f'_b(x), \quad \forall\, x \in X.$$

Therefore $l_{(y_1,\alpha_1),c_1} \in supp(f'_b, H_L)$.

$(i) \Longrightarrow (iii)$. It is clear that $\eta_y \geq 0$. Since $f(\eta_y y) \geq \eta_y$, so $l_{(y,\eta_y),a} \in supp(f_a, H_L)$. Also, $l_{(y,\eta_y),a} \in supp(f'_b, H_L)$, thus by Lemma 5.1 there exists a maximal element $l_{(y',\alpha'),c'} \in supp(f'_b, H_L)$ such that

$$l_{(y,\eta_y),a}(x) \leq l_{(y',\alpha'),c'}(x), \quad \forall\, x \in X, \tag{5.1}$$

where $y' = \frac{\varepsilon_y y}{f'(\varepsilon_y y)}$, $\alpha' = f'(\varepsilon_y y)$ *and* $c' = b$. Set $x = 0$ in (5.1), then $a \leq b$, and set $x = \eta_y y$ in (5.1), then

$$\eta_y + a \leq l_{(y',\alpha')}(\eta_y y) + b,$$
$$\eta_y + a \leq l_{(\frac{\varepsilon_y y}{f'(\varepsilon_y y)}, f'(\varepsilon_y y))}(\eta_y y) + b,$$
$$\eta_y + a \leq l(\eta_y y, y, \varepsilon_y) + b.$$

Since $l_{(y,\varepsilon_y)} \in supp(f', L)$, therefore we get

$$\eta_y + a \leq l(\eta_y y, y, \varepsilon_y) + b \leq f'(\eta_y y) + b.$$

Hence $\eta_y + a \leq f'_b(\eta_y y)$.

$(iii) \Longrightarrow (i)$. Let $a \leq b$ and $\eta_y + a \leq f'(\eta_y y) + b$ for all $y \in X \setminus (-S)$ with $\eta_y \geq 0$. Consider $l_{(y,\alpha),c} \in supp(f_a, H_L)$, then $c \leq a$. By the assumption $a \leq b$, therefore $c \leq b$, and by Lemma 5.1 there exists a maximal element $l_{(y',\alpha'),c'} \in supp(f_a, H_L)$ such that $l_{(y,\alpha),c} \leq l_{(y',\alpha'),c'}$ on X, where $y' = \frac{\eta_y y}{f(\eta_y y)}$, $\alpha' = f(\eta_y y)$ and $c' = a$. Now, set $y'' = \frac{\eta_{y'} y'}{f'(\eta_{y'} y')}$ and $\alpha'' = f'(\eta_{y'} y')$. Thus

$$l_{(y'',\alpha'')}(\eta_{y'} y') = f'(\eta_{y'} y') = \alpha'' = f'(\alpha'' y''). \tag{5.2}$$

The inequality $\eta_y + a \leq f'(\eta_y y) + b$ for all $y \in X \setminus (-S)$ and (5.2) imply that $l_{(y'',\alpha''),b}(\eta_{y'} y') \geq \eta_{y'} + a$. So, it follows from Corollary 3.3 that

$$l_{(y',\eta_{y'}),a} \leq l_{(y'',\alpha''),b} \text{ on } X.$$

It is clear that $l_{(y'',\alpha''),b} \in supp(f'_b, H_L)$. Moreover, by [3, Corollary 4.1],

$$\begin{aligned} \eta_{y'} &= \max\{\alpha > 0 : f(\alpha y') \geq \alpha\} \\ &= \alpha' \\ &= f(\eta_y y). \end{aligned}$$

So

$$l_{(y,\alpha),c} \leq l_{(y',\alpha'),a} \leq l_{(y'',\alpha''),b} \leq f'_b \text{ on } X.$$

Hence $l_{(y,\alpha),c} \in supp(f'_b, H_L)$. □

In the sequel, necessary and sufficient conditions for the global minimum of the difference of two strictly positive valued affine ICR functions are given.

Let $f, f' : X \longrightarrow [0, +\infty)$ be strictly ICR functions and $h := f' - f$ be such that

$$h(x) = f'(x) - f(x), \quad \forall\, x \in X. \tag{5.3}$$

Theorem 5.3. *Let $\eta_y := \max\{\alpha > 0 : f(\alpha y) \geq \alpha\}$ for all $y \in X \setminus (-S)$, where $f : X \longrightarrow [0, +\infty]$ is a strictly positive valued ICR function such that $f(\eta_y y) = \eta_y$. Also, let $\varepsilon_y := \max\{\alpha > 0 : f'(\alpha y) \geq \alpha\}$ for all $y \in X \setminus (-S)$, where $f' : X \longrightarrow [0, +\infty]$ is a strictly positive valued ICR function such that $f'(\varepsilon_y y) = \varepsilon_y$. Then an element $x_0 \in X$ is a global minimizer of the function h (defined by (5.3)) if and only if $h(x_0) \leq 0$ and $f'(\eta_y y) \geq \eta_y + h(x_0)$ for all $y \in C$, where $C = \left\{y \in X \setminus (-S) : \eta_y \geq 0\right\}$.*

Proof. It is clear that x_0 is a global minimizer of the function h if and only if $h(x_0) \leq h(x)$ for all $x \in X$. By the definition of the function h and Proposition 5.2, it follows that

$$\begin{aligned} & h(x_0) \leq h(x), \quad \forall\, x \in X, \\ \Longleftrightarrow\ & f_{h(x_0)}(x) \leq f'(x), \quad \forall\, x \in X, \\ \Longleftrightarrow\ & supp(f_{h(x_0)}, H_L) \subseteq supp(f', H_L), \\ \Longleftrightarrow\ & h(x_0) \leq 0,\ f'(\eta_y y) \geq \eta_y + h(x_0), \quad \forall\, y \in C. \end{aligned}$$

□

Corollary 5.4. *Suppose that $h(x_0) < 0$ with $x_0 \in X$. Then, under the assumptions of Theorem 5.3, x_0 is a global minimizer of the function h if and only if*

$$h(x_0) = \inf_{y \in C} \left\{ f'(\eta_y y) - \eta_y \right\},$$

where $C := \left\{ y \in X \setminus (-S) : \eta_y \geq 0 \right\}.$

Proof. Let $h(x_0) = \inf_{y \in C} \left\{ f'(\eta_y y) - \eta_y \right\}$. Then $f'(\eta_y y) \geq \eta_y + h(x_0)$ for all $y \in C$. So, by Theorem 5.3, x_0 is a global minimizer of the function h. Conversely, let $x_0 \in X$ be a global minimizer of the function h. Also, assume that $\nu := \inf_{y \in C} \left\{ f'(\eta_y y) - \eta_y \right\}$ and $h(x_0) < \nu$ (note that it follows from Theorem 5.3 that $h(x_0) \leq \nu$). Consider $\nu' < 0$ such that $h(x_0) < \nu' \leq \nu$. Therefore, Proposition 5.2 implies that $supp(f_{\nu'}, H_L) \subseteq supp(f', H_L)$, i.e., $f_{\nu'} \leq f'$ on X, which yields a contradiction with the fact that $h(x_0) < \nu'$. □

Corollary 5.1. Let $f, f' : X \longrightarrow [0, +\infty]$ be strictly ICR functions and $f_a = f + a$ and $f'_b = f' + b$ be strictly positive valued affine ICR functions. Then, x_0 is a global minimizer of the function $f'_b - f_a$ if and only if x_0 is a global minimizer of the function $f' - f$.

Example 5.5. Consider two strictly ICR functions $f, f' : \mathbb{R} \to [0, +\infty)$ are defined by

$$f(x) := \begin{cases} 0, & x < 0 \\ x^{\frac{1}{2}}, & x \geq 0 \end{cases} \quad and \quad f'(x) := \begin{cases} 0, & x < 0 \\ x^{\frac{2}{3}}, & x \geq 0 \end{cases}.$$

It is easy to observe that

$$\eta_y := \max\{\alpha > 0 : f(\alpha y) \geq \alpha\} = y,$$

and

$$\varepsilon_y := \max\{\beta > 0 : f'(\beta y) \geq \beta\} = y^2,$$

for all $y > 0$. Also, $f(\eta_y y) \geq \eta_y$ and $f'(\varepsilon_y y) = \varepsilon_y$. By Corollary 5.4, x_0 is a global minimizer of the function $h = f' - f$ if and only if $h(x_0) = \inf_{y>0} \left\{ y^{\frac{4}{3}} - y \right\} = \frac{-27}{256}$ if and only if $x_0 = (\frac{3}{4})^6$.

Example 5.6. Let $X := \{f : [0, 1] \longrightarrow \mathbb{R}$: f is a continuous function $\}$ and

$$S := \{f \in X : f(x) \geq 0, \quad \forall\, x \in [0, 1]\}.$$

It is clear that S is a closed convex pointed cone in X. Consider two functions

$$\varphi,\ \psi : X \longrightarrow [0, +\infty),$$

are defined by

$$\varphi(f) := \sqrt{f^+(y_0)} \quad and \quad \psi(f) = \sup_{0 \leq x \leq 1} \sqrt{f^+(x)},$$

for all $f \in X$ ($y_0 \in [0,1]$ is fixed), where $f^+(x) := \max\{f(x),\ 0\}$ for all $x \in [0,1]$. It is clear that φ and ψ are strictly positive valued ICR functions, and it is easy to see that

$$\eta_f := \max\{\alpha > 0 : \varphi(\alpha f) \geq \alpha\} = f^+(y_0),$$

and

$$\varepsilon_f := \max\{\alpha > 0 : \psi(\alpha f) \geq \alpha\} = \left(\sup_{0\leq x\leq 1} \sqrt{f^+(x)}\right)^2,$$

for all $f \in X \setminus (-S)$. Also, $\varphi(\eta_f f) \geq \eta_f$ and $\psi(\varepsilon_f f) = \varepsilon_f$ for all $f \in X \setminus (-S)$. It is not difficult to check that

$$\inf_{f\in X\setminus(-S)} \left\{\psi(\eta_f f) - \eta_f\right\} = 0.$$

By Corollary 5.4, f_0 is a global minimizer of the function $h = \psi - \varphi$ if and only if $h(f_0) = 0$ if and only if

$$\sup_{0\leq x\leq 1} \sqrt{f_0^+(x)} = \sqrt{f_0^+(y_0)}$$

if and only if y_0 is a maximizer of the function $\sqrt{f_0^+(x)}$.

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