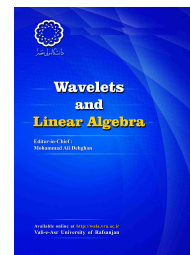


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On Some Special Classes of Sonnenschein Matrices

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ABSTRACT

In this paper we consider the special classes of Sonnenschein matrices, namely the Karamata matrices $K[\alpha, \beta] = (a_{n,k})$ with the entries

$$a_{n,k} = \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v},$$

and calculate their row and column sums and give some applications of these sums.

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1. Introduction

Let $f(z)$ be an analytic function in $D_f = \{z \in \mathbb{C} : |z| < r, r \geq 1\}$ with $f(1) = 1$. The matrix $S = S_f = (a_{n,k})$, where $(a_{n,k})$ are defined by $[f(z)]^n = \sum_{k=0}^{\infty} a_{n,k} z^k$ is called a Sonnenschein matrix

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[7, 9]. The special choice

$$f(z) = \frac{\alpha + (1 - \alpha - \beta)z}{1 - \beta z}, \quad z \in \mathbb{C} \setminus \left\{ \frac{1}{\beta} \right\},$$

where α and β are complex numbers, gives the Karamata matrix $K[\alpha, \beta]$ and its coefficients are given by [3, 4]

$$a_{n,k} = \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v}.$$

In particular, $K[1 - \alpha, 0]$ and $K[0, 1 - \alpha]$, give the Euler [1] matrix E^α , and the Taylor [10] matrix T^α , respectively.

In this paper we are going to compute the row and column sums of the Karamata matrix $K[\alpha, \beta]$ for the cases $\alpha, \beta \in (0, 1)$ and to give some applications for these sums. Our results are based on the following binomial coefficients identities:

- (i) $\frac{1}{(1-x)^{k+1}} = \frac{1}{k!} \sum_{n=0}^{\infty} (n+k) \dots (n+1) x^n, \quad |x| < 1.$
- (ii) $\binom{t}{v} = 0, \quad (v > t \text{ or } v < 0).$

We begin with the following lemma which is essential in the text.

Lemma 1.1. *Let $n, k, v \in \mathbb{N} \cup \{0\}$ and $\alpha < 1$. Then*

$$\sum_{n=v}^{\infty} \binom{n}{v} \binom{n+k-v-1}{k-v} \alpha^{n-v} = \frac{\alpha \binom{k}{v} + (1 - \alpha) \binom{k-1}{v-1}}{(1 - \alpha)^{k+1}}.$$

Proof. Using the identity (i), we have

$$\begin{aligned} \sum_{n=v}^{\infty} \binom{n}{v} \binom{n+k-v-1}{k-v} \alpha^{n-v} &= \sum_{n=0}^{\infty} \binom{n+v}{v} \binom{n+k-1}{k-v} \alpha^n \\ &= \frac{1}{v!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+v)(n+k-1)!}{n!} \alpha^n \\ &= \frac{1}{v!(k-v)!} \sum_{n=1}^{\infty} \frac{(n+k-1)!}{(n-1)!} \alpha^n + \frac{1}{(v-1)!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+k-1)!}{n!} \alpha^n \\ &= \frac{\alpha}{v!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \alpha^n + \frac{1}{(v-1)!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+k-1)!}{n!} \alpha^n \\ &= \frac{k!}{v!(k-v)!} \frac{\alpha}{(1-\alpha)^{k+1}} + \frac{(k-1)!}{(v-1)!(k-v)!} \frac{1}{(1-\alpha)^k} \\ &= \frac{\alpha \binom{k}{v} + (1-\alpha) \binom{k-1}{v-1}}{(1-\alpha)^{k+1}}, \end{aligned}$$

as desired. □

Theorem 1.2. For the Karamata matrix $K[\alpha, \beta]$, the sum of the first column is $\frac{1}{1-\alpha}$, the sum of all other columns are $\frac{1-\beta}{1-\alpha}$, and the sum of all rows are 1.

Proof. For $k = 0$, obviously $\sum_{n=0}^{\infty} a_{n,0} = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$. If $k \geq 1$, applying Lemma 1.1 together with the identity (i), we have

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n,k} &= \sum_{n=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\ &= \sum_{v=0}^k \sum_{n=0}^{\infty} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\ &= \sum_{v=0}^k (1-\alpha-\beta)^v \beta^{k-v} \sum_{n=v}^{\infty} \binom{n}{v} \binom{n+k-v-1}{k-v} \alpha^{n-v} \\ &= \sum_{v=0}^k (1-\alpha-\beta)^v \beta^{k-v} \frac{\alpha \binom{k}{v} + (1-\alpha) \binom{k-1}{v-1}}{(1-\alpha)^{k+1}} \\ &= \frac{\alpha}{(1-\alpha)^{k+1}} \sum_{v=0}^k \binom{k}{v} (1-\alpha-\beta)^v \beta^{k-v} \\ &\quad + \frac{1-\alpha}{(1-\alpha)^{k+1}} \sum_{v=1}^k \binom{k-1}{v-1} (1-\alpha-\beta)^v \beta^{k-v} \\ &= \frac{\alpha(1-\alpha)^k}{(1-\alpha)^{k+1}} + \frac{1-\alpha}{(1-\alpha)^{k+1}} (1-\alpha-\beta) \sum_{v=1}^k \binom{k-1}{v-1} (1-\alpha-\beta)^v \beta^{k-v} \\ &= \frac{\alpha(1-\alpha)^k + (1-\alpha)(1-\alpha-\beta)(1-\alpha)^{k-1}}{(1-\alpha)^{k+1}} \\ &= \frac{1-\beta}{1-\alpha}. \end{aligned}$$

For the row sums of the Karamata matrix, fix $n \in \mathbb{N} \cup \{0\}$, using identities (i) and (ii), we have

$$\begin{aligned} \sum_{k=0}^{\infty} a_{n,k} &= \sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\ &= \sum_{k=0}^{\infty} \sum_{v=0}^n \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\ &= \sum_{v=0}^n \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \sum_{k=v}^{\infty} \binom{n+k-v-1}{k-v} \beta^{k-v} \\ &= \sum_{v=0}^n \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \beta^k \\ &= (1-\beta)^n \frac{1}{(1-\beta)^n} = 1. \end{aligned}$$

□

2. Applications

For $0 < p < \infty$, the Karamata sequence space is defined by

$$\mathcal{K}_p^{\alpha,\beta} = \left\{ (x_n) \in \mathbb{C} : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p < \infty \right\}.$$

More precisely, $\mathcal{K}_p^{\alpha,\beta}$ is the set of all sequences such that $K[\alpha, \beta]$ -transforms of them are in the space ℓ_p . In particular, $\mathcal{K}_p^{0,0} = \ell_p$ and $\mathcal{K}_p^{1-\alpha,0} = e_p^\alpha$, the Euler sequence space of order α [2], which is defined as below:

$$e_p^\alpha = \left\{ (x_n) \in \mathbb{C} : \sum_{n=0}^{\infty} \left| \sum_{k=0}^n \binom{n}{k} (1 - \alpha)^{n-k} \alpha^k x_k \right|^p < \infty \right\}.$$

Lemma 2.1. *The set $\mathcal{K}_p^{\alpha,\beta}$ becomes a linear space with the coordinatewise addition and scalar multiplication, which is the semi normed space with the semi norm*

$$\|x\|_{\mathcal{K}_p^{\alpha,\beta}} := \left(\sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p \right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$. Further, it is a p -semi normed space with the p -semi norm $\|x\| = \|x\|_{\mathcal{K}_p^{\alpha,\beta}}^p$, whenever $0 < p < 1$.

An easy calculation shows that the absolute property does not hold on the space $\mathcal{K}_p^{\alpha,\beta}$, that is, $\|x\|_{\mathcal{K}_p^{\alpha,\beta}} \neq \| |x| \|_{\mathcal{K}_p^{\alpha,\beta}}$ for at least one sequence in the space $\mathcal{K}_p^{\alpha,\beta}$, and this says us that $\mathcal{K}_p^{\alpha,\beta}$ is a sequence space of non-absolute type, where $|x| = (|x_k|)$. It is immediate by the well-known inclusion $\ell_p \subset \ell_q$ that the inclusion $\mathcal{K}_p^{\alpha,\beta} \subset \mathcal{K}_q^{\alpha,\beta}$ holds whenever $p \leq q$.

Theorem 2.2. *The inclusion $\ell_p \subseteq \mathcal{K}_p^{\alpha,\beta}$ holds for $1 \leq p < \infty$.*

Proof. Let $x \in \ell_p$. Applying Theorem 1.2 with Hölder's inequality for $1 < p < \infty$, we obtain

$$\begin{aligned} & \left| \sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p \\ & \leq \left(\sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} |x_k|^p \right) \\ & \quad \times \left(\underbrace{\sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v}}_{= 1} \right)^{p-1} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} |x_k|^p. \tag{2.1}$$

Applying (2.1) with Theorem 1.2, we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p \\ & \leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} |x_k|^p \\ & \leq \sum_{k=0}^{\infty} |x_k|^p \sum_{n=0}^{\infty} \sum_{v=0}^k \binom{n}{v} (1 - \alpha - \beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\ & = |x_0|^p \left(\frac{1}{1-\alpha} \right) + |x_1|^p \left(\frac{1-\beta}{1-\alpha} \right) + |x_2|^p \left(\frac{1-\beta}{1-\alpha} \right) + |x_3|^p \left(\frac{1-\beta}{1-\alpha} \right) + \dots \\ & \leq \left(\frac{1}{1-\alpha} \right) \sum_{k=0}^{\infty} |x_k|^p, \end{aligned}$$

which yields us that

$$\|x\|_{\mathcal{K}_p^{\alpha,\beta}} \leq \left(\frac{1}{1-\alpha} \right)^{1/p} \|x\|_{\ell_p}. \tag{2.2}$$

Therefore, $x \in \mathcal{K}_p^{\alpha,\beta}$, i.e. $\ell_p \subseteq \mathcal{K}_p^{\alpha,\beta}$. By similar discussions, it may be easily proved that the inequality (2.2) holds in the case $p = 1$ and so we omit the detail. This completes the proof. \square

Theorem 2.3. Let $0 < p < 1$, and $x = (x_n)$ be a non-negative sequence in $\mathcal{K}_p^{\alpha,\beta}$. Then x belongs to the space ℓ_p , and we have

$$\|x\|_{\ell_p} \leq \left(\frac{1-\beta}{1-\alpha} \right)^{-1/p} \|x\|_{\mathcal{K}_p^{\alpha,\beta}}. \tag{2.3}$$

Proof. The proof is similar to that of Theorem 2.2. The only difference is due to the first part in which the Hölder’s inequality for $0 < p < 1$, should be applied and therefore the inequality sign must be reversed. \square

We refer the readers to [5, 6, 8] to give more information about sequence spaces.

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