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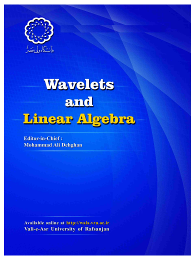

The Sign-Real Spectral Radius for Real Tensors

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Abstract

In this paper a new quantity for real tensors, the sign-real spectral radius, is defined and investigated. Various characterizations, bounds and some properties are derived. In certain aspects our quantity shows similar behavior to the spectral radius of a nonnegative tensor. In fact, we generalize the Perron Frobenius theorem for nonnegative tensors to the class of real tensors.

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1. Introduction and Notation

Tensors have numerous applications in many branches of mathematics and physics. It arises in diverse fields such as signal and image processing, data analysis, nonlinear continuum mechanics,

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higher order statistics, as well as independent component analysis [1, 5, 6, 8, 9, 15, 18, 19, 25, 27]. The concept of eigenvalues for tensors was first introduced and studied by Qi [21] and Lim [17] independently in 2005, and initiated the rapid developments of the spectral theory of tensors. In late studies of numerical multilinear algebra [3, 17, 21], eigenvalue problems for tensors have been brought to special attention. In particular, the Perron Frobenius theorem for nonnegative tensors is related to measuring higher order connectivity in linked objects [16] and hypergraphs [11]. Also, the largest eigenvalue problem for nonnegative tensors has attracted special attention. Perron Frobenius theorem for nonnegative tensors that includes existence, uniqueness of Perron root and positivity of the Perron vector is established, Collectively in [4, 12, 17]. Ng et. al. [20] gave a method to find the largest eigenvalue of a nonnegative irreducible tensor. Yang and Yang [26] defined the spectral radius of a tensor and gave further results for the Perron Frobenius theorem and proved that the spectral radius is the largest eigenvalue of any nonnegative tensor and all eigenvalues with the spectral radius as their modulus distribute uniformly on the circle.

It was in 1997, S. Rump introduced and investigated a new quantity for real matrices, the sign-real spectral radius, [23]. This quantity was shown to share certain properties with the Perron root of nonnegative matrices and has intimate connections to the componentwise distance to the nearest singular matrix. Componentwise distances, perturbation bounds, and error bounds have received quite some attention in recent years (see [2, 13]). Any lower bound on the sign-real spectral radius implies an upper bound on the componentwise distance to the nearest singular matrix [23]. This was the original motivation for defining and investigating the sign-real spectral radius. It turns out that the sign-real spectral radius is interesting in itself and, in certain aspects, shows similar behavior to the Perron root of a nonnegative matrix, for example, the inheritance property on going to principal submatrices [23]. Now, the question arises whether this quantity can be expressed and checked for real tensors? In the other words, can we extend the Perron Frobenius theorem for nonnegative tensors to the class of real tensors? In this paper, we will give a positive answer to this question.

This paper is organized as follows. In Section 2, we recall the definitions and theorems concerning the eigenvalues of the tensor. In Section 3, the sign-real spectral radius is defined for tensors then the properties of this quantity, which is similar to the spectral radius for nonnegative ones will be investigated. In Section 4, an extension of the Perron Frobenius theorem for nonnegative tensors to real tensors is given. In fact we show that for every mth-order n-dimensional tensor $\mathbb{A}$, the sign-real spectral radius is a real eigenvalue of $S\mathbb{A}$ for some signature matrix S and the associated eigenvector is nonnegative (see Theorem 4.2). Also it will be shown that the sign-real spectral radius is equal to the spectral radius for nonnegative tensors.

We first add a comment on the notation that is used. Vectors are written as $(x, y, ...)$, matrices correspond to $(A, B, ...)$ and tensors are written as $(\mathbb{A}, \mathbb{B}, ...)$. The entry with row index i and column index j in a matrix A, i.e. $(A)_{ij}$ is symbolized by a_{ij} (also $(\mathbb{A})_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m}$). The symbol $|\,.\,|$ used on a matrix A (tensor $\mathbb{A}$) means that $(|\mathbb{A}|)_{ij} = \left|a_{ij}\right|$ ($(|\mathbb{A}|)_{i_1 i_2 \ldots i_m} = \left|a_{i_1 i_2 \ldots i_m}\right|$). $\mathfrak{R}$ and C denote the real and complex field respectively.

2. Preliminaries and some conclusions concerning the eigenvalue

A real mth-order n-dimensional tensor $\mathbb{A}$ consists of n^m real entries:

$$a_{i_1 i_2 \ldots i_m} \in \Re,$$

where $i_j \in \{1, ..., n\}$ for $j \in \{1, ..., m\}$. We denote the set of all real mth-order n-dimensional tensors by $\Re^{[m,n]}$. For a vector $x \in \Re^n$, we use x_i to denote its components, and $x^{[m]}$ to denote a vector in $\Re^n$ such that

$$x_i^{[m]} = x_i^m,$$

for all i. $\mathbb{A}x^{m-1}$ for a vector $x \in \Re^n$ denotes a vector in $\Re^n$, whose ith component is

$$\left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m}. \tag{2.1}$$

Definition 2.1. A pair $(\lambda, x) \in C \times (C^n \setminus \{0\})$ is called an eigenvalue and an eigenvector of $\mathbb{A} \in \Re^{[m,n]}$, if they satisfy

$$\mathbb{A}x^{m-1} = \lambda x^{[m-1]}. \tag{2.2}$$

Furthermore, we say λ is an H-eigenvalue with the corresponding H-eigenvector x (or (λ, x) is an H-eigenpair) of $\mathbb{A}$ if they are both real.

In the case $m = 2$, (2.2) reduces to the definition of eigenvalues and corresponding eigenvectors of a square matrix. This definition was introduced by Qi [21]. We denote the set of eigenvalues of $\mathbb{A}$ with $\sigma(\mathbb{A})$. An mth-order n-dimensional tensor $\mathbb{A}$ is called nonnegative if $a_{i_1 i_2 \ldots i_m} \geq 0$. We call an mth-order n-dimensional tensor the unit tensor, denoted by $\mathbb{I}$, if its entries are $\delta_{i_1 i_2 \ldots i_m}$ with $\delta_{i_1 i_2 \ldots i_m} = 1$ if and only if $i_1 = \cdots = i_m$ and the others are zero.

Theorem 2.2. *[4] If $\mathbb{A}$ is a nonnegative tensor of order m dimension n, then there exist $\lambda_0 \geq 0$ and a nonnegative vector $x_0 \neq 0$ such that*

$$\mathbb{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}.$$

The spectral radius of tensor $\mathbb{A}$ is defined by Yang and Yang in [26] as follows:

Definition 2.3. The spectral radius of tensor $\mathbb{A}$ is defined as

$$\rho(\mathbb{A}) = \max\left\{|\lambda| \ : \ \lambda \ is\, an\, eigenvalue\, of\ \mathbb{A}\right\}. \tag{2.3}$$

They proved that the spectral radius of a nonnegative tensor, is an eigenvalue of it.

Definition 2.4. [14] Let $\mathbb{A}$ be an mth-order n-dimensional tensor with $m \geq 2$ and $x = (x_1, ..., x_n)^T$. Then the determinant of $\mathbb{A}$, denoted by $det(\mathbb{A})$, is the resultant of the ordered system of homogeneous equations $\mathbb{A}x^{m-1} = 0$ (i.e. the system of homogeneous equations $\left(\mathbb{A}x^{m-1}\right)_i = 0$ for $i = 1, ..., n$), where $\mathbb{A}x^{m-1}$ is as defined in (2.1).

By using the properties of the resultants, it can be shown [14] that the above definition for determinant is equivalent to the following definition.

Definition 2.5. [14] Let $\mathbb{A}$ be an mth-order n-dimensional tensor with $m \geq 2$. Then $det(\mathbb{A})$ is the unique polynomial on the entries of $\mathbb{A}$ satisfying the following three conditions:

(1) $det(\mathbb{A}) = 0$ if and only if the system of homogeneous equations $\mathbb{A}x^{m-1} = 0$ has a nonzero solution.

(2) $det(\mathbb{I}) = 1$, where $\mathbb{I}$ is the unit tensor.

(3) $det(\mathbb{A})$ is an irreducible polynomial on the entries of $\mathbb{A}$, when the entries $a_{i_1...i_m}$ $(1 \leq i_1, ..., i_m \leq n)$ of $\mathbb{A}$ are all viewed as independent different variables.

By using the definition of determinants, we can define the characteristic polynomial of a tensor $\mathbb{A}$ as the determinant $det(\lambda\mathbb{I} - \mathbb{A})$, where $\mathbb{I}$ is the unit tensor. It is easy to see from the definitions that, λ is an eigenvalue of $\mathbb{A}$ if and only if it is a root of the characteristic polynomial of $\mathbb{A}$.

Definition 2.6. [24] Let $\mathbb{A}$ and $\mathbb{B}$ be order $m \geq 2$ and order $k \geq 1$, dimension n tensors, respectively. The product $\mathbb{AB}$ is the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n with entries:

$$c_{i\alpha_1...\alpha_{m-1}} = \sum_{i_2,...,i_m=1}^{n} a_{ii_2...i_m} b_{i_2\alpha_1}...b_{i_m\alpha_{m-1}}$$
$$\left(i \in [n] := \{1, ..., n\}, \ \alpha_1, ..., \alpha_{m-1} \in [n]^{k-1}\right). \tag{2.4}$$

It is easy to check from the definition that $I_n\mathbb{A} = \mathbb{A} = \mathbb{A}I_n$, where I_n is the identity matrix of order n.

Theorem 2.7. *[24] The tensor product defined in above has the following properties.*
(i) $(\mathbb{A} + \mathbb{B})\mathbb{C} = \mathbb{AC} + \mathbb{BC}$.
(ii) $A(\mathbb{B} + \mathbb{C}) = A\mathbb{B} + A\mathbb{C}$, *where A is an $n \times n$ matrix.*
(iii) $(\alpha\mathbb{A})\mathbb{B} = \alpha(\mathbb{AB})$, *for any* $(\alpha \in C)$.
(iv) $\mathbb{A}(\alpha\mathbb{B}) = \alpha^{m-1}(\mathbb{AB})$, *for any* $(\alpha \in C)$.

Theorem 2.8. *[24] (The associative law of the tensor product): Let $\mathbb{A}$ (and $\mathbb{B}$, $\mathbb{C}$) be an order $m+1$ (and order $k+1$, order $r+1$), dimension n tensor, respectively. Then we have*

$$\mathbb{A}(\mathbb{BC}) = (\mathbb{AB})\mathbb{C}.$$

Lemma 2.9. *[24] Let $\mathbb{A}$ be an mth-order n-dimensional tensor, $\mathbb{I}$ be the mth-order n-dimensional unit tensor, and P and Q are two matrices of order n. Then we have:*

$$\det(P\mathbb{A}Q) = \det(P\mathbb{I}Q)\det(\mathbb{A}).$$

Definition 2.10. [24] Let $\mathbb{A}$ and $\mathbb{B}$ be two order k tensors with dimension n and m, respectively. Define the direct product $\mathbb{A} \otimes \mathbb{B}$ to be the following tensor of order k and dimension nm (the set of subscripts is taken as $[n] \times [m]$ in the lexicographic order):

$$(\mathbb{A} \otimes \mathbb{B})_{(i_1,j_1),(i_2,j_2),...,(i_k,j_k)} = a_{i_1i_2...i_k} b_{j_1j_2...j_k}.$$

Theorem 2.11. *[24] Let $\mathbb{A}$ and $\mathbb{B}$ be two order $k+1$ tensors with dimension n and m, respectively. Let $\mathbb{C}$ and $\mathbb{D}$ be two order $r+1$ tensors with dimension n and m, respectively. Then we have:*

$$(\mathbb{A} \otimes \mathbb{B})(\mathbb{C} \otimes \mathbb{D}) = (\mathbb{A}\mathbb{C}) \otimes (\mathbb{B}\mathbb{D}).$$

Theorem 2.12. *[24] Let $\mathbb{A}$ and $\mathbb{B}$ be two order k tensors with dimension n and m, respectively. Suppose that we have $\mathbb{A}u^{k-1} = \lambda u^{[k-1]}$, and $\mathbb{B}v^{k-1} = \mu v^{[k-1]}$, and we also write $w = u \otimes v$. Then we have:*

$$(\mathbb{A} \otimes \mathbb{B}) w^{k-1} = (\lambda\mu) w^{[k-1]}.$$

Definition 2.13. A diagonal tensor is a tensor that only the entries of which all the indices are equal can be different from zero.

Lemma 2.14. *[26] Let $\mathbb{A}, \mathbb{B} \in \mathfrak{R}^{[m,n]}$. If there is a diagonal nonsingular matrix D such that $\mathbb{A} = \mathbb{B}.D^{-(m-1)}.\overbrace{D...D}^{m-1}$, then they have the same eigenvalues, where*

$$a_{i_1 i_2 \ldots i_m} = b_{i_1 i_2 \ldots i_m} d_{i_1 i_1}^{-(m-1)} d_{i_2 i_2} \ldots d_{i_m i_m}. \tag{2.5}$$

Definition 2.15. [14] Let $\mathbb{A} \in \mathfrak{R}^{[m,n]}$. Suppose that $a_{i_1 i_2 \ldots i_m} = 0$ if $\min\{i_2, ..., i_m\}$ is less than i_1, then $\mathbb{A}$ is called an upper triangular tensor. Suppose that $a_{i_1 i_2 \ldots i_m} = 0$ if $\max\{i_2, ..., i_m\}$ is greater than i_1, then $\mathbb{A}$ is called a lower triangular tensor. If $\mathbb{A}$ is either upper or lower triangular, then $\mathbb{A}$ is called a triangular tensor. In particular, a diagonal tensor is a triangular tensor.

Theorem 2.16. *[14] Suppose that $\mathbb{A} \in \mathfrak{R}^{[m,n]}$ is a triangular tensor. Then*

$$\sigma(\mathbb{A}) = \left\{a_{ii\ldots i} : i = 1, ..., n\right\},$$

and the algebraic multiplicity of $a_{ii\ldots i}$ is $(m-1)^{n-1}$ for all $i = 1, ..., n$. Also we have

$$\det(\mathbb{A}) = \prod_{i=1}^{n} (a_{i\ldots i})^{(m-1)^{(n-1)}}$$

Definition 2.17. [17] A tensor $\mathbb{A} \in \mathfrak{R}^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, ..., n\}$ such that

$$a_{i_1,\ldots,i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, ..., i_m \notin I,$$

If $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

Theorem 2.18. *Let $\mathbb{A} \in \mathfrak{R}_+^{[m,n]}$ be an irreducible tensor. Define the function $f_{\mathbb{A}}$ from $\mathfrak{R}_+^n$ to the set of nonnegative numbers by*

$$f_{\mathbb{A}}(x) = \min_{x_i \neq 0} \frac{\left(\mathbb{A}x^{m-1}\right)_i}{x_i^{m-1}},$$

for all $x \neq 0$, $x \geq 0$. If y is nonnegative, nonzero and ρ is the largest real number such that $\mathbb{A}y^{m-1} - \rho y^{[m-1]} \geq 0$, then $\rho = f_{\mathbb{A}}(y)$.

Proof. By definition of $f_{\mathbb{A}}$ we have $\mathbb{A}y^{m-1} - f_{\mathbb{A}}(y)\, y^{[m-1]} \geq 0$, and there exists an integer s, $1 \leq s \leq n$, such that $y_s \neq 0$ and sth coordinate of $\mathbb{A}y^{m-1} - f_{\mathbb{A}}(y)\, y^{[m-1]}$ is zero. Therefore if $\rho > f_{\mathbb{A}}(y)$, then the sth coordinate of $\mathbb{A}y^{m-1} - \rho y^{[m-1]}$ is negative. The result follows. □

3. The sign-real spectral radius for real tensors

The purpose of this paper is to extend Perron Frobenius theorem for nonnegative tensors to general real tensors. The key of this extension is the following nonlinear eigenvalue problem:

$$\max\left\{|\lambda| : \left|\mathbb{A}x^{m-1}\right| = \left|\lambda x^{[m-1]}\right| \ , x \neq 0\right\}. \tag{3.1}$$

For an m-order n-dimensional nonnegative tensor $\mathbb{A}$ we can in (3.1) clearly eliminate the absolute values and obtain the spectral radius:

$$\begin{aligned}\rho(\mathbb{A}) &= \max\left\{|\lambda| : \left|\mathbb{A}x^{m-1}\right| = \left|\lambda x^{[m-1]}\right|, \lambda \in C, \ 0 \neq x \in C^n\right\} \\ &= \max\left\{0 \le \lambda \in \Re \ : \ \mathbb{A}x^{m-1} = \lambda x^{[m-1]} \ , 0 \le x \in \Re^n \ , x \neq 0\right\}.\end{aligned} \tag{3.2}$$

The reason comes from the Perron Frobenius theorem for nonnegative tensors (see Theorem 2.2), and

$$\begin{aligned}\left|\left(\lambda x^{[m-1]}\right)_i\right| = |\lambda|\left|x_i^{m-1}\right| \ &= \ \left|\left(\mathbb{A}x^{m-1}\right)_i\right| = \left|\sum_{i_2\ldots i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2}\ldots x_{i_m}\right| \\ &\le \ \sum_{i_2\ldots i_m=1}^{n} a_{ii_2\ldots i_m}\left|x_{i_2}\right|\ldots\left|x_{i_m}\right| \\ &= \ \left(\mathbb{A}|x|^{m-1}\right)_i,\end{aligned}$$

thus $\mathbb{A}|x|^{m-1} - |\lambda|\left|x^{[m-1]}\right| \ge 0$. By Theorem (2.18) we have $|\lambda| \le f_{\mathbb{A}}(|x|) \le \rho(\mathbb{A})$.

For the extension to general real tensors we consciously restrict attention to real eigenvalues (if it exists [see Definition 3.2])(and eigenvectors), that is we consider the quantity

$$\max\left\{|\lambda| : \left|\mathbb{A}x^{m-1}\right| = \left|\lambda x^{[m-1]}\right|, \quad \mathbb{A} \in \Re^{[m,n]}, \quad \lambda \in \Re, \quad 0 \neq x \in \Re^n\right\}. \tag{3.3}$$

A real diagonal matrix S with diagonal entries of modulus one is called a real signature matrix. Real signature matrices are the set of diagonal orthogonal matrices, which are in the real case 2^n matrices with diagonal entries ± 1. In our entrywise notation of absolute value, real signature matrix S is characterized by $|S| = I$ where I denotes the identity matrix. For brevity we denote the set of all signature matrices by φ. For a real vector $x \in \Re^n$, there is always a signature matrix $S \in M_n(\Re)$ with $Sx = |x|$. If all entries of x are nonzero, S is unique. Hence, for our nonlinear eigenvalue problem (3.1) there are signature matrices S_1 and S_2 with $S_1\mathbb{A}x^{m-1} = \left|\mathbb{A}x^{m-1}\right|$ and $S_2\lambda x^{[m-1]} = \left|\lambda x^{[m-1]}\right|$, such that

$$\left|\mathbb{A}x^{m-1}\right| = \left|\lambda x^{[m-1]}\right| \Leftrightarrow S_1\mathbb{A}x^{m-1} = S_2\lambda x^{[m-1]}. \tag{3.4}$$

Therefore the quantity in (3.3) is for $\mathbb{A} \in \Re^{[m,n]}$ and $S = S_2S_1$ the same as

$$\max\left\{|\lambda| : S\mathbb{A}x^{m-1} = \lambda x^{[m-1]}, \quad \lambda \in \Re, \ \ 0 \neq x \in \Re^n \ , \ \ S \in M_n(\Re), \quad |S| = I\right\}. \tag{3.5}$$

And this integrated view also extends to (3.3), the Perron root (3.2), because there is exactly one nonnegative real signature matrix, namely the identity matrix, and the Perron vector and the Perron root are known to be nonnegative. Refer to [4]. This leads us to the following unified definition of the quantities (3.3).

Definition 3.1.

$$\rho_0^s(\mathbb{A}) := \max\left\{|\lambda| : S\mathbb{A}x^{m-1} = \lambda x^{[m-1]}, \lambda \in \Re, 0 \neq x \in \Re^n, S \in M_n(\Re), |S| = I\right\}.$$

This quantity is called the sign-real spectral radius for real tensors. In 2015 Ding and Wei introduced sign-complex spectral radius, Which proves the accuracy of this definition [10].

We note that the index zero in ρ_0^s referred to Rohn's definition of the real spectral radius of a real matrix [22], we have also introduced for real tensors:

Definition 3.2. Let $\mathbb{A} \in \Re^{[m,n]}$, the real spectral radius is defined by

$$\rho_0(\mathbb{A}) := \max\left\{|\lambda| \ : \ \lambda \in \sigma(\mathbb{A}) \cap \Re\right\},$$

where $\rho_0(\mathbb{A}) = 0$ if $\mathbb{A}$ has no real eigenvalues.

It easily follows that

$$\rho_0^s(\mathbb{A}) = \max_{S\in\varphi} \rho_0(S\mathbb{A}). \tag{3.6}$$

Note. Let $\mathbb{A}$ be a real mth-order n-dimensional tensor and $S \in \varphi$ then by Definition (2.6) S$\mathbb{A}$ and $\mathbb{A}$S are also real mth-order n-dimensional tensors with entries

$$(S\mathbb{A})_{i\alpha_1} = \sum_{i_1=1}^{n} s_{ii_1} a_{i_1\alpha_1} \qquad \left(i \in [n], \ \alpha_1 \in [n]^{m-1}\right).$$

If $\alpha_1 = i_2 ... i_m$ then we have

$$(S\mathbb{A})_{ii_2...i_m} = s_{ii} a_{ii_2...i_m},$$

and

$$(\mathbb{A}S)_{i\alpha_1...\alpha_{m-1}} = \sum_{i_2,...,i_m=1}^{n} a_{ii_2...i_m} s_{i_2\alpha_1} ... s_{i_m\alpha_{m-1}} \qquad \left(i \in [n], \ \alpha_1, ..., \alpha_{m-1} \in [n]\right).$$

Since $S \in \varphi$ we have

$$(\mathbb{A}S)_{i\alpha_1...\alpha_{m-1}} = a_{i\alpha_1...\alpha_{m-1}} s_{\alpha_1\alpha_1} ... s_{\alpha_{m-1}\alpha_{m-1}}.$$

Theorem 3.3. *Let $\mathbb{A} \in \Re^{[m,n]}$, then $\rho_0^s(\alpha\mathbb{A}) = |\alpha| \ \rho_0^s(\mathbb{A})$, for all $\alpha \in \Re$.*

Proof. If $\alpha = 0$ the assertion is clear. Now let $\alpha \neq 0$.

$$\begin{aligned}
\rho_0^s\big(\alpha\mathbb{A}\big) = \max_{T\in\varphi}\rho_0\big(T\alpha\mathbb{A}\big) &= \max_{T\in\varphi}\max\left\{\left|\lambda\right| : \lambda \in \sigma\left(T\alpha\mathbb{A}\right)\cap\mathfrak{R}\right\} \\
&= \max_{T\in\varphi}\max\left\{\left|\lambda\right| : \lambda \in \sigma\left(\alpha T\mathbb{A}\right)\cap\mathfrak{R}\right\} \\
&= \max_{T\in\varphi}\max\left\{\left|\lambda\right| : \frac{\lambda}{\alpha} \in \sigma\left(T\mathbb{A}\right)\cap\mathfrak{R}\right\} \\
&= \max_{T\in\varphi}\max\left\{\left|\alpha\gamma\right| : \gamma \in \sigma\left(T\mathbb{A}\right)\cap\mathfrak{R}\right\} \\
&= \left|\alpha\right|\max_{T\in\varphi}\max\left\{\left|\gamma\right| : \gamma \in \sigma\left(T\mathbb{A}\right)\cap\mathfrak{R}\right\} = \left|\alpha\right|\rho_0^s\left(\mathbb{A}\right).
\end{aligned}$$

□

Theorem 3.4. *For upper or lower triangular tensor* $\rho_0^s\left(\mathbb{A}\right) = \max_i |a_{i\ldots i}|$.

Proof. Since $\mathbb{A}$ is triangular tensor therefore $S\mathbb{A}$ is also triangular tensor, thus by Theorem (2.16) the assertion is clear. □

Note. Since $TT = I$ and $T\mathbb{A} = T\left(\mathbb{A}T\right)T$ then we have $\sigma\left(T\mathbb{A}\right) = \sigma\left(\mathbb{A}T\right)$ therefore $\rho_0\left(T\mathbb{A}\right) = \rho_0\left(\mathbb{A}T\right)$ for any $T \in \varphi$. Hence we can use the definition below for sign-real spectral radius,

$$\rho_0^s\left(\mathbb{A}\right) = \max_{T\in\varphi}\rho_0\left(T\mathbb{A}\right) = \max_{T\in\varphi}\rho_0\left(\mathbb{A}T\right).$$

Note. Let $\mathbb{A}, \mathbb{B} \in \mathfrak{R}^{[m,n]}$. In general $\rho_0^s\left(\mathbb{A}\mathbb{B}\right) \neq \rho_0^s\left(\mathbb{B}\mathbb{A}\right)$ and $\rho_0^s\left(\mathbb{A}\right) \neq \rho_0^s\left(Q^T\mathbb{A}Q\right)$ for any orthogonal matrix Q.

Definition 3.5. A signature tensor $\mathbb{S}$ of order m and dimension n, is a diagonal tensor with diagonal entries $+1$ or -1. we denote the set of all signature tensors of order m and dimension n, by ϕ_n.

Theorem 3.6. *Let* $\mathbb{A}, \mathbb{B} \in \mathfrak{R}^{[m,n]}$ *and* $\mathbb{S}, \mathbb{T} \in \phi_n$. *For direct product of tensors we have*
(i). $\rho_0^s\left(\mathbb{A}\otimes\mathbb{S}\right) = \rho_0^s\left(\mathbb{S}\otimes\mathbb{A}\right) = \rho_0^s\left(\mathbb{S}\otimes\mathbb{A}\otimes\mathbb{T}\right) = \rho_0^s\left(\mathbb{A}\right)$.
(ii). $\rho_0^s\left(\mathbb{A}\otimes\mathbb{B}\right) = \rho_0^s\left(\mathbb{B}\otimes\mathbb{A}\right)$.
(iii). $\rho_0^s\left(\mathbb{A}\right)\rho_0^s\left(\mathbb{B}\right) \leq \rho_0^s\left(\mathbb{A}\otimes\mathbb{B}\right)$.

Proof. By definition there exists $\mathbb{T} \in \phi_{n^2}$ and $\mathbb{T}_1, \mathbb{T}_2 \in \phi_n$ such that $\mathbb{T} = \mathbb{T}_1 \otimes \mathbb{T}_2$ thus

$$\begin{aligned}
\rho_0^s\left(\mathbb{A}\otimes\mathbb{S}\right) &= \max_{\mathbb{T}\in\phi_{n^2}}\rho_0\Big(\mathbb{T}\left(\mathbb{A}\otimes\mathbb{S}\right)\Big) = \max_{\mathbb{T}_1,\mathbb{T}_2\in\phi_n}\rho_0\Big(\left(\mathbb{T}_1\otimes\mathbb{T}_2\right)\left(\mathbb{A}\otimes\mathbb{S}\right)\Big) \\
&= \max_{\mathbb{T}_1,\mathbb{T}_2\in\phi_n}\rho_0\Big(\left(\mathbb{T}_1\mathbb{A}\right)\otimes\left(\mathbb{T}_2\mathbb{S}\right)\Big) = \rho_0^s\left(\mathbb{A}\right).
\end{aligned}$$

Theorem (2.12) and $\sigma(\mathbb{T}_2\mathbb{S}) \subseteq \{\pm 1\}$ implies the last equality.
Also similarly we have $\rho_0^s(\mathbb{S} \otimes \mathbb{A}) = \rho_0^s(\mathbb{A})$. Therefore the third equation is clear.
(**ii**). There exists $T \in \phi_{n^2}$ such that

$$\begin{aligned}
\rho_0^s(\mathbb{A} \otimes \mathbb{B}) &= \rho_0\Big(\mathbb{T}(\mathbb{A} \otimes \mathbb{B})\Big) = \rho_0\Big((\mathbb{T}_1 \otimes \mathbb{T}_2)(\mathbb{A} \otimes \mathbb{B})\Big) \\
&= \rho_0\Big((\mathbb{T}_1\mathbb{A}) \otimes (\mathbb{T}_2\mathbb{B})\Big) = \rho_0\Big((\mathbb{T}_2\mathbb{B}) \otimes (\mathbb{T}_1\mathbb{A})\Big) \\
&= \rho_0\Big((\mathbb{T}_2 \otimes \mathbb{T}_1)(\mathbb{B} \otimes \mathbb{A})\Big) = \rho_0\Big(\mathbb{S}(\mathbb{B} \otimes \mathbb{A})\Big) \leq \rho_0^s(\mathbb{B} \otimes \mathbb{A}),
\end{aligned}$$

thus

$$\rho_0^s(\mathbb{A} \otimes \mathbb{B}) \leq \rho_0^s(\mathbb{B} \otimes \mathbb{A}).$$

If we just exchange $\mathbb{A}$, $\mathbb{B}$ we have

$$\rho_0^s(\mathbb{B} \otimes \mathbb{A}) \leq \rho_0^s(\mathbb{A} \otimes \mathbb{B}).$$

(**iii**). Let there exists $\mathbb{T}_1 \in \phi_n$ such that $\lambda \in \sigma(\mathbb{T}_1\mathbb{A})$, $\rho_0^s(\mathbb{A}) = |\lambda|$ and also there exists $\mathbb{T}_2 \in \phi_n$ such that $\mu \in \sigma(\mathbb{T}_2\mathbb{B})$ and $\rho_0^s(\mathbb{B}) = |\mu|$ then by Theorem (2.12) we have

$$\lambda\mu \in \sigma\Big((\mathbb{T}_1\mathbb{A}) \otimes (\mathbb{T}_2\mathbb{B})\Big),$$

$$\lambda\mu \leq \rho_0\Big((\mathbb{T}_1\mathbb{A}) \otimes (\mathbb{T}_2\mathbb{B})\Big) \leq \max_{\mathbb{S}\in\phi_{n^2}} \rho_0\Big(\mathbb{S}\Big((\mathbb{T}_1\mathbb{A}) \otimes (\mathbb{T}_2\mathbb{B})\Big)\Big) = \rho_0^s\Big((\mathbb{T}_1\mathbb{A}) \otimes (\mathbb{T}_2\mathbb{B})\Big).$$

Therefore

$$\Big|\lambda\mu\Big| \leq \rho_0^s\Big((\mathbb{T}_1\mathbb{A}) \otimes (\mathbb{T}_2\mathbb{B})\Big) = \rho_0^s\Big((\mathbb{T}_1 \otimes \mathbb{T}_2)(\mathbb{A} \otimes \mathbb{B})\Big) = \rho_0^s(\mathbb{A} \otimes \mathbb{B}).$$

This finishes the proof. □

4. Extension of the Perron Frobenius theory to real tensors

In this section we state and prove the main conclusion of this paper which is a generalization of Theorem (2.2) to the class of real tensors. First, we mention one widely used theorem in the following.

Lemma 4.1. *Suppose that $\mathbb{A} \in \mathfrak{R}^{[m,n]}$. Then for every $S \in \varphi$ there is $T \in \varphi$ such that $\left|S\,\mathbb{A}x^{m-1}\right| = (TS\,\mathbb{A})\,x^{m-1}$. In particular $\left|\mathbb{A}x^{m-1}\right| = (T\mathbb{A})\,x^{m-1}$.*

Proof. By definition we have

$$\begin{aligned}
\left(\left|S\mathbb{A}x^{m-1}\right|\right)_i &= \left|\sum_{i_2,\dots,i_m=1}^{n} (S\mathbb{A})_{ii_2\dots i_m} x_{i_2}\dots x_{i_m}\right| \\
&= \left|\sum_{i_2,\dots,i_m=1}^{n} S_{ii}\mathbb{A}_{ii_2\dots i_m} x_{i_2}\dots x_{i_m}\right| \\
&= \left|S_{ii}\sum_{i_2,\dots,i_m=1}^{n} \mathbb{A}_{ii_2\dots i_m} x_{i_2}\dots x_{i_m}\right| \\
&= \left|\left(\mathbb{A}x^{m-1}\right)_i\right|.
\end{aligned}$$

Also

$$\begin{aligned}
\left((TS\mathbb{A})\,x^{m-1}\right)_i &= \sum_{i_2,\dots,i_m=1}^{n} T_{ii}S_{ii}\mathbb{A}_{ii_2\dots i_m} x_{i_2}\dots x_{i_m} \\
&= (T_{ii})\,(S_{ii})\left(\sum_{i_2,\dots,i_m=1}^{n} \mathbb{A}_{ii_2\dots i_m} x_{i_2}\dots x_{i_m}\right) \\
&= T_{ii}S_{ii}\left(\mathbb{A}x^{m-1}\right)_i.
\end{aligned}$$

Therefore we can find $T \in \varphi$ such that the assertion is true. □

Theorem 4.2. *Suppose that $\mathbb{A} \in \mathfrak{R}^{[m,n]}$. Then for every $T \in \varphi$ there exists some $S \in \varphi$ such that $S\mathbb{A}$ has an eigenvector in that orthant corresponding to a real nonnegative eigenvalue, i.e.,*

$$\forall\, T \in \varphi\ \exists\, S \in \varphi\ \ \exists\, 0 \neq x \in \mathfrak{R}^n :$$

$$x \geq 0 \ \textit{and}\ S\mathbb{A}(Tx)^{m-1} = \lambda(Tx)^{[m-1]}\ \textit{for some}\ \ 0 \leq \lambda \in \mathfrak{R}.$$

Proof. Let $T \in \varphi$ be given. If there exists $0 \neq x \in \mathfrak{R}^n$, $x \geq 0$ such that $\mathbb{A}(Tx)^{m-1} = 0$ then $\lambda = 0$ and $S = I$. This finishes the proof. (Note if $0 \neq x$ then $Tx \neq 0$.)
Now we assume that $\mathbb{A}(Tx)^{m-1} \neq 0$ for all nonzero $x \geq 0$. We define

$$E = \left\{ x = (x_1, x_2, ..., x_n) \in \mathfrak{R}^n \ :\ x_i \geq 0\ ,\ 1 \leq i \leq n\ ,\ \sum_{i=1}^{n} x_i = 1 \right\},$$

which is a nonempty, compact, and convex set. Now we define the function $f : E \to E$ such that

$$(f(x))_i = \frac{\left(\left|\mathbb{A}(Tx)^{m-1}\right|\right)_i^{\frac{1}{m-1}}}{\sum_{j=1}^{n}\left(\left(\left|\mathbb{A}(Tx)^{m-1}\right|\right)_j^{\frac{1}{m-1}}\right)}, \qquad , \qquad 1 \leq i \leq n,$$

that is well defined on E and continuous. By Brouwer's fixed-point theorem, there exists $x_0 \in E$ such that $f(x_0) = x_0$. Therefore

$$\left(\left|\mathbb{A}(Tx_0)^{m-1}\right|\right)_i^{\frac{1}{m-1}} = \sum_{j=1}^{n}\left(\left(\left|\mathbb{A}(Tx_0)^{m-1}\right|\right)_j^{\frac{1}{m-1}}\right)(x_0)_i, 1 \le i \le n,$$

$$T_{ii}\left(\left|\mathbb{A}(Tx_0)^{m-1}\right|\right)_i^{\frac{1}{m-1}} = \sum_{j=1}^{n}\left(\left(\left|\mathbb{A}(Tx_0)^{m-1}\right|\right)_j^{\frac{1}{m-1}}\right)T_{ii}(x_0)_i, 1 \le i \le n,$$

$$T_{ii}^{m-1}\left(\left|\mathbb{A}(Tx_0)^{m-1}\right|\right)_i = \left(\sum_{j=1}^{n}\left(\left(\left|\mathbb{A}(Tx_0)^{m-1}\right|\right)_j^{\frac{1}{m-1}}\right)\right)^{m-1}(Tx_0)_i^{m-1}, 1 \le i \le n,$$

It follows that

$$T^{m-1}\left|\mathbb{A}(Tx_0)^{m-1}\right| = \lambda_0(Tx_0)^{[m-1]}.$$

By previous lemma, there exists $S \in \varphi$ such that

$$S\,\mathbb{A}(Tx_0)^{m-1} = \lambda_0(Tx_0)^{[m-1]},$$

where $\lambda_0 = \left(\sum_{j=1}^{n}\left(\left(\left|\mathbb{A}(Tx_0)^{m-1}\right|\right)_j^{\frac{1}{m-1}}\right)\right)^{m-1}$. $\square$

This theorem shows that there is always some $S \in \varphi$ such that S$\mathbb{A}$ has a real eigenvalue, which means that $\rho_0^s(\mathbb{A})$ is always equal to a real eigenvalue of some $S\mathbb{A}$ and also this theorem shows that for every $\mathbb{A} \in \Re^{[m,n]}$ there exists $S \in \varphi$ such that $S\mathbb{A}$ has an H-eigenvalue.

Theorem 4.3. *Suppose that* $\mathbb{A} \in \Re^{[m,n]}$, $0 \neq x \in \Re^n$ *and* $0 \le r \in \Re$. *Then*

$$\left|\mathbb{A}x\right| \ge r\left|x^{[m-1]}\right| \quad \Rightarrow \quad \rho_0^s(\mathbb{A}) \ge r.$$

Proof. Define diagonal D by

$$D_{ii} = \begin{cases} \dfrac{r\left|x_i^{m-1}\right|}{(\mathbb{A}x)_i} & (\mathbb{A}x)_i \neq 0 \\ 0 & otherwise \end{cases}$$

If $(\mathbb{A}x)_i = 0$ then $|(\mathbb{A}x)_i| \ge r\left|x^{[m-1]}\right|$ implies $rx_i^{m-1} = 0$. Thus $D\ |\mathbb{A}x| = r\left|x^{[m-1]}\right|$ where $0 \le D_{ii} \le 1$. There are signture matrices S_1, S_2 such that $S_1\mathbb{A}x = |\mathbb{A}x|$ and $x^{[m-1]} = S_2\left|x^{[m-1]}\right|$. Now $DS_1\mathbb{A}x = rS_2\,x^{[m-1]}$ hence $S_2DS_1\mathbb{A}x = rx^{[m-1]}$. For $D_1 = S_2DS_1$ it follows $D_1\mathbb{A}x = rx^{[m-1]}$. Now $0 \neq x$ yield

$$\det\left(r\mathbb{I} - D_1\mathbb{A}\right) = 0, \qquad -1 \le (D_1)_{ii} \le 1. \tag{4.1}$$

We construct a signature matrix S with $\det(r\mathbb{I} - S\mathbb{A}) \leq 0$. For fixed index i, $1 \leq i \leq n$, define $D_2 = D_2(\alpha)$ by

$$(D_2)_{jj} := \begin{cases} (D_1)_{jj} & j \neq i \\ \\ \alpha & j = i \end{cases}$$

By using (4.1) and especially $-1 \leq (D_1)_{ii} \leq 1$ it follows that

$$\det\left(r\mathbb{I} - D_2\mathbb{A}\right) \leq 0,$$

by choosing appropriate $\alpha = 1$ or $\alpha = -1$. Repeating this argument for all indices i, $1 \leq i \leq n$, we obtain a signature matrix S such that $\det(r\mathbb{I} - S\mathbb{A}) \leq 0$.
This is the value of the characteristic polynomial $P(t) = \det(t\mathbb{I} - S\mathbb{A})$ of $S\mathbb{A}$ at the nonnegative point $t = r$. Since the characteristic polynomial $P(t)$ is a one variable monic polynomial, thus it tends to $+\infty$ as t tends to $+\infty$. Thus $P(t)$ must cross the real axis for some $s \geq r$. Now

$$r \leq s \leq \rho_0(S\mathbb{A}) \leq \rho_0^s(\mathbb{A}).$$

□

Similar to the Collatz Wielandt theorem, Chang et al. extended the minimax characterization of the spectral radius $\rho(\mathbb{A})$ in [4]. Now we generalize the maxmin theorem of nonnegative tensors to the class of real tensors.

Theorem 4.4. *Let $\mathbb{A} \in \mathfrak{R}^{[m,n]}$ then*

$$\rho_0^s(\mathbb{A}) = \max_{x \in \mathfrak{R}^n \setminus \{0\}} \min_{x_i \neq 0} \left| \frac{\left(\mathbb{A}x^{m-1}\right)_i}{x_i^{m-1}} \right|.$$

Proof. Let S be a signature matrix such that $S\mathbb{A}y^{m-1} = \lambda y^{[m-1]}$, $0 \neq y \in \mathfrak{R}^n$ with $|\lambda| = \rho_0^s(\mathbb{A})$. Then

$$\min_{y_i \neq 0} \left| \frac{\left(\mathbb{A}y^{m-1}\right)}{y_i^{m-1}} \right| = |\lambda| = \rho_0^s(\mathbb{A})$$

and

$$\rho_0^s(\mathbb{A}) \leq \max_{x \in \mathfrak{R}^n \setminus \{0\}} \min_{x_i \neq 0} \left| \frac{\left(\mathbb{A}x^{m-1}\right)}{x_i^{m-1}} \right|.$$

Conversly, Theorem 4.3 implies for any nonzero vector x

$$\rho_0^s(\mathbb{A}) \geq \min_{x_i \neq 0} \left| \frac{\left(\mathbb{A}x^{m-1}\right)}{x_i^{m-1}} \right|.$$

The theorem is proved. □

Note. If $\mathbb{A} \in \mathfrak{R}^{[m,n]}$ and $\mathbb{A} \geq 0$ then $\rho_0^s(\mathbb{A}) = \rho(\mathbb{A})$.

Following are more properties of the sign-real spectral radius, showing similarities to Perron Frobenius theory for nonnegative tensors.

Lemma 4.5. *Suppose that* $\mathbb{A}, \mathbb{B} \in \mathfrak{R}^{[m,n]}$ *be diagonal, and* D *be an* $n \times n$ *diagonal matrix. Then*
(i). $\det(\mathbb{A}\mathbb{B}) = \det(\mathbb{A})\det(\mathbb{B})$.
(ii). $\det(D\mathbb{A}) = \det(D\mathbb{I})\det(\mathbb{A})$, *where* $\mathbb{I}$ *is a unit tensor.*

Proof. By Theorem (2.16) and Lemma (2.9) the proof is clear. □

Theorem 4.6. *Suppose that* $\mathbb{A}$ *be a real mth-order n-dimensional tensor. Then there are* $S, T \in \varphi$ *and* $0 \neq x \in \mathfrak{R}^n$ *with* $x \geq 0$ *and*

$$(S\mathbb{A}T)\, x^{m-1} = \rho_0^s(\mathbb{A})\, x^{[m-1]}.$$

Proof. Let $\lambda \in \sigma(S\mathbb{A})$ then there is $0 \neq x \in \mathfrak{R}^n$ such that $S\mathbb{A}x^{m-1} = \lambda x^{[m-1]}$ and $|\lambda| = \rho_0^s(\mathbb{A})$. Now $\left|S\mathbb{A}x^{m-1}\right| = \rho_0^s(\mathbb{A})\left|x^{[m-1]}\right|$. By Theorem (4.2) there exists $T_1 \in \varphi$ such that

$$(T_1 S\mathbb{A})\, x^{m-1} = \rho_0^s(\mathbb{A})\left|x^{[m-1]}\right|,$$

by using the properties of signature matrix we have

$$(T_1 S\mathbb{A})\, S_2 S_2 x^{m-1} = \rho_0^s(\mathbb{A})\left|x^{[m-1]}\right|,$$

by Theorem (4.2)

$$(T_1 S\mathbb{A})\, S_2 T_2 \left|x^{m-1}\right| = \rho_0^s(\mathbb{A})\left|x^{[m-1]}\right|.$$

Define $S := T_1 S$ and $T := S_2 T_2$. This finishes the proof. □

Theorem 4.7. *If* $\rho_0^s(\mathbb{A}) = 0$ *then* $\det(\mathbb{A}) = 0$.

Proof. By Theorem (4.6) there exists $S \in \varphi$ such that

$$\det\left(\rho_0^s(\mathbb{A})\,\mathbb{I} - S\mathbb{A}\right) = 0.$$

By Lemma (4.5) we have $0 = \det(-S\mathbb{A}) = \det(-S\mathbb{I})\det(\mathbb{A})$. Since $\det(-S\mathbb{I}) \neq 0$ therefore $\det(\mathbb{A}) = 0$. □

Theorem 4.8. *For every* $\mathbb{A} \in \mathfrak{R}^{[m,n]}$ *and diagonal matrix* D *we have*

$$\rho_0^s(\mathbb{A}) = \max\left\{\rho_0(D\mathbb{A}) \;:\; |D| \leq I\right\}.$$

Proof. Since every $S \in \varphi$ is a diagonal matrix D such that $|D| \leq I$ thus for every $S \in \varphi$ we have

$$\rho_0(S\mathbb{A}) \leq \max\left\{\rho_0(D\mathbb{A}) \;:\; |D| \leq I\right\},$$

therefore

$$\rho_0^s(\mathbb{A}) \leq \max\left\{\rho_0(D\mathbb{A}) \;:\; |D| \leq I\right\}.$$

On the other hand since there exists $T \in \varphi$ such that $|T| \leq I$ and $\rho_0^s(\mathbb{A}) = \rho_0(T\mathbb{A})$ thus the proof is complete. □

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