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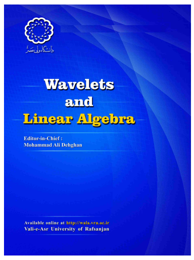

On the two-wavelet localization operators on homogeneous spaces with relatively invariant measures

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Abstract

In the present paper, we introduce the two-wavelet localization operator for the square integrable representation of a homogeneous space with respect to a relatively invariant measure. We investigate some properties of the two-wavelet localization operator and show that it is a compact operator and is contained in a Schatten p-class.

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1. Introduction and Preliminaries

Two-wavelet localization operators in the setting of homogeneous spaces with respect to a left invariant measure are studied in [11, 2]. We have studied the localization operators in the setting of homogeneous spaces with an admissible wavelet in [5]. In this manuscript we continue our investigation on localization operators with two admissible wavelets in a completely different and more general approach, by considering a relatively invariant measure on the homogeneous space G/H. A reason for the extension of the results from the one wavelet case to the two wavelet case comes from an extra degree of flexibility in signal analysis and imaging when the localization operators are used as time-varying filters (see [2, 3]).
To make the point clear, let us first review some basic concepts of strongly quasi invariant measures on homogeneous spaces (for more details see [8, 9, 7]).

Let G be a locally compact group and H be a closed subgroup of G. Consider G/H as a homogeneous space on which G acts from the left. Let μ be a Radon measure acting in G/H. A strongly quasi invariant Borel measure μ on G/H is translation-continuous if there exists a positive real valued continuous function λ on $G \times G/H$ such that

$$d\mu_g(kH) = \lambda(g, kH)d\mu(kH),$$

for all $g, k \in G$. If the functions $\lambda(g, .)$ reduce to constants, then μ is called a relatively invariant measure under G and if $\lambda(g, .) = 1$, for all $g \in G$, the measure μ is said to be G-invariant. A rho-function for the pair (G, H) is defined to be a continuous function $\rho : G \to (0, \infty)$ which satisfies

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g) \quad (g \in G, h \in H),$$

where Δ_G, Δ_H are the modular functions on G and H, respectively. It is well known that (see [8]), any pair (G, H) admits a rho-function and for each rho-function ρ there is a strongly quasi invariant measure μ on G/H such that

$$\frac{d\mu_g}{d\mu}(kH) = \frac{\rho(gk)}{\rho(k)} \quad (g, k \in G).$$

As has been shown in [8], every strongly quasi invariant measure on G/H, arises from a rho-function and all such measures are strongly equivalent. That is, μ and μ' are strongly quasi invariant measures on G/H, then $\frac{d\mu'}{d\mu}$ is a positive continuous function.

The paper is organized as follows. In section 2, we introduce a two-wavelet localization operator on a homogeneous space with a relatively invariant measure. We show that it is a bounded linear operator. Section 3 investigates the compactness and the Schatten p-class properties of the two-wavelet localization operators. The section is concluded by examples supporting our arguments.

2. Boundedness of Two- Wavelet Localization Operators on G/H

In this section we define the localization operators for pairs of admissible wavelets in the setting of homogeneous spaces with relatively invariant measures and show that they are bounded operators. For this, we need to review some basic concepts from [4, 6] concerning the square integrable representations in the case of homogeneous spaces with relatively invariant measures.

A continuous unitary representation of a homogeneous space G/H is a map σ from G/H into the group $U(\mathcal{H})$ of all unitary operators on some nonzero Hilbert space $\mathcal{H}$, for which the function $gH \mapsto \langle \sigma(gH)x, y\rangle$ is continuous , for each $x, y \in \mathcal{H}$ and

$$\sigma(gkH) = \sigma(gH)\sigma(kH), \quad \sigma(g^{-1}H) = \sigma(gH)^*,$$

for each $g, k \in G$. Moreover, a closed subspace M of $\mathcal{H}$ is said to be invariant with respect to σ if $\sigma(gH)M \subseteq M$, for all $g \in G$. A continuous unitary representation σ is said to be irreducible if the only invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$ (in the sequel we always mean by a representation, a continuous unitary representation). An irreducible representation σ of G/H on $\mathcal{H}$ is said to be *square integrable* if there exists a nonzero element $\zeta \in \mathcal{H}$ such that

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} \left|\langle \zeta, \sigma(gH)\zeta\rangle\right|^2 d\mu(gH) < \infty, \tag{2.1}$$

where μ is a relatively invariant measure on G/H which arises from a rho function $\rho : G \to (0, \infty)$. If ζ satisfies (2.1), it is called an *admissible vector*. An admissible vector $\zeta \in \mathcal{H}$ is said to be *admissible wavelet* if $\|\zeta\| = 1$. In this case, we define the wavelet constant c_ζ as

$$c_\zeta := \int_{G/H} \frac{\rho(e)}{\rho(g)} \left|\langle \zeta, \sigma(gH)\zeta\rangle\right|^2 d\mu(gH). \tag{2.2}$$

We call c_ζ the *wavelet constant* associated to the admissible wavelet ζ. For a given representation σ, two vector $\zeta, x \in \mathcal{H}$ and $g \in G$ define the linear operator $W_\zeta : \mathcal{H} \to C(G/H)$ by

$$(W_\zeta x)(gH) = \frac{1}{\sqrt{c_\zeta}} \left(\frac{\rho(e)}{\rho(g)}\right)^{1/2} \langle x, \sigma(gH)\zeta\rangle.$$

The linear operator W_ζ is called the continuous wavelet transform and it is a bounded linear operator from $\mathcal{H}$ into $L^2(G/H)$ when σ is a square integrable representation of G/H on $\mathcal{H}$ and ζ is an admissible wavelet for σ. Note that we assume H is a compact subgroup of G.
The reconstruction formula and orthogonality relations for square integrable representation σ of homogeneous spaces G/H with relatively invariant measure have been studied in [4, 6]. For the reader's convenience we state them here which are used in our results.

Theorem 2.1. *(reconstruction formula) Let σ be a square integrable representation of G/H on $\mathcal{H}$. If ζ is an admissible wavelet for σ, then*

$$\langle x, y\rangle = \frac{1}{c_\zeta} \int_{G/H} \frac{\rho(e)}{\rho(g)} \langle x, \sigma(gH)\zeta\rangle \, \langle \sigma(gH)\zeta, y\rangle \, d\mu(gH), \tag{2.3}$$

where c_ζ is as in (2.2) and μ is a relatively invariant measure that arises from a rho-function ρ.

Theorem 2.2. *(Orthogonality Relations) Let σ be a square integrable representation of G/H on $\mathcal{H}$ and ζ, ξ be two admissible wavelets for σ. Then*
for all x ,y in $\mathcal{H}$,

$$\frac{1}{c_{\zeta,\xi}} \int_{G/H} \frac{\rho(e)}{\rho(g)} \langle x, \sigma(gH)\zeta\rangle \, \langle \sigma(gH)\xi, y\rangle \, d\mu(gH) = \langle x, y\rangle, \tag{2.4}$$

where

$$c_{\zeta,\xi} = \int_{G/H} \frac{\rho(e)}{\rho(g)} \langle \zeta, \sigma(gH)\zeta\rangle \, \langle \sigma(gH)\xi, \zeta\rangle \, d\mu(gH), \tag{2.5}$$

in which $c_{\zeta,\xi}$ is called two-wavelet constant.

Now, we are ready to define two-wavelet localization operators in the setting of homogeneous spaces with a relatively invariant measure and we establish their boundedness property.

Definition 2.3. Let σ be a square integrable representation of G/H and ζ, ξ be two admissible wavelets for σ with respect to a relatively invariant measure μ on G/H. The operator $L_{\zeta,\xi,\psi} : \mathcal{H} \to \mathcal{H}$ defined as follows:

$$\langle L_{\zeta,\xi,\psi} x, y\rangle = \frac{1}{c_{\zeta,\xi}} \int_{G/H} \frac{\rho(e)}{\rho(g)} \psi(gH) \langle x, \sigma(gH)\zeta\rangle \, \langle \sigma(gH)\xi, y\rangle \, d\mu(gH), \tag{2.6}$$

for all $\psi \in L^p(G/H)$, $x, y \in \mathcal{H}$ and $c_{\zeta,\xi}$ is two-wavelet constant defined as in (2.5). The linear operator $L_{\zeta,\xi,\psi}$ is called two-wavelet localization operator.

We intend to show that $L_{\zeta,\xi,\psi}$ is a bounded linear operator for $\psi \in L^p(G/H)$, $1 \le p \le \infty$. But first, we show that the localization operator $L_{\zeta,\xi,\psi}$ is bounded, for $\psi \in L^\infty(G/H)$.

Proposition 2.4. *Let $\psi \in L^\infty(G/H)$. Then $L_{\zeta,\xi,\psi}$ is a bounded linear operator and*

$$\|L_{\zeta,\xi,\psi}\| \le \frac{(c_\zeta c_\xi)^{1/2}}{|c_{\zeta,\xi}|} \|\psi\|_\infty,$$

in which ζ, ξ are two admissible wavelets.

Proof. Using the reconstruction formula (2.3) and the Schwarz inequality we get,

$$\Big|\langle L_{\zeta,\xi,\psi} x, y\rangle\Big|$$

$$\le \frac{1}{|c_{\zeta,\xi}|} \int_{G/H} \frac{\rho(e)}{\rho(g)} \, \Big|\psi(gH)\Big| \, \Big|\langle x, \sigma(gH)\zeta\rangle\Big| \, \Big|\langle \sigma(gH)\xi, y\rangle\Big| \, d\mu(gH)$$

$$\le \frac{1}{|c_{\zeta,\xi}|} \|\psi\|_\infty \Big(\int_{G/H} \frac{\rho(e)}{\rho(g)} \, \Big|\langle x, \sigma(gH)\zeta\rangle\Big|^2 d\mu(gH)\Big)^{1/2}$$

$$\Big(\int_{G/H} \frac{\rho(e)}{\rho(g)} \, \Big|\langle \sigma(gH)\xi, y\rangle\Big|^2 d\mu(gH)\Big)^{1/2}$$

$$\le \frac{1}{|c_{\zeta,\xi}|} c_\zeta^{1/2} c_\xi^{1/2} \|x\| \, \|y\| \, \|\psi\|_\infty,$$

for all $x, y \in \mathcal{H}$. Then $\|L_{\zeta,\xi,\psi}\| \le \frac{(c_\zeta c_\xi)^{1/2}}{|c_{\zeta,\xi}|} \|\psi\|_\infty$. □

Now, let $\psi \in L^1(G/H)$ in which G/H is equipped with a G-invariant measure μ'. Note that, since H is compact, G/H admits such a G-invariant measure.

Proposition 2.5. *Let $\psi \in L^1(G/H)$. Then $L_{\zeta,\xi,\psi}$ is a bounded linear operator and*

$$||L_{\zeta,\xi,\psi}|| \leq \frac{\rho(e)}{|c_{\zeta,\xi}|}||\psi||_1.$$

Proof. Consider G/H with a G-invariant measure μ' which arises from the rho-function $\rho' \equiv 1$. Since μ, μ' are strongly equivalent, there exists a positive function τ on G/H such that

$$\frac{d\mu}{d\mu'} = \tau, \ \ \rho(g) = \tau(gH),$$

where μ is a relatively invariant measure which arises from ρ. Thus

$$\begin{aligned}
&\Big|\langle L_{\zeta,\xi,\psi}x, y\rangle\Big| \\
&\leq \frac{1}{|c_{\zeta,\xi}|}\int_{G/H}\frac{\rho(e)}{\rho(g)}\Big|\psi(gH)\Big| \ \Big|\langle x, \sigma(gH)\zeta\rangle\Big| \ \Big|\langle\sigma(gH)\xi, y\rangle\Big| \, d\mu(gH) \\
&\leq \frac{1}{|c_{\zeta,\xi}|}\int_{G/H}\frac{\rho(e)}{\tau(gH)}\Big|\psi(gH)\Big| \ \Big|\langle x, \sigma(gH)\zeta\rangle\Big| \ \Big|\langle\sigma(gH)\xi, y\rangle\Big|\tau(gH)d\mu'(gH) \\
&\leq \frac{1}{|c_{\zeta,\xi}|}\int_{G/H}\rho(e)\Big\|x\Big\| \ \Big\|y\Big\| \ \Big|\psi(gH)\Big|d\mu'(gH) \\
&\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\Big\|\psi\Big\|_1\Big\|x\Big\| \ \Big\|y\Big\|.
\end{aligned}$$

where $||\psi||_1 = \int_{G/H}|\psi(gH)|d\mu'(gH)$. Then $||L_{\zeta,\xi,\psi}|| \leq \frac{\rho(e)}{|c_{\zeta,\xi}|}||\psi||_1$. □

Finally, we show that if $\psi \in L^p(G/H)$, $1 < p < \infty$, then $L_{\zeta,\xi,\psi}$ is a bounded linear operator.

Theorem 2.6. *Let $\psi \in L^p(G/H)$, $1 < p < \infty$. Then there exists a unique bounded linear operator $L_{\zeta,\xi,\psi} : \mathcal{H} \to \mathcal{H}$ such that*

$$||L_{\zeta,\xi,\psi}|| \leq \frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}\Big(c_\zeta c_\xi\Big)^{1/2(1-1/p)}||\psi||_p, \tag{2.7}$$

where $L_{\zeta,\xi,\psi}$ is given for a simple function ψ on G/H for which

$$\mu\Big(\Big\{gH \in G/H; \ \ \psi(gH) \neq 0\Big\}\Big) < \infty. \tag{2.8}$$

Proof. Let $\Gamma : \mathcal{H} \to L^2(\mathbb{R}^n)$ be a unitary operator and $\psi \in L^1(G/H)$. Then the linear operator $\tilde{L}_{\psi,\zeta,\xi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$\tilde{L}_{\psi,\zeta,\xi} = \Gamma L_{\psi,\zeta,\xi}\Gamma^{-1}, \tag{2.9}$$

is bounded and $\|\tilde{L}_{\psi,\zeta,\xi}\| \leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\|\psi\|_1$. If $\psi \in L^\infty(G/H)$, then the linear operator $\tilde{L}_{\psi,\zeta,\xi}$ on $L^2(\mathbb{R}^n)$ defined as (2.9) is bounded and $\|\tilde{L}_{\psi,\zeta,\xi}\| \leq \frac{(c_\zeta c_\xi)^{1/2}}{|c_{\zeta,\xi}|}\|\psi\|_\infty$.
Denote by $\mathfrak{A}$ the set of all simple functions ψ on G/H which satisfy (2.8). Let $g \in L^2(\mathbb{R}^n)$ and Φ_g be a linear transformation from $\mathfrak{A}$ into the set of all Lebesgue measurable function on $\mathbb{R}^n$ defined as $\Phi_g(\psi) = \tilde{L}_{\psi,\zeta,\xi}(g)$. Then for all $\psi \in L^1(G/H)$

$$\left\|\Phi_g(\psi)\right\|_2 = \left\|\tilde{L}_{\psi,\zeta,\xi}(g)\right\|_2 \leq \left\|\tilde{L}_{\psi,\zeta,\xi}\right\| \left\|g\right\|_2 \leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\left\|\psi\right\|_1 \left\|g\right\|_2.$$

Similarly for all $\psi \in L^\infty(G/H)$,

$$\left\|\Phi_g(\psi)\right\|_2 \leq \frac{\left(c_\zeta c_\xi\right)^{1/2}}{|c_{\zeta,\xi}|}\left\|\psi\right\|_\infty \left\|g\right\|_2.$$

By the Riesz Thorin Interpolation Theorem we get,

$$\left\|\tilde{L}_{\psi,\zeta,\xi}(g)\right\|_2 = \left\|\Phi_g(\psi)\right\|_2 \leq \frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}\left(c_\zeta c_\xi\right)^{1/2(1-1/p)}\left\|\psi\right\|_p \left\|g\right\|_2.$$

So,

$$\left\|\tilde{L}_{\psi,\zeta,\xi}\right\| \leq \frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}\left(c_\zeta c_\xi\right)^{1/2(1-1/p)}\left\|\psi\right\|_p.$$

for each $\psi \in \mathfrak{A}$.
Now, let $\psi \in L^p(G/H)$, for all $1 < p < \infty$. Then there exists a sequence $\{\psi_k\}_{k=1}^\infty$ of functions in $\mathfrak{A}$ such that ψ_k is convergent to ψ in $L^p(G/H)$ as $k \to \infty$. Also, $\{\tilde{L}_{\psi_k,\zeta,\xi}\}$ is a Cauchy sequence in $B(L^2(G/H))$. Indeed,

$$\left\|\tilde{L}_{\psi_k,\zeta,\xi} - \tilde{L}_{\psi_m,\zeta,\xi}\right\| \leq \frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}\left(c_\zeta c_\xi\right)^{1/2(1-1/p)}\left\|\psi_k - \psi_m\right\|_p \to 0.$$

By the completeness of $B(L^2(\mathbb{R}^n))$, there exists a bounded linear operator $\tilde{L}_{\psi,\zeta,\xi}$ on $L^2(\mathbb{R}^n)$ such that $\tilde{L}_{\psi_k,\zeta,\xi}$ converges to $\tilde{L}_{\psi,\zeta,\xi}$ in $B(L^2(\mathbb{R}^n))$, in which

$$\left\|\tilde{L}_{\psi,\zeta,\xi}\right\| \leq \frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}\left(c_\zeta c_\xi\right)^{1/2(1-1/p)}\left\|\psi\right\|_p.$$

Thus the linear operator $L_{\psi,\zeta,\xi}$ is bounded, where $L_{\psi,\zeta,\xi} = \Gamma^{-1}\tilde{L}_{\psi,\zeta,\xi}\Gamma$, and

$$\left\|L_{\psi,\zeta,\xi}\right\| \leq \frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}\left(c_\zeta c_\xi\right)^{1/2(1-1/p)}\left\|\psi\right\|_p. \tag{2.10}$$

For the proof of uniqueness, let $\psi \in L^p(G/H)$, $1 < p < \infty$, and suppose that $P_{\psi,\zeta,\xi}$ is another bounded linear operator satisfying (2.10). Let $\Theta : L^p(G/H) \to B(\mathcal{H})$ be the linear operator defined by

$$\Theta(\psi) = L_{\psi,\zeta,\xi} - P_{\psi,\zeta,\xi}, \quad \psi \in L^p(G/H).$$

Then by (2.10),

$$\left\|\Theta(\psi)\right\| \le 2\frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}\left(c_\zeta c_\xi\right)^{1/2(1-1/p)}\left\|\psi\right\|_p.$$

Moreover $\Theta(\psi)$ is equal to the zero operator on $\mathcal{H}$ for all $\psi \in \mathfrak{A}$. Thus, $\Theta : L^p(G/H) \to B(\mathcal{H})$ is a bounded linear operator that is equal to zero on the dense subspace $\mathfrak{A}$ of $L^p(G/H)$. Therefore $L_{\psi,\zeta,\xi} = P_{\psi,\zeta,\xi}$ for all $\psi \in L^p(G/H)$. □

3. Compactness and Schatten *p*-class Properties of $L_{\psi,\zeta,\xi}$

In this section we show that the two-wavelet localization operators on homogeneous spaces with relatively invariant measure defined in (2.6) are compact and they are in Schatten p-class. For the reader's convenience we introduce basic preliminaries on Schatten p-class and for more details see [10, 11] .

Let $T : \mathcal{H} \to \mathcal{H}$ be a compact operator. Then the linear operator $|T| : \mathcal{H} \to \mathcal{H}$ is positive and compact. Assume that $\{\xi_n, n = 1, 2, ...\}$ is an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors of $|T|$ and $s_n(T)$ is the eigenvalue of $|T|$ corresponding to the eigenvector $\{\xi_n, n = 1, 2, ...\}$. The eigenvalue $s_n(T), n = 1, 2, ...$ is called the singular value of T. A compact operator T is in Schatten p-class $S_p, 1 \le p < \infty$, if $\sum_{n=1}^{\infty}(s_n(T))^p < \infty$. It can be shown that $S_p, 1 \le p < \infty$ is a Banach space in which the norm $\|,\|_{S_p}$ is defined by

$$\left\|T\right\|_{S_p} = \Big(\sum_{n=1}^{\infty}(s_n(T))^p\Big)^{1/p}, \quad T \in S_p.$$

Let S_∞ be the C^*-algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$. Then the norm $\|,\|_{S_\infty}$ is the same as the operator norm in $B(\mathcal{H})$. The Banach spaces S_1 and S_2 are known as the trace and the Hilbert-Schmidt classes, respectively. Note that S_2 is a Hilbert space. It is worthwhile to note that if the operator T on $\mathcal{H}$ is a compact operator such that for all orthonormal sets $\{\xi_n\}_{n=1}^{\infty}$ and $\{\zeta_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ $\sum_{n=1}^{\infty}|\langle T\xi_n, \zeta_n\rangle| < \infty$, then T is in S_1. For any $T \in S_1$ and any orthonormal basis $\{\zeta_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ we write

$$tr(T) = \sum_{n=1}^{\infty}\langle T\zeta_n, \zeta_n\rangle,$$

which is called the trace of T. Moreover, if T is a bounded operator such that $\sum_{n=1}^{\infty}\|T\xi_n\|^2 < \infty$, for all orthonormal bases $\{\xi_n\}_{n=1}^{\infty}$ for $\mathcal{H}$, then T is in Hilbert Schmidt class S_2 (see [11, Section 2]). Throughout this section $L^p(G/H)$ denotes the Lebesgue space $L^p(G/H, \mu')$, where μ' is a G-invariant measure on G/H.

The following theorem show that the two-wavelet localization operator $L_{\psi,\zeta,\xi}$, for two admissible wavelets ζ, ξ and $\psi \in L^p(G/H), 1 \le p < \infty$ is compact.

Theorem 3.1. *For $\psi \in L^p(G/H), 1 \le p < \infty$, the two-wavelet localization operator $L_{\psi,\zeta,\xi}$ is compact.*

Proof. Let $\psi \in L^p(G/H, \mu')$. There exists $\psi_n \in C_c(G/H)$ such that $\|\psi_n - \psi\|_p \to 0$. Let $\{\zeta_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then by Fubini's theorem and the Schwarz inequality, we have

$$\sum_{k=1}^{\infty}\left|\langle L_{\psi_n,\zeta,\xi}\zeta_k, L_{\psi_n,\zeta,\xi}\zeta_k\rangle\right|$$

$$\leq \frac{1}{|c_{\zeta,\xi}|}\sum_{k=1}^{\infty}\int_{G/H}\frac{\rho(e)}{\rho(g)}\left|\psi_n(gH)\right|\left|\langle\zeta_k,\sigma(gH)\zeta\rangle\right|\ \left|\langle\sigma(gH)\xi, L_{\psi_n,\zeta,\xi}\zeta_k\rangle\right|d\mu(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\int_{G/H}\left|\psi_n(gH)\right|\Big(\sum_{k=1}^{\infty}\left|\langle\zeta_k,\sigma(gH)\zeta\rangle\right|^2\Big)^{1/2}\Big(\sum_{k=1}^{\infty}\left|\langle L^*_{\psi,\zeta,\xi}\sigma(gH)\xi,\zeta_k\rangle\right|^2\Big)^{1/2}d\mu'(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\left\|L^*_{\psi,\zeta,\xi}\right\|\ \left\|\psi_n\right\|_1 < \infty.$$

Thus $L_{\psi_n,\zeta,\xi}$ is in S_2 and it implies that $L_{\psi_n,\zeta,\xi}$ is compact. Since $\|L_{\psi,\zeta,\xi}-L_{\psi_n,\zeta,\xi}\| \leq \frac{\rho(e)^{1/p}}{|c_{\zeta,\xi}|}(c_\zeta c_\xi)^{1/2(1-1/p)}\|\psi-\psi_n\|_p$. Then the localization operator $L_{\psi,\zeta,\xi}$ is compact. □

Proposition 3.2. *If $\psi \in L^1(G/H)$, then $L_{\psi,\zeta,\xi}$ is in S_1 and*

$$tr(L_{\psi,\zeta,\xi}) = \frac{\rho(e)}{c_{\zeta,\xi}}\langle\zeta,\xi\rangle\left\|\psi\right\|_1.$$

Proof. Let $\{\zeta_k\}_{k=1}^{\infty}$ and $\{\xi_k\}_{k=1}^{\infty}$ be any two orthonormal sets of $\mathcal{H}$. Then

$$\sum_{k=1}^{\infty}\left|\langle L_{\psi,\zeta,\xi}\zeta_k,\xi_k\rangle\right|$$

$$\leq \frac{1}{|c_{\zeta,\xi}|}\sum_{k=1}^{\infty}\int_{G/H}\frac{\rho(e)}{\rho(g)}\left|\psi(gH)\right|\left|\langle\zeta_k,\sigma(gH)\zeta\rangle\right|\quad\left|\langle\sigma(gH)\xi,\xi_k\rangle\right|d\mu(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\int_{G/H}\left|\psi(gH)\right|\Big(\sum_{k=1}^{\infty}\left|\langle\zeta_k,\sigma(gH)\zeta\rangle\right|^2\Big)^{1/2}\Big(\sum_{k=1}^{\infty}\left|\langle\sigma(gH)\xi,\xi_k\rangle\right|^2\Big)^{1/2}d\mu'(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\int_{G/H}\left|\psi(gH)\right|d\mu'(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\ \left\|\psi\right\|_1 < \infty.$$

So $L_{\psi,\zeta,\xi} \in S_1$ and

$$\begin{aligned} tr(L_{\psi,\zeta,\xi}) &= \sum_{k=1}^{\infty}\langle L_{\psi,\zeta,\xi}\zeta_k, \zeta_k\rangle \\ &= \frac{1}{c_{\zeta,\xi}} \textstyle\sum_{k=1}^{\infty} \displaystyle\int_{G/H} \frac{\rho(e)}{\rho(g)}\psi(gH)\langle \zeta_k, \sigma(gH)\zeta\rangle \, \langle \sigma(gH)\xi, \zeta_k\rangle \, d\mu(gH) \\ &= \frac{\rho(e)}{c_{\zeta,\xi}} \int_{G/H} \psi(gH) \sum_{k=1}^{\infty}\langle \zeta_k, \sigma(gH)\zeta\rangle \, \langle \sigma(gH)\xi, \zeta_k\rangle \, d\mu'(gH) \\ &= \frac{\rho(e)}{c_{\zeta,\xi}} \int_{G/H} \psi(gH)\langle \sigma(gH)\zeta, \sigma(gH)\xi\rangle d\mu'(gH) \\ &= \frac{\rho(e)}{c_{\zeta,\xi}}\langle \zeta, \xi\rangle \int_{G/H} \psi(gH) d\mu'(gH), \end{aligned}$$

where $\{\zeta_k\}_{k=1}^{\infty}$ is any orthonormal basis for $\mathcal{H}$. □

Proposition 3.3. *If* $\psi \in L^1(G/H)$, *then* $\|L_{\psi,\zeta,\xi}\|_{S_1} \leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\|\psi\|_1$.

Proof. By Proposition 3.2 the localization operator $L_{\psi,\zeta,\xi}$ is in S_1. Using the canonical form [11, Theorem 2.2] for compact operator $L_{\psi,\zeta,\xi}$, we get

$$L_{\psi,\zeta,\xi}x = \sum_{k=1}^{\infty} s_k(L_{\psi,\zeta,\xi})\langle x, \zeta_k\rangle \xi_k, \tag{3.1}$$

where $s_k(L_{\psi,\zeta,\xi}), k = 1, 2, ...$ are the positive singular values of $L_{\psi,\zeta,\xi}$, the set $\{\zeta_k, k = 1, 2, ...\}$ is an orthonormal basis for $N(L_{\psi,\zeta,\xi})^{\perp}$ and $\{\xi_k, k = 1, 2, ...\}$ is an orthonormal set in $\mathcal{H}$. Then (3.1) implies that

$$\sum_{j=1}^{\infty}\langle L_{\psi,\zeta,\xi}\zeta_j, \xi_j\rangle = \sum_{j=1}^{\infty} s_j(L_{\psi,\zeta,\xi}).$$

So $\|L_{\psi,\zeta,\xi}\|_{S_1} = \sum_{j=1}^{\infty}\langle L_{\psi,\zeta,\xi}\zeta_j, \xi_j\rangle$. Now Fubini's theorem, Parseval's identity, Bessel's inequality and Schawrtz's inequality imply that

$$\Big|\sum_{j=1}^{\infty}\langle L_{\psi,\zeta,\xi}\zeta_j,\xi_j\rangle\Big|$$

$$\leq \sum_{j=1}^{\infty}\frac{\rho(e)}{|c_{\zeta,\xi}|}\int_{G/H}\frac{1}{\rho(g)}\Big|\psi(gH)\Big|\ \Big|\langle\zeta_j,\sigma(gH)\zeta\rangle\Big|\ \Big|\langle\sigma(gH)\xi,\xi_j\rangle\Big|\,d\mu(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\int_{G/H}\Big|\psi(gH)\Big|\Big(\sum_{j=1}^{\infty}\Big|\langle\zeta_j,\sigma(gH)\zeta\rangle\Big|^2\Big)^{1/2}\Big(\sum_{j=1}^{\infty}\Big|\langle\sigma(gH)\xi,\xi_j\rangle\Big|^2\Big)^{1/2}d\mu'(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\int_{G/H}\Big|\psi(gH)\Big|\,d\mu'(gH)$$

$$\leq \frac{\rho(e)}{|c_{\zeta,\xi}|}\Big\|\psi\Big\|_1.$$

□

Note that by Proposition 2.4 the two-wavelet localization operator $L_{\zeta,\xi,\psi}$ is a bounded linear operator for $\psi\in L^\infty(G/H)$ and

$$\Big(\Big\|L_{\zeta,\xi,\psi}\Big\| = \Big)\Big\|L_{\zeta,\xi,\psi}\Big\|_{S_\infty}\leq\frac{(c_\zeta c_\xi)^{1/2}}{|c_{\zeta,\xi}|}\Big\|\psi\Big\|_\infty.$$

Now Riesz Thorin Interpolation Theorem [10] implies that the localization operator $L_{\psi,\zeta,\xi}$ for $1\leq p\leq\infty$ is in S_p. More precisely we have the following theorem.

Theorem 3.4. *Let $\psi\in L^p(G/H), 1\leq p\leq\infty$. Then the localization operator $L_{\psi,\zeta,\xi}$ is in S_p and*

$$\Big\|L_{\psi,\zeta,\xi}\Big\|_{S_p}\leq\frac{\rho(e)^{1/p}(c_\zeta.c_\xi)^{1/2(1-1/p)}}{|c_{\zeta,\xi}|}\Big\|\psi\Big\|_p$$

We conclude with some examples concerning localization operators on some homogeneous spaces.

Example 3.5. Let G be the Weyl-Heisenberg group $(WH)^n$ and $H=\{(0,0,t),\ \ t\in\mathbb{R}/2\pi\mathbb{Z}\}$. The Euclidean space $\mathbb{R}^n\times\mathbb{R}^n$ is as homogenous space of $(WH)^n$ and $\frac{(WH)^n}{H}=\mathbb{R}^n\times\mathbb{R}^n$ admits the Lebesgue measure. The representation

$$\sigma:\mathbb{R}^n\times\mathbb{R}^n\to U\Big(L^2(\mathbb{R}^n)\Big),\ \ \Big(\sigma(q,p)\varphi\Big)(x)=e^{i(px-qp)}\varphi(x-q),$$

where $x\in\mathbb{R}^n,\varphi\in L^2(\mathbb{R}^n)$ is square integrable (see [11]). For $\psi\in L^p(\mathbb{R}^n)$, the localization operator $L_{\psi,\zeta,\xi}$ with two admissible wavelets ζ,ξ is defined by

$$\begin{aligned}\langle L_{\psi,\zeta,\xi}f,g\rangle &= \frac{1}{c_{\zeta,\xi}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi(q,p)\langle f,\sigma(q,p)\zeta\rangle\ \langle\sigma(q,p)\xi,g\rangle\,dqdp\\ &= \frac{1}{c_{\zeta,\xi}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\psi(q,p)\langle f,\zeta_{q,p}\rangle\ \langle\xi_{q,p},g\rangle\,dqdp,\end{aligned}$$

for $f, g \in L^2(\mathbb{R}^n)$, where

$$\begin{aligned} c_{\zeta,\xi} &= \frac{1}{\|\zeta\|^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle \zeta, \sigma(q,p)\zeta \rangle \langle \sigma(q,p)\xi, \zeta \rangle \, dqdp \\ &= \frac{1}{\|\zeta\|^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle \zeta, \zeta_{q,p} \rangle \langle \xi_{q,p}, \zeta \rangle \, dqdp, \end{aligned}$$

where $\zeta_{q,p}(x) = e^{ipx}\zeta(x-q), \quad x \in \mathbb{R}^n$.

Example 3.6. Consider the similitude group, $SIM(n) = \mathbb{R}^n \times_\tau (\mathbb{R}^+ \times SO(n))$, i.e. the semidirect product of $\mathbb{R}^n$ and $\mathbb{R}^+ \times SO(n)$ with respect to

$$\tau : \mathbb{R}^+ \times SO(n) \to Aut(\mathbb{R}^n), \quad \tau_(a, w)\vec{b} = aw\vec{b}.$$

Evidently, $\mathbb{R}^n$ can be considered as a homogeneous space of $SIM(n)$. Let $d^n b$ be the Lebesgue measure on $\mathbb{R}^n$ which arises from rho function $\rho : SIM(n) \to (0, \infty)$, such that $\rho(\vec{b}, a, w) = a^n$ [1]. The representation

$$\sigma : \mathbb{R}^n \to U(L^2(\mathbb{R}^n)), \quad (\sigma(\zeta)\varphi)(x) = e^{ix\zeta}\varphi(x), \quad x, \zeta \in \mathbb{R}^n$$

is square integrable. For two admissible wavelets $\varphi, \psi \in L^2(\mathbb{R}^n)$ and $\hat{\varphi}, \hat{\psi} \in L^4(\mathbb{R}^n)$, the two-wavelet constant $c_{\varphi,\psi}$ is as follows

$$\begin{aligned} c_{\varphi,\psi} &= \frac{1}{\|\psi\|^2} \int_{\mathbb{R}^n} \frac{1}{a^n} \langle \psi, \sigma(\zeta)\psi \rangle \langle \varphi, \sigma(\zeta)\psi \rangle \, d^n\zeta \\ &= \frac{1}{\|\psi\|^2} \int_{\mathbb{R}^n} \frac{1}{a^n} \Big(\int_{\mathbb{R}^n} \psi(x) \overline{e^{ix\zeta}\varphi(x)} d^n x \Big) \Big(\int_{\mathbb{R}^n} \varphi(x) \overline{e^{ix\zeta}\varphi(x)} d^n x \Big) d^n\zeta \\ &= \frac{1}{\|\psi\|^2} \int_{\mathbb{R}^n} \frac{1}{a^n} \widehat{\varphi\bar{\varphi}(\zeta)} . \widehat{\varphi\bar{\psi}(\zeta)} d^n\zeta \\ &= \frac{1}{a^n} \langle \widehat{\varphi\bar{\varphi}}, \widehat{\varphi\bar{\psi}} \rangle. \end{aligned}$$

Moreover, for $F \in L^\infty(\mathbb{R}^n)$ and two admissible wavelet φ, ψ, the localization operator $L_{F,\varphi,\psi}$ is given by

$$\begin{aligned} \langle L_{F,\varphi,\psi} f, g \rangle &= \frac{1}{c_{\varphi,\psi}} \int_{\mathbb{R}^n} \frac{1}{a^n} F(\zeta) \langle f, \sigma(\zeta)\varphi \rangle \langle \sigma(\zeta)\psi, g \rangle \, d^n\zeta \\ &= \frac{1}{c_{\varphi,\psi}} \int_{\mathbb{R}^n} \frac{1}{a^n} F(\zeta) \widehat{f\bar{\varphi}}(\zeta) \overline{\widehat{\psi\bar{g}}}(\zeta) d^n\zeta \end{aligned}$$

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