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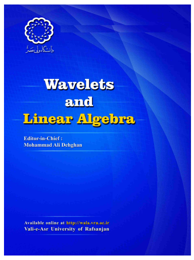

On zero product determined Banach algebras

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ABSTRACT

Let $\mathcal{A}$ be a Banach algebra with a left approximate identity. In this paper, under each of the following conditions, we prove that $\mathcal{A}$ is zero product determined.

(i) For every continuous bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{X}$, where $\mathcal{X}$ is a Banach space, there exists $k > 0$ such that $\|\phi(a,b)\| \leq k\|ab\|$, for all $a, b \in \mathcal{A}$.

(ii) $\mathcal{A}$ is generated by idempotents.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a arbitrary Banach space. Then the continuous bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad \phi(a,b) = 0, \quad a, b \in \mathcal{A}. \tag{1.1}$$

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Moreover, $\mathcal{A}$ is said to be *zero product determined* if every continuous zero product preserving bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ is implemented by a linear map $T : \mathcal{A} \longrightarrow \mathcal{X}$, i.e., $\phi(a, b) = T(ab)$ for all $a, b \in \mathcal{A}$.

Characterizing homomorphisms, derivations and multipliers on Banach algebras, matrix algebras and C^*-algebras through the action on zero products have been studied by many authors, see for example [1, 2, 3, 4, 6, 7, 8] and the references therein.

In [2], Bresar proved that if $\mathcal{A}$ is a unital Banach algebra and $\mathcal{A} = \mathcal{J}(\mathcal{A})$, where $\mathcal{J}(\mathcal{A})$ is the subalgebra of $\mathcal{A}$ generated by all idempotents, then $\mathcal{A}$ is zero product determined. Bresar *et.al.* proved in [4] that the matrix algebra $M_n(\mathcal{A})$ of $n \times n$ matrices over a unital algebra $\mathcal{A}$ is zero product determined. In [6], Ghahramani studied the (centeralizers) multipliers on Banach algebras through identity products by consideration of bilinear mapping satisfying a related condition.

Motivated by (1.1) the following concept was introduced in [1].

Definition 1.1. [1] A Banach algebra $\mathcal{A}$ *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space, the condition (1.1) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in \mathcal{A}$.

Under mild assumptions, the condition $\phi(ab, c) = \phi(a, bc)$ for every $a, b, c \in \mathcal{A}$, implies that $\mathcal{A}$ is zero product determined. For example, if $\mathcal{A}$ is unital, then just take $c = e_{\mathcal{A}}$ and note that T can be defined according to $T(a) = \phi(a, e_{\mathcal{A}})$.

However, throughout the paper we focus for nonunital Banach algebras, but we will assume the existence of a left approximate identity.

Recall that a left (right) approximate identity for $\mathcal{A}$ is a net $\{e_\lambda\}_{\lambda \in I}$ in $\mathcal{A}$ such that $e_\lambda a \longrightarrow a$ ($ae_\lambda \longrightarrow a$) for all $a \in \mathcal{A}$. For example, it is known that the C^*-algebras have an approximate identity bounded by one [5].

In this paper, we introduce the property $(\mathbb{P})$, which closely related to the property $(\mathbb{B})$, and prove that the property $(\mathbb{B})$ follows from the property $(\mathbb{P})$. We show that every Banach algebra $\mathcal{A}$ with a left approximate identity is zero product determined if either $\mathcal{A}$ has the property $(\mathbb{P})$ or it is generated by idempotents.

2. The property $(\mathbb{P})$

We commence with the next concept which is closely related to the property $(\mathbb{B})$.

Definition 2.1. A Banach algebra $\mathcal{A}$ *has the property* $(\mathbb{P})$ if for every continuous bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{X}$, where $\mathcal{X}$ is an arbitrary Banach space, there exists $k > 0$ such that

$$\|\phi(a, b)\| \leq k\|ab\|, \tag{2.1}$$

for all $a, b \in \mathcal{A}$.

A crucial result of this section is the following theorem which states that the property $(\mathbb{P})$ implies the property $(\mathbb{B})$.

Theorem 2.2. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then $\mathcal{A}$ has the property $(\mathbb{B})$.*

Proof. Let $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ be a continuous bilinear mapping. Since $\mathcal{A}$ has the property $(\mathbb{P})$, ϕ preserves zero products. Let $\mathcal{A}_1$ denote the algebra $\mathcal{A}$ with an identity adjoined. Define

$$\phi_1 : \mathcal{A}_1 \times \mathcal{A} \longrightarrow \mathcal{X} \quad by \quad \phi_1(e_{\mathcal{A}}, a) = \lim_\lambda \phi(e_\lambda, a).$$

Then the limit exists, since $\mathcal{A}$ has the property $(\mathbb{P})$. Now for every $f \in \mathcal{X}^*$ and $a, b \in \mathcal{A}$, let $F : \mathbb{C} \longrightarrow \mathbb{C}$ via

$$F(z) = f \circ \phi_1(e^{-za}, e^{za}b).$$

Then F is an entire function and for $z \in \mathbb{C}$,

$$|F(z)| \leq \|f\|\|\phi_1(e^{-za}, e^{za}b)\| \leq k\|f\|\|b\|.$$

Thus, by Liouville's Theorem F is constant, and hence the coefficient of z in the power series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-z)^n}{n!}\frac{z^m}{m!} f \circ \phi_1(a^n, a^m b) = f \circ \phi_1(e^{-za}, e^{za}b)$$

is zero, i.e.,

$$f(\phi_1(e_{\mathcal{A}}, ab) - \phi_1(a, b)) = 0.$$

Since f was arbitrary, $\phi_1(e_{\mathcal{A}}, ab) = \phi_1(a, b)$ and hence

$$\lim_\lambda \phi(e_\lambda, ab) = \phi_1(e_{\mathcal{A}}, ab) = \phi_1(a, b) = \phi(a, b). \tag{2.2}$$

Therefore

$$\phi(ab, c) = \lim_\lambda \phi(e_\lambda, abc) = \phi(a, bc),$$

for all $a, b, c \in \mathcal{A}$. Consequently, $\mathcal{A}$ has the property $(\mathbb{B})$. □

Corollary 2.3. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then $\mathcal{A}$ is zero product determined.*

Proof. Let $\{e_\lambda\}$ be a left approximate identity in $\mathcal{A}$. Define the continuous linear mapping $T : \mathcal{A} \longrightarrow \mathcal{X}$ via $T(a) = \lim_\lambda \phi(e_\lambda, a)$. Then by using Theorem 2.2, we obtain

$$T(ab) = \phi(a, b),$$

for all $a, b \in \mathcal{A}$. □

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra with an approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then $\mathcal{A}$ is commutative.*

Proof. Consider the continuous bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ defined by $\phi(a, b) = ba$. It follows from Theorem 2.2 that $cba = \phi(ab, c) = \phi(a, bc) = bca$, for all $a, b, c \in \mathcal{A}$. Taking $a = e_\lambda$, we get $bc = cb$ for all $b, c \in \mathcal{A}$. So $\mathcal{A}$ is commutative. □

Corollary 2.5. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity $\{e_\lambda\}$. If $\mathcal{A}$ has the property $(\mathbb{P})$, then every bilinear mapping $\phi : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{X}$ is symmetric, that is, $\phi(a, b) = \phi(b, a)$ for all $a, b \in \mathcal{A}$.*

Corollary 2.6. *Suppose that $\mathcal{A}$ is a Banach algebra with a left approximate identity, and let $f : \mathcal{A} \longrightarrow \mathcal{A}$ be a continuous linear mapping. If $\mathcal{A}$ has the property $(\mathbb{P})$, then f is a left multiplier.*

Proof. Let us define a continuous bilinear mapping $\phi : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$ by $\phi(a, b) = af(b)$. Then by using Theorem 2.2, we obtain

$$abf(c) = \phi(ab, c) = \phi(a, bc) = af(bc),$$

for all $a, b, c \in \mathcal{A}$. Replacing a by e_λ, we get $bf(c) = f(bc)$, for all $b, c \in \mathcal{A}$. □

Corollary 2.7. *Let $\mathcal{A}$ be a Banach algebra with a left approximate identity $\{e_\lambda\}$, and let $f \in \mathcal{A}^*$. If for all $a \in \mathcal{A}$,*

$$|f(a)| \leq kr(a),$$

where $r(a)$ is the spectral radius of a, then the bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{C}$ defined by $\phi(a, b) = f(ba)$ is symmetric.

Proof. For all $a, b \in \mathcal{A}$, we have

$$|\phi(a, b)| = |f(ba)| \leq kr(ba) = kr(ab) \leq k||ab||.$$

Therefore

$$\phi(b, a) = f(ab) = \lim_\lambda \phi(e_\lambda, ab) = \phi(a, b),$$

as claimed. □

3. Subalgebras generated by idempotents

By the subalgebra of an algebra $\mathcal{A}$ generated by a subset E of $\mathcal{A}$ we mean the linear subspace of $\mathcal{A}$ spanned by the set of all finite products of elements in E.

Theorem 3.1. *If the Banach algebra $\mathcal{A}$ is generated by idempotents, then $\mathcal{A}$ has the property $(\mathbb{B})$.*

Proof. Let $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ be a continuous bilinear mapping such that $ab = 0$ implies that $\phi(a, b) = 0$. Suppose that p is an idempotent in $\mathcal{A}$. Then for all $a, c \in \mathcal{A}$,

$$ap(1 - p)c = a(1 - p)pc = 0.$$

Hence

$$\phi(ap, (1 - p)c) = \phi(a(1 - p), pc) = 0.$$

Therefore

$$\phi(ap, c) = \phi(ap, pc) = \phi(a, pc),$$

for every idempotent $p \in \mathcal{A}$ and for all $a, c \in \mathcal{A}$. Take

$$S = \{t \in \mathcal{A} : \ \phi(at, c) = \phi(a, tc), \ \ a, c \in \mathcal{A}\}.$$

Now let $b \in \mathcal{A}$ be an arbitrary element. Then $b = \sum_{i=1}^{k} p_{i_1} p_{i_2} ... p_{i_k}$, for some idempotents $p_{i_1}, ..., p_{i_k} \in \mathcal{A}$. Since S is a subalgebra of $\mathcal{A}$ and contains all idempotents,

$$\phi(ab, c) = \phi(a, bc), \tag{3.1}$$

for all $a, b, c \in \mathcal{A}$. This finishes the proof. □

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra with a bounded left approximate identity. If $\mathcal{A}$ is generated by idempotents, then it is zero product determined.*

Combining Theorem 3.1 and [9, Theorem 1] we deduce the next result.

Corollary 3.3. *Let $\mathcal{A}$ be a Banach algebra which is generated by k of its elements and by the identity element. Then, for all $n \geq k + 2$, the algebra $M_n(\mathcal{A})$ of all $n \times n$ matrices with entries in $\mathcal{A}$, has the property (B).*

Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear mapping between Banach algebras. Then we say that f preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad f(a)f(b) = 0, \ \ (a, b \in \mathcal{A}).$$

It is obvious that homomorphisms from $\mathcal{A}$ into $\mathcal{B}$ preserve zero products.

Corollary 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous linear mapping preserving zero products. If $\mathcal{A}$ is generated by idempotents, then*

$$f(ab)f(c) = f(a)f(bc).$$

Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are unital and $f(e_{\mathcal{A}}) = e_{\mathcal{B}}$, then f is a homomorphism.

Proof. Define a continuous bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}$ by $\phi(a, b) = f(a)f(b)$. If $ab = 0$, then $\phi(a, b) = f(a)f(b) = 0$ and so the equality (3.1) holds. Thus,

$$f(ab)f(c) = \phi(ab, c) = \phi(a, bc) = f(a)f(bc),$$

for all $a, b, c \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are unital and $f(e_{\mathcal{A}}) = e_{\mathcal{B}}$, then setting $c = e_{\mathcal{A}}$, we conclude that f is a homomorphism. □

Corollary 3.5. *Suppose that $\mathcal{A}$ is a topologically simple Banach algebra containing a nontrivial idempotent. Then $\mathcal{A}$ has the property* $(\mathbb{B})$.

Proof. Let p be a nontrivial idempotent ($p \neq 0$ and $p \neq 1$) in $\mathcal{A}$. Then p cannot be contained in the centre of $\mathcal{A}$. This implies that the subalgebra M of $\mathcal{A}$ generated by idempotents contains a nonzero ideal of $\mathcal{A}$ by [2, Lemma 2.1], from which it follows that M is dense in $\mathcal{A}$. Thus, $\mathcal{A}$ has the property $(\mathbb{B})$ by Theorem 3.1. □

A topological space X is called *totally disconnected* if for every distinct $x, y \in X$, there exist disjoint open sets U and V such that $x \in U$, $y \in V$ and $X = U \cup V$.

Next we prove that the Banach algebra $C_0(X)$ is generated by idempotents if and only if X is totally disconnected.

Theorem 3.6. *The Banach algebra $C_0(X)$, for a locally compact Hausdorff space X, is generated by idempotents if and only if X is totally disconnected.*

Proof. Suppose that $C_0(X)$ is generated by idempotents and let $x, y \in X$. Then by Urysohn's Lemma there exists $f \in C_0(X)$ such that $f(x) = 1$ and $f(y) = 0$. Since every element of the self-adjoint subalgebra generated by idempotents is the form

$$F = \sum_{i=1}^{k} \alpha_i f_i, \tag{3.2}$$

for some $f_i \in C_0(X)$ and $\alpha_i \in \mathbb{C}$, there is a sequence (F_n) of elements of the form (3.2) such that $F_n \longrightarrow f$ uniformly on X. Hence,

$$\lim_n F_n(x) = 1, \quad \textit{and} \qquad \lim_n F_n(y) = 0.$$

So there exists a number N such that $|F_N(x)| > 1/2$ and $|F_N(y)| < 1/2$. Take

$$U = F_N^{-1}(\{z \in \mathbb{C} : \ |z| > 1/2\}) \ \textit{and} \quad V = F_N^{-1}(\{z \in \mathbb{C} : \ |z| < 1/2\}).$$

Then $U \cap V = \emptyset$, $x \in U$, $y \in V$ and $X = U \cup V$. Therefore, X is totally disconnected.

Conversely, suppose that X is totally disconnected. Let $x \neq y$, $x \in U$, $y \in V$, where U and V are disjoint open sets and $X = U \cup V$. Then the continuous function

$$f(x) = \begin{cases} 1 & x \in U, \\ \\ 0 & x \in V, \end{cases}$$

separates x and y. Consequently, by the Stone-Weierstrass theorem the closed self-adjoint subalgebra generated by the idempotents is $C_0(X)$. □

It follows from [1, Theorem 2.11] that every C^*-algebra $\mathcal{A}$ is zero product determined. Now as an upcoming consequence of Theorem 3.6 and Corollary 3.2 we have the following result.

Corollary 3.7. *The Banach algebra $C([0, 1])$ is zero product determined.*

Example 3.8. *Let $\mathcal{A} = C([0, 1])$. Then by Corollary 3.7, $\mathcal{A}$ is zero product determined. Thus, for every continuous bilinear mapping $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}$ satisfying (1.1), there exist a linear mapping $T : \mathcal{A} \longrightarrow \mathcal{X}$ such that $\phi(a, b) = T(ab)$ for every $a, b \in \mathcal{A}$. Since T is continuous, there exist $k > 0$ such that*

$$\|\phi(a, b)\| = \|T(ab)\| \leq k\|ab\|, \qquad a, b \in \mathcal{A}.$$

Hence $\mathcal{A}$ has the property $(\mathbb{P})$, for each continuous bilinear mapping ϕ satisfying (1.1).

The following example shows that the converse of Theorem 2.2 is false, in general.

Example 3.9. *Let K denote the cross $[-1, 1] \cup i[-1, 1]$ and let $C^{2\times2}(K)$ denote the algebra of all continuous 2×2 matrix functions on a compact set K. Then by [9, Theorem 4], the Banach algebra $\mathcal{A} = C^{2\times2}(K)$ is generated by two idempotents and by the identity function. Therefore it has the property $(\mathbb{B})$ by Theorem 3.1. Since $\mathcal{A}$ is not commutative, it fails to satisfy the property $(\mathbb{P})$ by Corollary 2.4.*

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