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Approximate biprojectivity of Banach algebras with respect to their character spaces

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Abstract

In this paper we introduce approximate ϕ-biprojective Banach algebras, where ϕ is a non-zero character. We show that for SIN group G, the group algebra $L^1(G)$ is approximately ϕ-biprojective if and only if G is amenable, where ϕ is the augmentation character. Also we show that the Fourier algebra $A(G)$ over a locally compact G is always approximately ϕ-biprojective.

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1. Introduction

Helemskii studied Banach algebras using the homological theory. There is an important notions in homological theory, namely biprojectivity. A Banach algebra A is biprojective, if there exists a bounded A-bimodule morphism $\rho : A \to A \otimes_p A$ such that $\pi_A \circ \rho(a) = a$ for all $a \in A$. It is known that the group algebra $L^1(G)$ over a locally compact group G is biprojective if and only if G compact and also the if the Fourier algebra $A(G)$ is biprojective, then G is discrete, see [10].

In [21] the first author with A. Pourabbas introduced a notion of biprojectivity related to a character. In fact a Banach algebra A is called ϕ-biprojective, if there exists a bounded A-bimodule morphism

$$\rho : A \to A \otimes_p A$$

such that

$$\phi \circ \pi_A \circ \rho(a) = \phi(a) \qquad (a \in A).$$

We showed that for a locally compact group G the Segal algebra $S(G)$ is ϕ-biprojective if and only if G is compact. Also the Fourier algebra $A(G)$ is ϕ-biprojective if and only if G is discrete, see [15] and [21].

An approximate notion of the homological theory was given by Zhang. A Banach algebra A is *approximate biprojective* if there exists a net of A-bimodule morphisms $\rho_\alpha : A \to A \otimes_p A$ such that

$$\pi_A \circ \rho_\alpha(a) \to a \quad (a \in A).$$

Approximate biprojectivity of some semigroup algebras and some related Triangular Banach algebras was studied in [1], [19] and [20].

Inspired by Zhang definition and also by replacing the "A-bimodule morphism" with "approximate A-bimodule morphism" in the definition of approximate biprojectivity, we give an approximate version of ϕ-biprojectivity here.

Definition 1.1. Let A be a Banach algebra and $\phi \in \Delta(A)$. Then A is called *approximate ϕ-biprojective* if there exists $(\rho_\alpha)_{\alpha\in I}$ a net of bounded linear maps from A into $A \otimes_p A$, $(\rho_\alpha)_{\alpha\in I}$ such that

(i) $a \cdot \rho_\alpha(b) - \rho_\alpha(ab) \xrightarrow{||\cdot||} 0$,

(ii) $\rho_\alpha(ba) - \rho_\alpha(b) \cdot a \xrightarrow{||\cdot||} 0$,

(iii) $\phi \circ \pi_A \circ \rho_\alpha(a) - \phi(a) \to 0$,

for every $a, b \in A$. Also we say that A is approximately character biprojective if A is approximate ϕ-biprojective for each $\phi \in \Delta(A)$.

In this paper, first we study the general properties of approximate ϕ-biprojectivity. The hereditary properties of this notion were investigated. We show that for a SIN group G, the Segal algebra $S(G)$ is approximate ϕ-biprojective if and only if G is amenable, where ϕ is the augmentation character on $S(G)$ and the measure algebra $M(G)$ is approximate character biprojective if and only if G is discrete and amenable. Finally some examples of Banach algebras among Triangular Banach algebras were given which we show that these matrix algebras are not approximate ϕ-biprojective and some examples which reveal the differences of our new notion and the classical ones.

2. Approximate ϕ-biprojectivity

This section is devoted to investigate the general properties of approximate ϕ-biprojectivity for Banach algebras.

Let A be a Banach algebra and $\phi \in \Delta(A)$. A Banach algebra A is called approximately left $\phi-$amenable, if there exists a (not necessarily bounded) net (m_α) in A such that

$$am_\alpha - \phi(a)m_\alpha \to 0, \quad \phi(m_\alpha) \to 1, \qquad (a \in A).$$

The right case similarly defined. For further information see [2].

Proposition 2.1. *Suppose that A is a Banach algebra and $\phi \in \Delta(A)$. Let A be approximate ϕ-biprojective which has an element a_0 such that $aa_0 = a_0a$ for all $a \in A$ and $\phi(a_0) = 1$. Then A is approximate left and approximate right ϕ-amenable.*

Proof. Let $(\rho_\alpha)_{\alpha\in I}$ be as in Definition 1.1. Suppose that a_0 be an element in A such that $aa_0 = a_0a$ and $\phi(a_0) = 1$ for every $a \in A$. Set $n_\alpha = \rho_\alpha(a_0)$. It is clear that (n_α) is a net in $A \otimes_p A$ such that

$$\begin{aligned} a\cdot n_\alpha - n_\alpha\cdot a &= a\cdot\rho_\alpha(a_0) - \rho_\alpha(a_0)\cdot a \\ &= a\cdot\rho_\alpha(a_0) - \rho_\alpha(aa_0) + \rho_\alpha(aa_0) - \rho_\alpha(a_0a) + \rho_\alpha(a_0a) - \rho_\alpha(a_0)\cdot a \to 0 \end{aligned}$$

for every $a \in A$. Also we have

$$\phi\circ\pi_A(n_\alpha) - 1 = \phi\circ\pi_A\circ\rho_\alpha(a_0) - \phi(a_0) \to 0.$$

Define $T : A\otimes_p A \to A$ by $T(a\otimes b) = \phi(b)a$ for each $a, b \in A$. It is clear that T is a bounded linear map which satisfies

$$T(a\cdot x) = aT(x), \quad T(x\cdot a) = \phi(a)T(x), \quad \phi\circ T = \phi\circ\pi_A, \quad (a \in A, x \in A\otimes_p A).$$

Set $m_\alpha = T(n_\alpha)$. One can show that

$$am_\alpha - \phi(a)m_\alpha = aT(n_\alpha) - \phi(a)T(n_\alpha) = T(a\cdot n_\alpha - n_\alpha\cdot a) \to 0, \quad (a \in A)$$

and

$$\phi(m_\alpha) = \phi\circ T(n_\alpha) = \phi\circ\pi_A(n_\alpha) \to 1.$$

Thus A is approximate left ϕ-amenable. Defining $T : A\otimes_p A \to A$ by $T(a\otimes b) = \phi(b)a$ and following the similar method we can see that $m_\alpha a - \phi(a)m_\alpha \to 0$ and $\phi(m_\alpha) \to 1$, for all $a \in A$. It follows that A is approximately right ϕ-amenable. □

Example 2.2. Let $T = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C}\}$. Equip T with matrix operations and the ℓ^1-norm. Then T becomes a Banach algebra. Define $\phi \in \Delta(T)$ by $\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = c$ for all $a, b, c \in \mathbb{C}$. We claim that T is not approximate ϕ-biprojective. Suppose in contradiction that T is approximate ϕ-biprojective. Since T posses an element which commute with all elements of T and does not belong to $\ker\phi$, Proposition 2.1 follows that T is approximately left ϕ-amenable. Set $I = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}$. Clearly I is a closed ideal of T which $\phi|_I \neq 0$. It gives that I is approximately left ϕ-amenable. Thus there exists a net (i_α) in I such that

$$ii_\alpha - \phi(i)i_\alpha \to 0, \quad \phi(i_\alpha) \to 1, \quad (i \in I).$$

So nets (a_α) and (b_α) in $\mathbb{C}$ exist such that $i_\alpha = \begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix}$. Thus any $i = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$ in I, gives that

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix} - b\begin{pmatrix} 0 & a_\alpha \\ 0 & b_\alpha \end{pmatrix} \to 0,$$

which follows that $ab_\alpha - ba_\alpha \to 0$, for each $a, b \in \mathbb{C}$. Since $b_\alpha \to 1$, by considering $a = 1$ and $b = 0$ at above, we have a contradiction.

We recall that a Banach algebra A is pseudo-amenable if there exists a net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$, for each $a \in A$, see [9].

Proposition 2.3. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is pseudo-amenable, then A is approximate ϕ-biprojective.*

Proof. Suppose that A is pseudo-amenable. Then there exists a net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$, for all $a \in A$. Define $\rho_\alpha(a) = a \cdot m_\alpha$. Clearly

$$\rho_\alpha(ab) - \rho_\alpha(a) \cdot b \to 0, \quad \rho_\alpha(ab) - a \cdot \rho_\alpha(b) \to 0, \quad \phi \circ \pi_A(m_\alpha) \to 1,$$

for all $a \in A$. □

Theorem 2.4. *Let A be a Banach algebra and $\phi \in \Delta(A)$. If A is biflat, then A is approximately ϕ-biprojective.*

Proof. Suppose that A is ϕ-biflat. Then there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$ for each $a \in A$. So a net (ρ_α) in $B(A, A \otimes_p A)$, the set of all bounded linear maps from A into $A \otimes_p A$, exists such that $\rho_\alpha \xrightarrow{W^*OT} \rho$, where W^*OT stands for the weak-star operator topology. Clearly π_A^{**} is a w^*-continuous map. Thus, for each $a \in A$ it gives that

$$\pi_A \circ \rho_\alpha(a) = \pi_A^{**} \circ \rho_\alpha(a) \xrightarrow{w^*} \pi_A^{**} \circ \rho(a) = a,$$

also we have

$$a \cdot \rho_\alpha(b) \xrightarrow{w^*} \rho(ab), \quad \rho_\alpha(ab) \xrightarrow{w^*} \rho(ab)$$

and

$$\rho_\alpha(a) \cdot b \xrightarrow{w^*} \rho(ab), \quad \rho_\alpha(ab) \xrightarrow{w^*} \rho(ab),$$

for every $a, b \in A$. Let $\epsilon > 0$ and take arbitrary finite subsets $F = \{a_1, a_2, ..., a_r\}$ and $G = \{x_1, x_2, ..., x_r\}$ of A. Define

$$\begin{aligned} M = \{(&a_1 \cdot T(x_1) - T(a_1x_1), a_2 \cdot T(x_2) - T(a_2x_2), ..., a_r \cdot T(x_r) - T(a_rx_r), \\ &T(x_1) \cdot a_1 - T(x_1a_1), T(x_2) \cdot a_2 - T(x_2a_2), ..., T(x_r) \cdot a_r - T(x_ra_r) \\ &\phi \circ \pi_A \circ T(x_1) - \phi(x_1), \phi \circ \pi_A \circ T(x_2) - \phi(x_2), ..., \phi \circ \pi_A \circ T(x_r) - \phi(x_r)) \\ &: T \in B(A, A \otimes_p A)\} \end{aligned} \tag{2.1}$$

It clear that M is a subset of $\prod_{i=1}^{2r}(A \otimes_p A) \oplus_1 \prod_{i=1}^{r} \mathbb{C}$. One can show that M is a convex set and $(0, 0, ..., 0)$ belongs to $\overline{M}^w = \overline{M}^{||\cdot||}$. So there exists element $\theta_{(F,G,\epsilon)}$ in $B(A, A \otimes_p A)$ such that

$$||a_i \cdot \theta_{(F,G,\epsilon)}(b_i) - \theta_{(F,G,\epsilon)}(a_ib_i)|| < \epsilon, \quad ||\theta_{(F,G,\epsilon)}(a_ib_i) - \theta_{(F,G,\epsilon)}(a_i) \cdot b_i|| < \epsilon$$

and

$$|\phi \circ \pi_A \circ \theta_{(F,G,\epsilon)}(a_i) - \phi(a_i)| < \epsilon,$$

for each $i \in \{1, 2, ..., r\}$. Therefore $(\theta_{(F,G,\epsilon)})_{(F,G,\epsilon)}$ satisfies

$$a \cdot \theta_{(F,G,\epsilon)}(b) - \theta_{(F,G,\epsilon)}(ab) \to 0, \quad \theta_{(F,G,\epsilon)}(ab) - \theta_{(F,G,\epsilon)}(a) \cdot b \to 0$$

and

$$|\phi \circ \pi_A \circ \theta_{(F,G,\epsilon)}(a) - \phi(a)| \to 0,$$

for each $a, b \in A$. It deduces that A is approximately ϕ-biprojective. □

Proposition 2.5. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that I is a closed ideal of A such that $\phi|_I \neq 0$. If A is approximate ϕ-biprojective, then I is approximate ϕ-biprojective.*

Proof. Suppose that $(\rho_\alpha)_\alpha$ is a net of bounded maps satisfies Definition 1.1. Pick i_0 in I such that $\phi(i_0) = 1$. Define $T_{i_0} : A \otimes_p A \to I \otimes_p I$ by $T_{i_0}(a \otimes b) = ai_0 \otimes i_0b$ for each $a, b \in A$. Clearly T_{i_0} is a continuous linear map. Set $\eta_\alpha = T_{i_0} \circ \rho_\alpha|_I : I \to I \otimes_p I$. So consider

$$i \cdot \eta_\alpha(j) - \eta_\alpha(ij) = T(i \cdot \rho_\alpha(j) - \rho_\alpha(ij)) \to 0$$

and

$$\eta_\alpha(ij) - \eta_\alpha(i) \cdot j = T(\rho_\alpha(ij) - \rho_\alpha(i) \cdot j) \to 0,$$

also

$$\phi \circ \pi_I \circ \eta_\alpha(i) - \phi(i) = \phi \circ \pi_I \circ T \circ \rho_\alpha(i) - \phi(i) = \phi \circ \pi_A \circ \rho_\alpha(i) - \phi(i) \to 0$$

for all $i, j \in I$. So I is approximately ϕ-biprojective. □

Suppose that A and B are Banach algebras. Let $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. Then $\phi \otimes \psi$ is a bounded linear map given by $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ for all $a \in A$ and $b \in B$. Clearly $\phi \otimes \psi \in \Delta(A \otimes_p B)$. It worth mentioning that $A \otimes_p B$ with the following actions is a Banach A-bimodule:

$$a_1 \cdot (a_2 \otimes b) = a_1a_2 \otimes b, \quad (a_2 \otimes b) \cdot a_1 = a_2a_1 \otimes b, \quad (a_1, a_2 \in A, b \in B).$$

Theorem 2.6. *Suppose that A and B are Banach algebras. Let $\phi \in \Delta(A)$ and $\psi \in \Delta(B)$. Also let A be unital and B has an idempotent x_0 with $x_0 \notin \ker\psi$. If $A \otimes_p B$ is approximate $\phi\otimes\psi$-biprojective, then A is approximate ϕ-biprojective.*

Proof. Suppose that $A \otimes_p B$ is approximately $\phi\otimes\psi$-biprojective. Then there exists $(\rho_\alpha) : A \otimes_p B \to (A \otimes_p B) \otimes_p (A \otimes_p B)$ a net of bounded linear maps such that

$$x \cdot \rho_\alpha(y) - \rho_\alpha(xy) \to 0, \quad \rho_\alpha(xy) - \rho_\alpha(x) \cdot y \to 0 \tag{2.2}$$

and

$$\phi \otimes \psi \circ \pi_{A\otimes_p B} \circ \rho_\alpha(x) - \phi \otimes \psi(x) \to 0 \tag{2.3}$$

for every $x, y \in A \otimes_p B$. We know that x_0 is an idempotent. Then for a_1 and a_2 in A, the following happens

$$a_1a_2 \otimes x_0 = (a_1 \otimes x_0)(a_2 \otimes x_0). \tag{2.4}$$

Since A is unital with the unit element e, we have

$$\begin{aligned}\rho_\alpha(a_1a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0)\\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0)+\\ &\quad (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0)\\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0)+\\ &\quad (a_1 \cdot (e \otimes x_0)) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0)\\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - (a_1 \otimes x_0) \cdot \rho_\alpha(a_2 \otimes x_0)+\\ &\quad (a_1 \cdot (e \otimes x_0)) \cdot \rho_\alpha(a_2 \otimes x_0) - a_1 \cdot \rho_\alpha(ea_2 \otimes x_0x_0)+\\ &\quad a_1 \cdot \rho_\alpha(ea_2 \otimes x_0x_0) - a_1 \cdot \rho_\alpha(a_2 \otimes x_0) \to 0.\end{aligned}$$

By (2.3) and (2.2) the following happens

$$\begin{aligned}\rho_\alpha(a_1a_2 \otimes x_0) - \rho_\alpha(a_1 \otimes x_0) \cdot a_2 &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - \rho_\alpha(a_1 \otimes x_0) \cdot a_2\\ &= \rho_\alpha((a_1 \otimes x_0)(a_2 \otimes x_0)) - \rho_\alpha(a_1 \otimes x_0) \cdot (a_2 \otimes x_0)\\ &\quad + \rho_\alpha(a_1 \otimes x_0) \cdot (a_2 \otimes x_0) - \rho_\alpha((a_1 \otimes x_0) \cdot (e \otimes x_0)a_2+\\ &\quad \rho_\alpha((a_1 \otimes x_0) \cdot (e \otimes x_0)a_2 - \rho_\alpha((a_1 \otimes x_0) \cdot a_2) \to 0,\end{aligned}$$

for all $a_1, a_2 \in A$. Define

$$T : (A \otimes_p B) \otimes_p (A \otimes_p B) \to A \otimes_p A$$

by

$$T((a \otimes b) \otimes (c \otimes d)) = \psi(bd)a \otimes c,$$

for each $a, c \in A, b, d \in B$.It is easy to see that T is continuous and linear. Also we have

$$\pi_A \circ T = (id \otimes \psi) \circ \pi_{A\otimes_p B},$$

where $id \otimes \psi(a \otimes b) = \psi(b)a$ for all $a \in A, b \in B$. Define $\eta_\alpha(a) = T \circ \rho_\alpha(a \otimes x_0)$.Clearly for each α, the operator $\eta_\alpha : A \to A \otimes_p A$ is bounded and linear. Note that

$$a \cdot \eta_\alpha(b) - \eta_\alpha(ab) \to 0, \quad \eta_\alpha(ab) - \eta_\alpha(a) \cdot b \to 0, \quad (a, b \in A).$$

We can show that

$$\begin{aligned}\pi_A \circ \eta_\alpha(a) - a = \pi_A \circ T \circ \rho_\alpha(a \otimes x_0) &= (id \otimes \psi) \circ \pi_{A\otimes_p B} \circ \rho_\alpha(a \otimes x_0) - (id \otimes \psi)(a \otimes x_0) \\ &\quad + (id \otimes \psi)(a \otimes x_0) - a \\ &= (id \otimes \psi)(\pi_{A\otimes_p B} \circ \rho_\alpha(a \otimes x_0) - a \otimes x_0) + 0 \to 0,\end{aligned}$$

for all $a \in A$. Therefore A is approximate ϕ-biprojective. □

3. Some applications for harmonic analysis

For a locally compact group G, a linear subspace $S(G)$ of $L^1(G)$ is called a Segal algebra, whenever

(i) $S(G)$ is dense in $L^1(G)$;

(ii) With a norm $\|\cdot\|_S$ $S(G)$ becomes a Banach space which $\|f\|_1 \leq \|f\|_S$ for every $f \in S(G)$;

(iii) For $f \in S(G)$ and $y \in G$, we have $L_y f \in S(G)$. Here the map $y \mapsto L_y(f)$ from G into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$;

(iv) $\|L_y(f)\|_S = \|f\|_S$ for all $f \in S(G)$ and $y \in G$,

see [17].

We denote $\widehat{G}$ for the dual of G, which consists of all non-zero continuous homomorphism ζ from G into the circle group $\mathbb{T}$. It is well-known that $\Delta(L^1(G)) = \{\phi_\zeta : \zeta \in \widehat{G}\}$, where $\phi_\zeta(f) = \int_G \overline{\zeta(x)} f(x)dx$ and dx is a left Haar measure on G, for more details, see [11, Theorem 23.7]. The map $\phi_1 : L^1(G) \to \mathbb{C}$ which is specified by

$$\phi_1(f) = \int_G f(x)dx$$

is called augmentation character. It is well known that the augmentation character induces a character on $S(G)$ is still denoted by ϕ_1, see [3].

A locally compact group G is called SIN group if it contains a fundamental family of compact invariant neighborhoods of the identity, see [5, p. 86].

Theorem 3.1. *Let G be a locally compact SIN-group. Then $S(G)$ is approximate ϕ_1-biprojective if and only if G is amenable.*

Proof. Suppose that G is a SIN group. Then by [14], $S(G)$ has a central approximate identity. It follows that there exists an element $f \in S(G)$ such that $gf = fg$ and $\phi_1(f) = 1$ for all $g \in S(G)$. Now by Proposition 2.1, approximate ϕ_1-biprojectivity of $S(G)$ gives that $S(G)$ is approximate left ϕ_1-amenable. Thus there is a net (m_α) in $S(G)$ such that

$$\|gm_\alpha - \phi_1(g)m_\alpha\|_S \to 0, \quad \phi_1(m_\alpha) \to 1 \quad (g \in S(G)).$$

We know that $\|\cdot\|_1 \leq \|\cdot\|_S$. Thus it follows that

$$\|gm_\alpha - \phi_1(g)m_\alpha\|_1 \to 0, \quad \phi_1(m_\alpha) \to 1 \quad (g \in S(G)).$$

Put $f_\alpha = fm_\alpha$. For each $y \in G$ and $g \in S(G)$, consider

$$\phi_1(\delta_y g) = \int_G \delta_y g(x)dx = \int_G g(y^{-1}x)dx = \int_G g(x)dx = \phi_1(g).$$

Noticing that here δ_y gives for the point mass at $\{y\}$. Using (iii) in the definition of the Segal algebras

$$\begin{aligned}\|\delta_y f_\alpha - f_\alpha\|_1 &= \|(\delta_y f)m_\alpha - fm_\alpha\|_1 \\ &\leq \|(\delta_y f)m_\alpha - m_\alpha\|_1 + \|m_\alpha - fm_\alpha\|_1 \\ &\leq \|(\delta_y f)m_\alpha - \phi_1(\delta_y f)m_\alpha\|_1 + \|\phi_1(\delta_y f)m_\alpha - m_\alpha\|_1 \\ &\quad + \|m_\alpha - \phi_1(f)m_\alpha\|_1 + \|\phi_1(f)m_\alpha - fm_\alpha\|_1 \to 0.\end{aligned} \tag{3.1}$$

On the other hand

$$\phi_1(f_\alpha) = \phi_1(fm_\alpha) = \phi_1(f)\phi_1(m_\alpha) \to 1.$$

We know that ϕ_1 is a character. Then ϕ_1 is a bounded linear functional. So $|f_\alpha| \leq \|f_\alpha\|_1$ implies that the net f_α stays away from 0. Hence we suppose that $\|f_\alpha\|_1 \geq \frac{1}{2}$. Set $g_\alpha = \frac{|f_\alpha|}{\|f_\alpha\|_1}$. It is clear that (g_α) is a bounded net in $L^1(G)$. Consider

$$\|\delta_y g_\alpha - g_\alpha\|_1 \leq 2\|\delta_y|f_\alpha| - |f_\alpha|\|_1 \leq 2\|\delta_y f_\alpha - f_\alpha\|_1 \to 0.$$

Therefore [18, Exercise 1.1.6], follows that G is amenable.

Conversely, suppose that G is amenable. Using [22, Corollary 3.2], amenability of G gives that $S(G)$ is pseudo-amenable. Then it is easy to see that $S(G)$ is approximate ϕ_1-biprojective. □

Suppose that G is a locally compact group G. Then the Fourier algebra on G is denoted by $A(G)$.

Lemma 3.2. *Suppose that G is a locally compact group. Then $A(G)$ is approximate ϕ-biprojective, for all $\phi \in \Delta(A(G))$.*

Proof. Using [13, Example 2.6], we know that $A(G)$ is left ϕ-amenable for each $\phi \in \Delta(A(G))$. Thus there is a bounded net $n_\alpha \in A(G)$ such that $an_\alpha - \phi(a)n_\alpha \to 0$ and $\phi(n_\alpha)=1$ for all $a \in A(G)$, see [13]. Since $A(G)$ with respect to the pointwise multiplication is a commutative Banach algebra $an_\alpha - \phi(a)n_\alpha = n_\alpha a - \phi(a)n_\alpha \to 0$. Define $\rho_\alpha : A \to A \otimes_p A$ by $\rho_\alpha(a) = a \cdot n_\alpha \otimes n_\alpha$ for all $a \in A(G)$. One can easily see that

$$\rho_\alpha(ab) - a \cdot \rho_\alpha(b) \to 0, \quad \rho_\alpha(ab) - \rho_\alpha(a) \cdot b \to 0, \quad \phi \circ \pi_{A(G)} \circ \rho_\alpha(a) \to \phi(a),$$

for all $a \in A(G)$. Therefore $A(G)$ is approximately ϕ-biprojective, for all $\phi \in \Delta(A(G))$. □

For a locally compact group G, $M(G)$ is denoted the measure algebra with respect to G. We know that $L^1(G)$ is a closed ideal of $M(G)$. So we can extend every character of $L^1(G)$ to $M(G)$. So the augmentation character ϕ_1 can be extended to $M(G)$ which we denote it by ϕ_1 again.

Theorem 3.3. *The measure algebra $M(G)$ is approximate ϕ_1-biprojective if and only if G is amenable.*

Proof. Let $M(G)$ be approximate ϕ_1-biprojective. We know that $M(G)$ is unital. Applying Proposition 2.1 gives that $M(G)$ is approximate left ϕ_1-amenable. Since $L^1(G)$ is a closed ideal of $M(G)$ and $\phi_1|_{L^1(G)} \neq 0$, by [13, Lemma 3.1] $L^1(G)$ is approximate left ϕ_1-amenable. By Theorem 3.1, G is amenable.

Conversely, Suppose that G is amenable. So by Johnson's theorem $L^1(G)$ is an amenable Banach algebra[18]. Hence $L^1(G)$ is left and right ϕ_1-amenable. So there exist bounded nets (a_α) and (b_α) in $L^1(G)$ such that

$$b_\alpha b - \phi_1(b)b_\alpha \to 0, \quad aa_\alpha - \phi_1(a)a_\alpha \to 0, \quad \phi_1(a_\alpha) = 1 \qquad (a, b \in L^1(G)).$$

Pick $i_0 \in L^1(G)$ such that $\phi_1(i_0) = 1$. Set $m_\alpha = i_0a_\alpha \otimes b_\alpha i_0 \in M(G) \otimes_p M(G)$. Thus

$$\begin{aligned} am_\alpha - m_\alpha a &= ai_0a_\alpha \otimes b_\alpha i_0 - \phi_1(a)i_0a_\alpha \otimes b_\alpha i_0 \\ &\quad + \phi_1(a)i_0a_\alpha \otimes b_\alpha i_0 - i_0a_\alpha \otimes b_\alpha i_0 a \\ &= (ai_0a_\alpha - \phi_1(a)i_0a_\alpha) \otimes b_\alpha i_0 + i_0a_\alpha \otimes (b_\alpha i_0 a - \phi_1(a)b_\alpha i_0) \to 0, \end{aligned} \tag{3.2}$$

and

$$\phi_1 \circ \pi_{M(G)}(i_0a_\alpha \otimes b_\alpha i_0) = \phi_1(i_0a_\alpha b_\alpha i_0) = \phi_1(a_\alpha)\phi_1(b_\alpha) = 1,$$

for each $a \in M(G)$. Define $\rho_\alpha : A \to A \otimes_p A$ by $\rho_\alpha(a) = a \cdot m_\alpha$. It is easy to see that

$$\rho_\alpha(ab) - a \cdot \rho_\alpha(b) \to 0, \quad \rho_\alpha(ab) - \rho_\alpha(a) \cdot b \to 0, \quad \phi \circ \pi_{M(G)} \circ \rho_\alpha(a) \to \phi(a),$$

for all $a \in M(G)$. It finishes the proof. □

Corollary 3.4. *The measure algebra $M(G)$ is approximate character biprojective if and only if G is discrete and amenable.*

Proof. Let $M(G)$ be approximate character biprojective. We know that $M(G)$ha s unit. Using Proposition 2.1, approximate character biprojectivity gives that $M(G)$ is approximate character amenable. By [2, Theorem 7.2] G is discrete and amenable.

Conversely, Suppose that G is discrete and amenable. By [9, Proposition 4.2] $M(G)$ is pseudo-amenable. Hence one can easily see that $M(G)$ is approximate character biprojective. □

In fact, in the proof of above Corollary we showed if a Banach algebra A is pseudo-amenable, then A is approximately ϕ-biprojective. In the following example we show that the converse of this fact is not true.

Example 3.5. Suppose that G is an infinite compact group. So the compactness of G gives that $\widehat{G} \subseteq L^\infty(G) \subseteq L^1(G)$. Therefore each $\rho \in \widehat{G}$ implies that

$$f\rho(x) = \int f(y)\rho(y^{-1}x)dy = \rho(x)\int f(y)\rho(y^{-1})dy = \rho(x)\int f(y)\overline{\rho(y)}dy = \phi_\rho(f)\rho(x)$$

and

$$\phi_\rho(\rho) = \int_G \rho(x)\overline{\rho(x)}dx = \int_G 1dx = 1, \quad (f \in L^1(G)),$$

where the normalized left Haar measure on G considered here, for all $x \in G$. We know that $\rho \in Ł^1(G)$. Then $f \mapsto \rho f$ and $f \mapsto f\rho$ become w^*-continuous maps on $L^1(G)^{**}$. So for $\tilde{\phi}_\rho \in \Delta(L^1(G)^{**})$ we have

$$\rho f = f\rho = \tilde{\phi}_\rho(f)\rho, \quad \phi_\rho(\rho) = \tilde{\phi}_\rho(\rho) = 1, \quad (f \in L^1(G)^{**}).$$

Define $\eta : L^1(G)^{**} \to L^1(G)^{**} \otimes_p L^1(G)^{**}$ by $\eta(f) = f \cdot \rho \otimes \rho$. Clearly

$$\eta(fg) = f \cdot \eta(g) = \eta(f) \cdot g$$

and

$$\tilde{\phi} \circ \pi_{L^1(G)^{**}} \circ \eta(f) = \tilde{\phi} \circ \pi_{L^1(G)^{**}}(f \cdot \rho \otimes \rho) = \tilde{\phi}(f)$$

for each $f, g \in L^1(G)^{**}$. It deduces that $L^1(G)^{**}$ is approximate ϕ-biprojective. Suppose in contradiction that $L^1(G)^{**}$ is pseudo-amenable. Hence by [9, Proposition 4.2] G is discrete and amenable. Then the compactness of G gives that G is finite which is impossible.

The semigroup S is called *inverse semigroup*, if for each $s \in S$ there exists $s^* \in S$ such that $ss^*s = s^*$ and $s^*ss^* = s$. An inverse semigroup S is called *Clifford semigroup* if for each $s \in S$ there exists $s^* \in S$ such that $ss^* = s^*s$. There exists a partial order on each inverse semigroup S, that is,

$$s \leq t \Leftrightarrow s = ss^*t \quad (s, t \in S).$$

Let $(S, \leq)$ be an inverse semigroup. For each $s \in S$, set $(x] = \{y \in S \,|\, y \leq x\}$. S is called *uniformly locally finite* if $\sup\{|(x]| : x \in S\} < \infty$. Suppose that S is an inverse semigroup and $e \in E(S)$, where $E(S)$ is the set of all idempotents of S. Then $G_e = \{s \in S | ss^* = s^*s = e\}$ is a maximal subgroup of S with respect to e. See [12] as a main reference of semigroup theory.

In the following theorem, we show that for certain semigroup algebras the notion of approximate character biprojectivity is equivalent with pseudo-amenability.

Theorem 3.6. *Let $S = \cup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is approximate character biprojective if and only if $\ell^1(S)$ pseudo-amenable.*

Proof. Let $\ell^1(S)$ be approximate character biprojective. By [16, Theorem 2.16], $\ell^1(S) \cong \ell^1 - \oplus_{e\in E(S)}\ell^1(G_e)$. Since $\ell^1(G_e)$ has a character ϕ_1(at least augmentation character), then this character extends to $\ell^1(S)$ which we denote it again by ϕ_1. Then $\ell^1(S)$ is approximate ϕ_1-biprojective. Because $\phi_1|_{\ell^1(G_e)} \neq 0$ and $\ell^1(G_e)$ is a closed ideal of $\ell^1(S)$, Proposition 2.5 gives that $\ell^1(G_e)$ is approximate ϕ_1-biprojective. We know that $\ell^1(G_e)$ is unital. So Proposition 2.1 gives that $\ell^1(G_e)$ is approximate left ϕ_1-amenable. So by [2, Theorem 7.1], G_e is amenable for all $e \in E(S)$. Thus by [6, Corollary 3.9] $\ell^1(S)$ is pseudo-amenable.
Converse is clear. □

Example 3.7. We give a Banach algebra which is approximate ϕ-biprojective but it is not ϕ-biprojective.

Let $\mathbb{N}_{\max}$ be a semigroup with the operation $m * n = \max\{m, n\}$, for all $m, n \in \mathbb{N}$. The maximal ideal space of this algebra is $\Delta(\ell^1(\mathbb{N}_{\max}))$. It consists of all maps $\phi_n : \ell^1(\mathbb{N}_{\max}) \to \mathbb{C}$ given by $\phi_n(\sum_{i=1}^{\infty} \alpha_i\delta_i) = \sum_{i=1}^{n} \alpha_i$ for every $n \in \mathbb{N} \cup \{\infty\}$, for more information see [4]. Define $m = w^* - \lim \delta_n \otimes \delta_n \in (\ell^1(\mathbb{N}_{\max}) \otimes_p \ell^1(\mathbb{N}_{\max}))^{**}$. Clearly $a \cdot m = m \cdot a$ and $\tilde{\phi_\infty} \circ \pi_{\ell^1(\mathbb{N}_{\max})}(m) = 1$ for all $a \in \ell^1(\mathbb{N}_{\max})$. Following the similar method as in the proof of Theorem 2.4, gives that $\ell^1(\mathbb{N}_{\max})$ is approximate ϕ_∞-biprojective. We assume toward a contradiction that $\ell^1(\mathbb{N}_{\max})$ is ϕ_∞-biprojective. Define

$$m_n = (\delta_n - \delta_{n+1}) \otimes (\delta_n - \delta_{n+1}) \in \ell^1(\mathbb{N}_{\max}) \otimes_p \ell^1(\mathbb{N}_{\max})$$

one can see that

$$am_n = m_n a, \quad \phi_n \circ \pi_{\ell^1(\mathbb{N}_{\max})}(m_n) = 1, \quad (a \in \ell^1(\mathbb{N}_{\max})).$$

By defining $\rho_n(a) = a \cdot m_n$, we can show that $\phi_n \circ \pi_{\ell^1(\mathbb{N}_{\max})} \circ \rho_n(a) = \phi(a)$ for all $a \in \pi_{\ell^1(\mathbb{N}_{\max})}$. It follows that $\pi_{\ell^1(\mathbb{N}_{\max})}$ is ϕ_n-biprojective for all $\phi_n \in \Delta(\ell^1(\mathbb{N}_{\max}))$. Therefore [15, Remark 3.6] and [15, Lemma 3.7] imply the maximal ideal space of $\ell^1(\mathbb{N}_\vee)$ is finite which is impossible, because the maximal ideal space of $\ell^1(\mathbb{N}_\vee)$ is $\mathbb{N} \cup \{\infty\}$.

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