Wavelets and Linear Algebra 8(2) (2022) 1-9

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Wavelets and Linear Algebra

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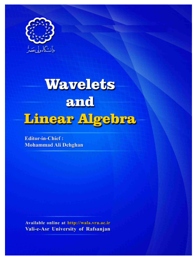

On Fractional Functional Calculus of Positive Operators

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Article Info

Article history:
Received 17 February 2021
Accepted 23 August 2021
Available online 14 March 2022
Communicated by Abbas Salemi

Keywords:
Hilbert space operator, Unbounded normal operator, Fractional functional calculus.

2000 MSC:
47A06, 47A60, 47B15

Abstract

Let $N \in B(H)$ be a normal operator acting on a real or complex Hilbert space H. Define $N^\dagger := N_1^{-1} \oplus 0 : \mathcal{R}(N) \oplus \mathcal{K}(N) \to H$, where $N_1 = N|_{\mathcal{R}(N)}$. Let the *fractional semigroup* $\mathfrak{Fr}(W)$ denote the collection of all words of the form $f_1^\diamond f_2^\diamond \cdots f_k^\diamond$ in which $f_j \in L^\infty(W)$ and $f^\diamond$ is either f or $f^\dagger$, where $f^\dagger = \chi_{\{f\neq 0\}}/(f + \chi_{\{f=0\}})$ and $L^\infty(W)$ is a certain normed functional algebra of functions defined on $\sigma_{\mathbb{F}}(W)$, besides that, $W = W^* \in B(H)$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ indicates the underlying scalar field. The *fractional calculus* $(f_1^\diamond f_2^\diamond \cdots f_k^\diamond)(W)$ on $\mathfrak{Fr}(W)$ is defined as $f_1^\diamond(W) f_2^\diamond(W) \cdots f_k^\diamond(W)$, where $f_j^\dagger(W) = (f_j(W))^\dagger$. The present paper studies sufficient conditions on f_j to ensure such fractional calculus are unbounded normal operators. The results will be extended beyond continuous functions.

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http://doi.org/10.22072/wala.2021.525358.1320

1. Introduction

In this preliminary section we study some basic properties of bounded linear operators between Hilbert spaces needed in the later developments; for reference to spectral theory, selfadjoint operators and functional calculus of bounded operators we suggest [3]-[7] and [9]-[10]. For every Hilbert space $H \neq \{0\}$ with underlying field $\mathbb{F}$, the spectrum of an operator $T \in B(H)$ are defined as follows:

$$\sigma_{\mathbb{F}}(T) \quad = \quad \mathbb{F}\backslash\{\lambda \in \mathbb{F} : \lambda I - T \text{ is bijective}\}. \tag{1.1}$$

An operator $T \in B(H)$ is called selfadjoint, if $T = T^*$. A selfadjoint operator T is called positive or, more precisely, nonnegative if it satisfies $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

In the rest of the paper, to avoid confusion of the notations for "closure of a set" as in $\bar{Z} = Z \cup Z'$ and the "conjugate of a complex number" as in $\bar{z} = \Re\mathfrak{e}(z) - i\, \Im\mathfrak{m}(z)$, the latter will be denoted by z^*. Throughout the paper, $\mathcal{D}(\cdot)$, $\mathcal{R}(\cdot)$, and $\mathcal{K}(\cdot)$ stand for the domain, the range and the kernel (null space) of some linear operator. Also, the symbol $\oplus$ represents the orthogonal direct sum of subspaces or unbounded operators. Sometimes, we replace $M \oplus N$ by $M \dot{+} N$ only to emphasise that the resulting direct sum of the two linear subspaces M and N may not be orthogonal.

In this paper, we study a class of unbounded normal operators. Given an arbitrary operator $A \in B(H)$ of the form $A = B \oplus 0$ with $\mathcal{K}(B) = \{0\}$, its Moor-Penrose inverse $A^\dagger$ will be an unbounded operator defined by

$$A^\dagger = B^{-1} \oplus 0\,; \quad \mathcal{D}(A^\dagger) = \mathcal{R}(A) \oplus \mathcal{K}(A). \tag{1.2}$$

(Note that $\mathcal{R}(A) = \mathcal{R}(B)$.) Identify $H \times K$ with the Hilbert space obtained from the external direct sum $H \oplus K$. For an operator T with graph $\mathcal{G}(T) \subset H \times K$, the inverse T^{-1}, the closure $\overline{T}$, the scalar multiple αT $(\alpha \in \mathbb{C})$ and the adjoint T^* of T are determined by the following graphs:

$$\begin{aligned}
&\mathcal{G}(T^{-1}) = \left\{[y, x]^T : [x, y]^T \in \mathcal{G}(T)\right\} \subset K \times H;\\
&\mathcal{G}(\overline{T}) = \overline{\mathcal{G}(T)} \subset H \times K;\\
&\mathcal{G}(\alpha T) = \left\{[x, \alpha y]^T : [x, y]^T \in \mathcal{G}(T)\right\} \subset H \times K;\\
&\mathcal{G}(T^*) = \mathcal{G}(-T^{-1})^\perp \subset K \times H.
\end{aligned}$$

The operators T^{-1}, $\bar{T}$ and T^* need not be single-valued even if T itself is a single-valued one. The operator T^* is single-valued if and only if T is densely defined. The operator T is closed if $T = \bar{T}$. Note that $T^* = (\bar{T})^* = \overline{T^*}$. Finally, if $T_1 : \mathcal{D}(T_1) \subset H \to K$ and $T_2 : \mathcal{D}(T_2) \subset K \to L$ are linear operators for some Hilbert spaces H, K and L. Then T_2T_1 is the linear operator with

$$\mathcal{D}(T_2T_1) := \{x \in \mathcal{D}(T_1) : \; T_1x \in \mathcal{D}(T_2)\} \text{ and } (T_2T_1)x = T_2(T_1x). \tag{1.3}$$

A densely defined closed operator T is called *normal* if $TT^* = T^*T$. In general, when we write $A = B$ for a pair of operators A and B, we mean $\mathcal{D}(A) = \mathcal{D}(B)$ and $Ax = Bx$ for all x in their common domain. Recall that if ϕ and ψ are given functions, the notation $\phi \subset \psi$ means that $\mathcal{D}(\phi) \subset \mathcal{D}(\psi)$ and $\phi = \psi|_{\mathcal{D}(\phi)}$. Also, in dealing with an unbounded linear operator T, the expression $I + T$ would always mean $I_{\mathcal{D}(T)} + T$.

For reader's convenience, we conclude the section by certain facts from [9] which will be necessary for our main results in the next section. We assume $N = T + i\tau \in B(H)$ is a normal operator, where (T, τ) is a commuting pair of selfadjoint operators on H over the field $\mathbb{F}$. (When $\mathbb{F} = \mathbb{R}$, assume $\tau = 0$.) The notation $\mathcal{A}_b(N) \subset \mathbb{F}^{\mathbb{F}}$ will denote the algebra of all Borel functions bounded on $\sigma_{\mathbb{F}}(N)$. Also, if D is a subset of a topological linear space, then $\bar{\vee}(D)$ will denote the closure of its linear span. (We define $\bar{\vee}(\emptyset) = \{0\}$.) For a vector $x \in H$, the smallest invariant subspace and the smallest reducing invariant subspace of N containing x can be, respectively, formulated as follows:

$$\begin{aligned} \mathcal{Z}(N; x) &:= \bar{\vee}\{N^m x : \ m = 0, 1, 2, \cdots\} \text{ and} \\ \mathcal{Z}(N, N^*; x) &:= \mathcal{Z}(T, \tau; x) = \bar{\vee}\{N^m N^{*n} x : \ m, n = 0, 1, 2, \cdots\} \\ &= \bar{\vee}\{T^m \tau^n x : \ m, n = 0, 1, 2, \cdots\}. \end{aligned}$$

Let $T = T^* \in B(H)$. It is known that the polynomial functional calculus can be extended to an isometric *-algebra isomorphism $f \mapsto f(T) : C_{\mathbb{F}}(\sigma_{\mathbb{F}}(T)) \to B(H)$ such that

$$\|f(T)\| = \|f\|_{\sigma_{\mathbb{F}}(T)}, \ \forall f \in C_{\mathbb{F}}(\sigma_{\mathbb{F}}(T)).$$

More generally, if $N \in B(H)$ is a complex normal operator or a real selfadjoint operator, one can extend this result to a normed linear space of Borel functions including the collection $\mathcal{A}_b(N)$ of all bounded Borel functions defined on $\sigma_{\mathbb{F}}(N)$.

Theorem 1.1. *([9]) Let $N = T + i\tau \in B(H)$ be a real selfadjoint or a complex normal operator. Then, for each $x \in H$, there exist a positive Borel measure $(\mathbb{F}, \mathfrak{B}, \mu_x^N)$ and a unitary operator $U_x : L^2_{\mathbb{F}}(\mu_x^N) \to \mathcal{Z}(N, N^*; x)$ such that for all $f \in \mathbb{F}[x]$, for all $\phi \in L^2_{\mathbb{F}}(\mu_x^N)$ and for all $y \in H$,*

$$U_x f = f(N)x, \ \ \|\mu_x^N\| = \mu_x^N(\sigma_{\mathbb{F}}(N)) = \|x\|^2, \ \text{ and} \tag{1.4}$$

$$\langle U_x\phi, y\rangle = \int \phi(s)\{[U_x^* P y](s)\}^* d\mu_x^N(s), \tag{1.5}$$

where $P : H \to H$ is the orthogonal projection onto $\mathcal{Z}(N, N^*; x)$. Moreover, the mapping $f \mapsto f(N) : \mathcal{A}_b(N) \to B(H)$ defined by $f(N)x = U_x f$ is a *-algebra homomorphism extending the continuous functional calculus, and, if $QN = NQ$ for some $Q \in B(H)$, then $Qf(N) = f(N)Q$.

We now equip $\mathcal{A}_b(N)$ with a norm with respect to which the functional calculus $f \mapsto f(T, \tau) : \mathcal{A}_b(N) \to B(H)$ is an isometric *-algebra isomorphism.

Definition 1.2. ([9]) Suppose the notation and the hypotheses of Theorem 1.1 are valid. For each $f \in \mathcal{A}_b(N)$ define $\nu(f) = \sup_{x \in H} \|f\|_{L^\infty(\mu_x^N)}$. Then ν is a seminorm on $\mathcal{A}_b(N)$ which induces a norm $\|\cdot\|_N$ on the completion $L^\infty(N)$ of the quotient space $\mathcal{A}_b(N)/\{f : \nu(f) = 0\}$.

Theorem 1.3. *([9]) With the notation established in Theorem 1.1 and Definition 1.2, $g(N) = 0$ if $\nu(g) = 0$. Moreover, the functional calculus $f \mapsto f(T, \tau) : L^\infty(N) \to B(H)$ induces an isometric *-algebra isomorphism.*

Let $W \in B(H)$ be a selfadjoint operator. If $g \in L^\infty(W)$, so is g^* and $g^*(W) = g(W)^*$, where g^* denotes the complex conjugate of the complex-valued function g, and $g(W)^*$ denotes the adjoint of the Hilbert space operator $g(W)$. Let the *fractional semigroup* $\mathfrak{Fr}(W)$ denote the collection of all words of the form $f_1^\diamond f_2^\diamond \cdots f_k^\diamond$ in which $f_j \in L^\infty(W)$ and $f^\diamond$ is either f or $f^\dagger$, where $f^\dagger = \chi_{\{f\neq 0\}}/(f + \chi_{\{f=0\}})$. In other words,

$$f^\dagger(x) := \frac{1}{f(x)}, \text{ if } f(x) \neq 0 \text{ and } f^\dagger(x) := 0, \text{ otherwise.}$$

Also, $L^\infty(W)$ is a certain normed functional algebra of functions defined on $\sigma_{\mathbb{F}}(W)$. The *fractional calculus* $(f_1^\diamond f_2^\diamond \cdots f_k^\diamond)(W)$ on $\mathfrak{Fr}(W)$ is defined as $f_1^\diamond(W)\, f_2^\diamond(W) \cdots f_k^\diamond(W)$, where $f_j^\dagger(W) = (f_j(W))^\dagger$.

In the following section, we study certain properties of such operator words and, in the third section, we find sufficient conditions on the functions f_i to assure the normality of such unbounded operators.

2. Noncommutativity

In this section, assume $W \in B(H)$ is a selfadjoint operator on a real or complex Hilbert space H and study the properties of the operator words $f_1^\diamond(W) f_2^\diamond(W) \cdots\ f_k^\diamond(W)$ as defined in the previous section. The fractional functional calculus $(f, g) \mapsto f(W)g(W)^\dagger$ (for $f, g \in C(\sigma(W))$) has been used by von Neumann in representaions such as $T^*T = (I - W)W^{-1}$ for any densely defined closed linear operator $T : \mathcal{D}(T) \subset H \to K$, where $W = (I + T^*T)^{-1} \in B(H)$ is the von Neumann generator of T. For more details about the von Neumann generator of an operatotor see [2]. First we need some notations. For $f, g \in L^\infty(W)$, define

$$T \;:=\; f(W)g(W)^\dagger, \tag{2.1}$$
$$\theta \;:=\; W|_{\mathcal{K}(g(W))} \quad \text{and} \quad \omega := W|_{\overline{\mathcal{R}}(g(W))}. \tag{2.2}$$

and verify that

$$\begin{cases} H &= H_1 \oplus H_2, \text{ where } H_1 = \mathcal{K}(g(W)), \text{ and } H_2 = \overline{\mathcal{R}}(g(W)); \\ f(W) &= f(\theta) \oplus f(\omega), \text{ and } f(W)^\dagger = f(\theta)^\dagger \oplus f(\omega)^\dagger; \\ g(W) &= 0 \oplus g(\omega), \text{ and } g^*(W) = 0 \oplus g^*(\omega); \\ g(W)^\dagger &= 0 \oplus g(\omega)^{-1} \text{ and } g^*(W)^\dagger = 0 \oplus g^*(\omega)^{-1}. \end{cases} \tag{2.3}$$

Note that if $\nu(g - h) = 0$ for some nowhere vanishing h defined on $\sigma_{\mathbb{F}}(W)$, then $g^\dagger = 1/h$ and $g(W)^\dagger = g(W)^{-1}$. Therefore, it is unamiguous to define $(1/g)(W) = g^\dagger(W) = g(W)^{-1}$. Furthermore, the following assertions are obvious.

$$\begin{aligned} &W = \theta \oplus \omega \;\text{, and, } \;\mathcal{R}(g(W)) = \mathcal{R}(g(\omega)); \qquad (2.4)\\ &T = 0 \oplus f(\omega)g(\omega)^{-1};\\ &\mathcal{D}(T) = \mathcal{D}(g(W)^\dagger) = \mathcal{K}(g(W)) \oplus \mathcal{R}(g(\omega));\\ &\mathcal{K}(T) = \mathcal{K}(g(W)) \oplus \mathcal{K}(f(\omega));\\ &\mathcal{R}(T) = \mathcal{R}(f(\omega)) = f(W)(\overline{\mathcal{R}}(g(W))). \end{aligned}$$

The verification of the less obvious fact $\mathcal{R}(g(W)) = \mathcal{R}(g(\omega))$ follows from the equality $g(W)(x_1 \oplus x_2) = g(W)x_2 = g(\omega)x_2$ for all $x_1 \oplus x_2 \in H_1 \oplus H_2$. If W is a bounded injective positive selfadjoint operator and if $\alpha > 0$, then $W^{-\alpha}$ means $(W^\alpha)^{-1}$ in its algebraic sense with $\mathcal{D}((W^\alpha)^{-1}) = \mathcal{R}(W^\alpha)$. (Note that $W^\alpha = h(W)$, where $h(t) := t^\alpha$ for $0 \le t \le \|W\|$.)

Proposition 2.1. *Let $W \in B(H)$ be a positive operator and, let $f, g \in L^\infty(W)$. The following assertions are true.*

(I) $f(W)g(W) = g(W)f(W)$*;*

(II) $f(W)g(W)^\dagger \subset g(W)^\dagger f(W)$*;*

(III) if $\sigma(W)$ has an accumulation point, then there exist f, g such that the linear operators $f(W)^\dagger g(W)^\dagger$ and $g(W)^\dagger f(W)^\dagger$ are not comparable .

Proof. Fact (I) is clear. For (II), let θ, ω be as in (2.2). Thus, $g(\omega)$ is injective and $g(\theta) = 0$. Therefore,

$$f(W)g(W)^\dagger = 0 \oplus f(\omega)g(\omega)^\dagger, \quad g(W)^\dagger f(W) = 0 \oplus g(\omega)^\dagger f(\omega) \tag{2.5}$$

in which $g(\omega)^\dagger = g(\omega)^{-1}$. Thus, it is sufficient to show that $f(\omega)g(\omega)^{-1} \subset g(\omega)^{-1}f(\omega)$. First we show that $\mathcal{R}(g(\omega)) \subset \mathcal{D}(g(\omega)^{-1}f(\omega))$, and this follows from the fact that $f(\omega)$ leaves $\mathcal{R}(g(\omega))$ invariant. To show $f(\omega)g(\omega)^{-1}x = g(\omega)^{-1}f(\omega)x$ for all $x = g(\omega)\xi \in \mathcal{R}(g(\omega))$, observe that

$$\begin{aligned} g(\omega)^{-1}f(\omega)x &= g(\omega)^{-1}f(\omega)g(\omega)\xi = g(\omega)^{-1}g(\omega)f(\omega)\xi \\ &= f(\omega)\xi = f(\omega)g(\omega)^{-1}g(\omega)\xi = f(\omega)g(\omega)^{-1}x. \end{aligned}$$

This proves (II). For (III), choose a strictly monoton sequence $\{\lambda_n\} \subset \sigma(W)$ converging to some $\lambda \in \sigma(W)$. Define $r_n = \min\{|\lambda_n - \lambda_{n-1}|, |\lambda_n - \lambda_{n+1}|\}/2$ for $n \ge 2$ and define $r_1 = |\lambda_2 - \lambda_1|/2$. Next, define $\varphi_n \in C_{\mathbb{F}}(\sigma(W))$ by

$$\varphi_n(t) = 2^{-n}(1 - r_n^{-1}|t - \lambda_n|)\, \mathcal{X}_{[\lambda_n - r_n\ ,\ \lambda_n + r_n]}\ , \quad n = 1, 2, \cdots .$$

Now, write

$$f(t) = \sum_{n=1}^{\infty} \varphi_{2n}(t) \ \text{, and, } \ g(t) = \sum_{n=1}^{\infty} \varphi_{2n-1}(t).$$

Since $f(W)g(W) = g(W)f(W) = 0$, it follows that $\mathcal{R}(g(W)) \subset \mathcal{K}(f(W))$, $\mathcal{R}(f(W)) \subset \mathcal{K}(g(W))$. Consider the direct sum $H = H_1 \oplus H_2 = (K_1 \oplus K_2) \oplus H_2$ in which

$$K_1 = \mathcal{K}(f(\theta)), \quad K_2 = \bar{\mathcal{R}}(f(\theta)) = \bar{\mathcal{R}}(f(W)). \tag{2.6}$$

Moreover, $W = \theta_1 \oplus \theta_2 \oplus \omega$ where $\theta_i = \theta\,|_{K_i}$ for $i = 1, 2$ and $f(\theta) = 0 \oplus f(\theta_2)$. Then

$$f(W)^\dagger = 0 \oplus f(\theta_2)^{-1} \oplus 0, \quad g(W)^\dagger = 0 \oplus 0 \oplus g(\omega)^{-1}.$$

Thus,

$$\begin{aligned} f(W)^\dagger g(W)^\dagger = 0 \quad &\text{on} \quad \mathcal{D}(f(W)^\dagger g(W)^\dagger) = K_1 \dot{+} K_2 \dot{+} \mathcal{R}(g(\omega)) \\ g(W)^\dagger f(W)^\dagger = 0 \quad &\text{on} \quad \mathcal{D}(g(W)^\dagger f(W)^\dagger) = K_1 \dot{+} \mathcal{R}(f(\theta_2)) \dot{+} H_2. \end{aligned}$$

Since $f(W) = 0 \oplus f(\theta_2) \oplus 0$ and since $2^{-2n} = f(\lambda_{2n}) \in \sigma(f(W))$, it follows that $2^{-2n} \in \sigma(f(\theta_2))$ and, hence, $0 \in \sigma(f(\theta_2))$. Similarly, $0 \in \sigma(g(\omega))$. Since $f(\theta_2)$ and $g(\omega)$ are injective, it follows from the Banach inverse mapping theorem that $\mathcal{R}(f(\theta_2)) \neq K_2$ and $\mathcal{R}(g(\omega)) \neq H_2$. Therefore, $\mathcal{D}(f(W)^\dagger g(W)^\dagger) \neq \mathcal{D}(g(W)^\dagger f(W)^\dagger)$ and, hence, $f(W)^\dagger g(W)^\dagger \neq g(W)^\dagger f(W)^\dagger$. □

Corollary 2.2. *Let $W \in B(H)$ be a positive operator and, for $f \in L^\infty(W)$, let $f(W)^\diamond$ denote either $f(W)$ or $f(W)^\dagger$. Define $S_W = \Pi_{i=1}^{k} f_{k+1-i}(W)^\diamond$ for some functions $f_1, f_2, \cdots, f_k \in L^\infty(W)$. Define $J^\dagger = \{j : f_j(W)^\diamond = f_j(W)^\dagger;\ 1 \le j \le k\}$ ordered by its hereditary order from $J := \{1 < 2 < 3 < \cdots < k\}$. Then*

$$[\Pi_{j\notin J^\dagger} f_j(W)][\Pi_{j\in J^\dagger} f_j(W)^\dagger] \subset S_W \subset [\Pi_{j\in J^\dagger} f_j(W)^\dagger][\Pi_{j\notin J^\dagger} f_j(W)]. \tag{2.7}$$

The hereditary order in $J^\dagger$ is important and may affect the result if changed.

Proof. In Proposition 2.1, we have proven (2.7) for two functions and the general case follows from induction on the number of consecutive interchanges. □

Theorem 2.3. *For the linear operator T defined in (2.1) the following assertions are true.*

(i) $T^* \supset g^*(W)^\dagger f^*(W) = 0 \oplus g^*(\omega)^{-1} f^*(\omega) \supset f^*(W) g^*(W)^\dagger = 0 \oplus f^*(\omega) g^*(\omega)^{-1}$;

(ii) If f and g are real-valued, then T is symmetric; i.e., $T \subset T^*$;

(iii) The operator T need not be normal.

(iv) $g(W)^\dagger g^*(W)^\dagger = |g|^2(W)^\dagger$*; in particular, $g(W)^\dagger$ and $g^*(W)^\dagger$ commute.*

Proof. Except for the inclusion $T^* \supset 0 \oplus g^*(\omega)^{-1} f^*(\omega)$, the remainder of (i)-(ii) are immediate consequences of Theorem 2.2. To prove this particular inclusion, note that $T = 0 \oplus \tau$ and $T^* = 0 \oplus \tau^*$, where $\tau = f(\omega) g(\omega)^{-1} : \mathcal{R}(g(\omega)) \subset H_2 \to H_2$. Therefore, we have to show that $\tau^* \supset g^*(\omega)^{-1} f^*(\omega)$. For the latter, let $u \in \mathcal{D}(g^*(\omega)^{-1} f^*(\omega))$. Hence, $f^*(\omega)u = g^*(\omega)v$ for some $v \in H_2$. Then, for $w = g(\omega)\xi \in \mathcal{D}(\tau)$,

$$\langle g^*(\omega)^{-1} f^*(\omega)u, w\rangle = \langle f^*(\omega)u, \xi\rangle = \langle u, f(\omega)\xi\rangle = \langle u, \tau w\rangle.$$

Since u and w were arbitrary selections, $g^*(\omega)^{-1} f^*(\omega) \subset \tau^*$.

For Part (iii), let $f(t) = g(t) \equiv t$ and assume W is an injective positive operator whose inverse W^{-1} is not bounded. Then $\mathcal{R}(W)$ is not closed and, hence, the densely defined operator $T = WW^{-1} = I_{\mathcal{R}(W)}$ is not closed. Thus T is not normal or selfadjoint. However, it is true that $T \subset T^* = I$ and $TT^* = T = T^*T$.

In view of (2.4), Part (iv) is equivalent with

$$g(\omega)^{-1} g^*(\omega)^{-1} = |g|^2(\omega)^{-1}.$$

If $u \in \mathcal{D}(g(\omega)^{-1} g^*(\omega)^{-1})$, then $u = g^*(\omega)v$ and $v = g(\omega)w$ for some $v, w \in H$. Then $u = g^*(\omega)g(\omega)w = |g|^2(\omega)w \in \mathcal{D}(|g|^2(\omega)^{-1})$ and $g(\omega)^{-1} g^*(\omega)^{-1} u = w = |g|^2(\omega)^{-1} u$. Thus, $g(\omega)^{-1} g^*(\omega)^{-1} \subset |g|^2(\omega)^{-1}$. For the inverse inclusion, let $u \in \mathcal{D}(|g|^2(\omega)^{-1})$. Then $u = |g|^2(\omega)\eta = g^*(\omega)g(\omega)\eta$ for some $\eta \in H$. Hence, $g(\omega)^{-1} g^*(\omega)^{-1} u = \eta = |g|^2(\omega)^{-1} u$. □

3. Commutativity

In this section, we study sufficient conditions which imply the commutativity of

$$f(W)g(W)^{\dagger} = g(W)^{\dagger}f(W)$$

and conclude the normality of the operator $T = f(W)g(W)^{\dagger}$ for Borel functions f, g. The proof is based on a condition involving the ranges of $f(W)$ and $g(W)$. The following key lemma provides sufficient conditions on f and g to ensure the normality of the operator T. The lemma will be often used throughout the remainder of the paper and may be referred to as the *key lemma*. The consequences of a special case of the key lemma and other results of the paper are used in [1] and [2] to prove the existence of dual frames for algebraic frames and to study the structure of unbounded linear operators.

Lemma 3.1. **(Key lemma)** *Let W be a bounded positive operator and let $f,\ g \in \mathcal{A}_b(W)$. The following assertions are true.*

(i) If $x \in \mathcal{R}(g(W))$ for some $x \in H$, then $f(W)x \in \mathcal{R}(g(W))$.

(ii) Assume $f(a) \neq 0$ for some $a \in \sigma_{\mathbb{F}}(W)$, and define h on $\sigma_{\mathbb{F}}(W)$ by

$$h(t) = \begin{cases} \frac{f(t)-f(a)}{g(t)}, & t \in \sigma(W)\backslash\{a\},\ g(t) \neq 0, \\ 0, & \textit{otherwise}. \end{cases}$$

If $h \in \mathcal{A}_b(W)$ and $f(W)x \in \mathcal{R}(g(W))$ for some $x \in H$, then $x \in \mathcal{R}(g(W)) = \mathcal{R}(g^*(W))$.

Proof. In part (i), there exists $\xi \in H$ such that $x = g(W)\xi$. Thus, Proposition 2.1 implies that $f(W)x = f(W)g(W)\xi = g(W)f(W)\xi \in \mathcal{R}(g(W))$.

For part (ii), with the hypotheses of the lemma, $f(W)x - f(a)x = g(W)h(W)x$. Now, choose $y \in H$ such that $f(W)x = g(W)y$. Then $x = g(W)[y - h(W)x]/f(a)$. The last equality follows from the fact that $\mathcal{R}(N) = \mathcal{R}(N^*)$ for any bounded normal operator N. □

Theorem 3.2. *Let W, f, g, a and ℓ be as in the Key Lemma 3.1. Let $T = f(W)g(W)^{\dagger}$. Then the various parts of Theorem 2.3 can be sharpened as follows.*

(i) $T = \bar{T} = g(W)^{\dagger}f(W)$.

(ii) $f^*(W)g(W)^{\dagger} = g(W)^{\dagger}f^*(W)$.

(iii) $T^* = f^*(W)g^*(W)^{\dagger} = g^*(W)^{\dagger}f^*(W)$, $\mathcal{D}(T^*) = \mathcal{R}(g^*(W)) = \mathcal{R}(g(W)) = \mathcal{D}(T)$.

(iv) T is normal and $T^*T = TT^* = |f|^2(W)|g|^2(W)^{\dagger} = |g|^2(W)^{\dagger}|f|^2(W)$.

(v) If f and g are real-valued, then T is selfadjoint.

Proof. As in the proof of Proposition 2.1, it is sufficient to replace W by $\omega = W|_{\bar{\mathcal{R}}(g(W))}$ and, hence, $\omega^{\dagger} = \omega^{-1}$. It was shown that $T = 0 \oplus f(\omega)g(\omega)^{-1}$, $\mathcal{D}(T) = \mathcal{K}(g(\omega)) \oplus \mathcal{R}(g(\omega))$ and $\mathcal{R}(T) = \mathcal{R}(f(\omega))$. We continue the proof with the notation of the proof of Theorem 2.3. Let $\tau = f(\omega)g(\omega)^{-1} : \mathcal{R}(g(\omega)) \subset H \to K$.

For Part (i), it is sufficient to show that τ is closed and $\mathcal{D}(g(\omega)^{-1}f(\omega)) \subset \mathcal{D}(\tau)$. Let $x_n \in \mathcal{R}(g(\omega))$ be such that $\lim_n x_n = x$ and $\lim_n \tau x_n = y$ for some $x \in H$ and some $y \in K$. We must show that $x \in \mathcal{R}(g(\omega))$ and $y = \tau x$. Observe that

$$\begin{aligned} f(\omega)x &= \lim_n f(\omega)x_n = \lim_n f(\omega)g(\omega)g(\omega)^{-1}x_n \\ &= \lim_n g(\omega)\tau x_n = g(\omega)\lim_n \tau x_n = g(\omega)y. \end{aligned}$$

By the key lemma, $x \in \mathcal{R}(g(\omega))$ and, by Part (i) of Theorem 2.2,

$$\tau x = f(\omega)g(\omega)^{-1}x = g(\omega)^{-1}f(\omega)x = g(\omega)^{-1}g(\omega)y = y.$$

Next, let $x \in \mathcal{D}(g(\omega)^{-1}f(\omega))$. Since $f(\omega)x \in \mathcal{R}(g(\omega))$, it follows from the Key Lemma 3.1, that $x \in \mathcal{R}(g(\omega)) = \mathcal{D}(\tau)$.

For Part (ii) also, it is sufficient to show that $\mathcal{D}(g(\omega)^{-1}f^*(\omega)) \subset \mathcal{D}(f^*(\omega)g(\omega)^{-1})$. Let $x \in \mathcal{D}(g(\omega)^{-1}f^*(\omega))$. Thus, $f^*(\omega)x \in \mathcal{R}(g(\omega)) = \mathcal{R}(g^*(\omega))$. The Key lemma implies that $x \in \mathcal{R}(g^*(\omega)) = \mathcal{R}(g(\omega)) = \mathcal{D}(f^*(\omega)g(\omega)^{-1})$.

In (iii), observe that $T^* = 0 \oplus \tau^*$ and let $u \in \mathcal{D}(\tau^*)$. Then $(-\tau^*u) \oplus u \perp \mathcal{G}(\tau)$ and, for all $\xi \in H_2$,

$$\begin{aligned} 0 &= \langle g(\omega)\xi \oplus \tau g(\omega)\xi, -\tau^*u \oplus u\rangle = \langle g(\omega)\xi \oplus f(\omega)g(\omega)^{-1}g(\omega)\xi, -\tau^*u \oplus u\rangle \\ &= \langle g(\omega)\xi, -\tau^*u\rangle + \langle f(\omega)\xi, u\rangle = \langle \xi, -g^*(\omega)\tau^*u + f^*(\omega)u\rangle. \end{aligned}$$

The Key Lemma applied to the pair f^* and g^* yields $f^*(\omega)u = g^*(\omega)\tau^*u$ and, hence, $u \in \mathcal{R}(g^*(\omega)) = \mathcal{D}(f^*(\omega)g^*(\omega)^{-1})$. Therefore, in view of Part (i), $\tau^*u = g^*(\omega)^{-1}f^*(\omega)u = f^*(\omega)g^*(\omega)^{-1}u$ which implies that $\tau^* \subset f^*(\omega)g^*(\omega)^{-1}$ with $\mathcal{D}(\tau^*) \subset \mathcal{R}(g^*(\omega)) = \mathcal{R}(g(\omega)) = \mathcal{D}(\tau)$. The inverse inclusions follow from Part (ii) of Theorem 2.2.

To show $TT^* = T^*T$, apply the last part of Theorem 2.2 as well as the previous parts of the present theorem to conclude that

$$\begin{aligned} TT^* &= f(\omega)g(\omega)^{-1}f^*(\omega)g^*(\omega)^{-1} = g(\omega)^{-1}f(\omega)g^*(\omega)^{-1}f^*(\omega) \\ &= f^*(\omega)f(\omega)g^*(\omega)^{-1}g(\omega)^{-1} = g(\omega)^{-1}g^*(\omega)^{-1}f(\omega)f^*(\omega) \\ &= |f|^2(\omega)|g|^2(\omega)^{-1} = |g|^2(\omega)^{-1}|f|^2(\omega). \end{aligned}$$

Writing similar expressions for T^*T, it follows that $TT^* = T^*T$. This proves Part (iv) from which Part (v) follows immediately. □

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