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On higher rank numerical hulls of normal matrices

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Abstract

In this paper, some algebraic and geometrical properties of the rank−k numerical hulls of normal matrices are investigated. A characterization of normal matrices whose rank−1 numerical hulls are equal to their numerical range is given. Moreover, using the extreme points of the numerical range, the higher rank numerical hulls of matrices of the form $A_1 \oplus iA_2$, where A_1 and A_2 are Hermitian, are investigated. The higher rank numerical hulls of the basic circulant matrix are also studied.

1. Introduction and preliminaries

Let $M_{n\times m}$ be the vector space of all $n \times m$ complex matrices. For the case $n = m$, $M_{n\times n}$ is denoted by M_n. Throughout the paper, k, m and n are considered as positive integers, and $k \leq n$.

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Moreover, I_k denotes the $k \times k$ identity matrix, and $\mathcal{I}_{n\times k} = \{X \in M_{n\times k} : X^*X = I_k\}$, that is the set of all $n \times k$ isometry matrices. Motivated by the study of convergence of iterative methods in solving linear systems, researchers studied the *polynomial numerical hull of order m* of a matrix $A \in M_n$ which is defined and denoted, e.g., see [11], by

$$V^m(A) = \left\{\lambda \in \mathbb{C} : \ |p(\lambda)| \le \|p(A)\| \text{ for all } p \in \mathbb{P}_m\right\},$$

where $\mathbb{P}_m$ is the set of all scalar polynomials of degree m or less, and $\|.\|$ is the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidean vector norm). This is a set designed to give more information than the spectrum alone can provide about the behavior of the matrix A under the action of polynomials and other functions. For the case $m = 1$, $V^1(A)$ reduces to the classical numerical range of A; i.e., $V^1(A) = W(A) := \{x^*Ax : x \in \mathbb{C}^n, \ x^*x = 1\}$, which is useful in studying and understanding matrices and operators, and has many applications in numerical analysis, differential equations, systems theory (see [8], [9, Chapter 1] and refrences cited there). It is known that $V^1(A)$ is convex with $conv(\sigma(A)) \subseteq V^1(A)$, where $conv(\sigma(A))$ denotes the convex hull of the spectrum $\sigma(A)$ of A, and the equality holds if A is a normal matrix. In the following proposition, we list some properties of the polynomial numerical hulls of matrices which will be useful in our discussion. For more information, see [1, 4, 5, 7].

Proposition 1.1. *Let $A \in M_n$. Then the following assertions are true:*

(*i*) $\{V^m(A) = \{\mu \in \mathbb{C} : (\mu, \ldots, \mu^m) \in conv(W(A, \ldots, A^m))\}\}$*, where for* $A_1, \ldots, A_m \in M_n$, $W(A_1, \ldots, A_m) = \{(x^*A_1x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}$ *is the joint numerical range;*

(*ii*) *If A is a normal matrix, then $\partial(W(A)) \cap V^2(A) \subseteq \sigma(A)$, where $\partial(.)$ denotes the boundary;*

(*iii*) *If A is a normal matrix whose spectrum consists of three non-colinear points and λ_0 is the orthocenter of the triangle $\sigma(A)$, then $V^2(A) = \sigma(A) \cup (\{\lambda_0\} \cap W(A))$;*

(*iv*) *Let $A = diag(\alpha, -\beta, \imath\gamma, \imath\theta)$, where α, β, γ and θ are distinct positive numbers. Then $V^2(A) = \sigma(A)$ if and only if $\sigma(A)$ is an orthocentric system;*

(*v*) *Let $A = A_1 \oplus \imath A_2$, where $A_1^* = A_1$ and A_2 is a semi-definite matrix. Then $V^3(A) = \sigma(A)$; and if A_1 and A_2 are semi-definite matrices, then $V^2(A) = \sigma(A)$.*

One of the other motivation concerns the study of higher rank numerical ranges of matrices which are useful in quantum error correction [12]. In this connection, the *rank-k numerical range* of $A \in M_n$ is defined and denoted by

$$\begin{aligned}\Lambda_k(A) &= \left\{\lambda \in \mathbb{C} \ : \ X^*AX = \lambda I_k \text{ for some } X \in \mathcal{I}_{n\times k}\right\} \\ &= \left\{\lambda \in \mathbb{C} \ : \ PAP = \lambda P \text{ for some rank-k orthogonal projection } P \in M_n\right\}.\end{aligned}$$

For $k = 1$, $\Lambda_1(A)$ coincides with $W(A)$. The sets $\Lambda_k(A)$ are convex for any $k \in \{1, \ldots, n\}$, and they are generally called *higher rank numerical ranges of A*. In the following proposition, the higher rank numerical ranges of normal matrices, which will be useful in our discussion, are characterized.

Proposition 1.2. *([10, Corollary 2.4]) Let $A \in M_n$ be a normal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ (counting multiplicities). Then*

$$\Lambda_k(A) = \bigcap_{1 \leqslant j_1 < \cdots < j_{n-k+1} \leqslant n} conv\Big(\{\lambda_{j_1}, \dots, \lambda_{j_{n-k+1}}\}\Big).$$

Recently, the notion of *rank-k numerical hull of order m* of a matrix $A \in M_n$ has been defined in [13], as a generalization of $V^m(A)$ and $\Lambda_k(A)$, and has been denoted by:

$$\mathcal{X}_k^m(A) = \Big\{\lambda \in \mathbb{C} : (\lambda, \lambda^2, \dots, \lambda^m) \in conv(\Lambda_k(A, A^2, \dots, A^m))\Big\},$$

where $\Lambda_k(A_1, \dots, A_m) = \{(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m : \exists X \in \mathcal{I}_{n\times k} \ s.t. \ X^*A_jX = \lambda_j I_k, \ j = 1, \dots, m\}$ is the *joint rank-k numerical range* of $(A_1, \dots, A_m) \in \underbrace{M_n \times \cdots \times M_n}_{m-times}$. It is clear that $\mathcal{X}_1^m(A) = V^m(A)$ and $\mathcal{X}_k^1(A) = \Lambda_k(A)$. The sets $\mathcal{X}_k^m(A)$, where $k \in \{1, 2, \dots, n\}$ and $m \in \mathbb{N}$, are generally called the *higher rank numerical hulls* of A. The *rank−k spectrum* of a matrix $A \in M_n$, as a generalization of the spectrum of A, is defined and denoted in [13] by $\sigma_k(A) = \{\lambda \in \mathbb{C} : dim(ker(\lambda I_n - A)) \geq k\}$. Next, we list some properties of the higher rank numerical hulls and the rank-k spectrum of matrices which will be useful in our discussion. One may see [13] for more details.

Proposition 1.3. *Let $A \in M_n$. Then the following assertions are true:*

(i) $\sigma_k(A) \subseteq \sigma_{k-1}(A) \subseteq \cdots \subseteq \sigma_1(A) = \sigma(A)$;

(ii) $\sigma_k(A) \subseteq \mathcal{X}_k^m(A) \subseteq \mathcal{X}_{k-1}^m(A) \subseteq \cdots \subseteq \mathcal{X}_1^m(A) = V^m(A) \subseteq V^{m-1}(A) \subseteq \cdots \subseteq V^1(A) = W(A)$;

(iii) $\sigma_k(A) \subseteq \mathcal{X}_k^m(A) \subseteq \mathcal{X}_k^{m-1}(A) \subseteq \cdots \subseteq \mathcal{X}_k^1(A) = \Lambda_k(A) \subseteq \Lambda_{k-1}(A) \subseteq \cdots \subseteq \Lambda_1(A) = W(A)$;

(iv) $\mathcal{X}_k^m(A) \subseteq V^m(A) \cap \Lambda_k(A)$;

(v) $\mathcal{X}_k^m(\alpha A + \beta I_n) = \alpha \mathcal{X}_k^m(A) + \beta$, *where* $\alpha, \beta \in \mathbb{C}$;

(vi) *If A is Hermitian and $m \geq 2$, then* $\mathcal{X}_k^m(A) = \sigma_k(A)$;

(vii) *If A is unitary, then* $\mathcal{X}_k^m(A) \cap \sigma(A) = \sigma_k(A)$;

(viii) *If $m, k \geq 2$, $n = 2k$, and A is a unitary matrix with distinct eigenvalues, then* $\mathcal{X}_k^m(A) = \emptyset$.

In this paper, we are going to study some algebraic and geometrical properties of the higher rank numerical hulls of normal matrices. For this, in Section 2, we find the conditions that under them $\mathcal{X}_k^m(A)$ is empty or a nonempty set in $\mathbb{C}$. We also characterize the higher rank numerical hulls of a unitary matrix such that its spectrum lies in a semicircle. Moreover, we give a characterization of normal matrices A whose $\mathcal{X}_1^m(A) = V^m(A) = W(A)$. In Section 3, by using the extreme points of the numerical range, we study the higher rank numerical hulls of matrices of the form $A_1 \oplus iA_2$, where A_1 and A_2 are Hermitian. In Section 4, we study the higher rank numerical hulls of the basic circulant matrix.

2. Some general properties

At first we show that there are matrices such that their higher rank numerical hulls are nonempty. It will also be useful in our discussion. Note that by Proposition 1.3(ii), $\sigma(A) \subseteq V^m(A)$, and hence $V^m(A)$ is a nonempty set.

Theorem 2.1. *Let $A \in M_n$. Then the following assertions are true:*

(i) *If $2k > n$, then $\mathcal{X}_k^m(A)$ is empty or a singleton set. Moreover,*

$$\sigma_k(A) \subseteq \mathcal{X}_k^m(A) \subseteq \sigma_{2k-n}(A);$$

(ii) *If $2k > n+1$ and A is a nonderogatory matrix, then $\mathcal{X}_k^m(A) = \emptyset$;*

(iii) $\mathcal{X}_k^m(I_k \otimes A) = V^m(A)$.

Proof. To prove (i), we assume that $\mathcal{X}_k^m(A) \neq \emptyset$ and it contains at least two points such as λ_0 and λ_1. Since $\mathcal{X}_k^m(A) \subseteq \Lambda_k(A)$, there exist $X = [x_1, x_2, \ldots, x_k]$ and $Y = [y_1, y_2, \ldots, y_k]$ in $\mathcal{I}_{n\times k}$ such that $X^*AX = \lambda_0 I_k$ and $Y^*AY = \lambda_1 I_k$. We assume that w is a unit vector in the intersection of column spaces of X and Y. So, there exist $\alpha_i, \beta_i \in \mathbb{C}$, $i = 1, \ldots, k$, such that $\sum_{i=1}^k \alpha_i x_i = w = \sum_{i=1}^k \beta_i y_i$, where $\sum_{i=1}^k |\alpha_i|^2 = 1 = \sum_{i=1}^k |\beta_i|^2$. Therefore, $w^*Aw = \sum_{i=1}^k |\alpha_i|^2 x_i^* A x_i = \lambda_0 \sum_{i=1}^k |\alpha_i|^2 = \lambda_0$. Also, by a similar way, we have $w^*Aw = \lambda_1$. So, $\lambda_0 = \lambda_1$, and hence, $\mathcal{X}_k^m(A)$ is a singleton set. The left inclusion in the other assertion in (i) is trivial. To prove the right inclusion, let $\lambda \in \mathcal{X}_k^m(A)$ be given. Then there exists a rank$-k$ orthogonal projection $P \in M_n$ such that $PAP = \lambda P$. So, the equality $A - \lambda I = P(A - \lambda I)(I - P) + (I - P)(A - \lambda I)$ shows that $rank(A - \lambda I) \leq rank(P(A - \lambda I)(I - P)) + rank((I - P)(A - \lambda I)) \leq 2\, rank(I - P) = 2(n - k)$. Hence, $dim(ker(A - \lambda I)) \geq n - (2n - 2k) = 2k - n$. So, $\lambda \in \sigma_{2k-n}(A)$, and this completes the proof of (i).

The result in (ii) easily follows from (i).

By [13, Proposition 2.4] and Proposition 1.3(ii), $V^m(A) \subseteq \mathcal{X}_k^m(I_k \otimes A) \subseteq V^m(I_k \otimes A)$. By [3, Theorem 3.5$(v)$], $V^m(I_k \otimes A) = V^m(A)$, and hence, the result in (iii) also holds. So, the proof is complete. □

In the following proposition, we characterize the higher rank numerical hulls of unitary matrices whose spectrum lies in a semicircle (including the end points). Recall that for such matrices, we have $V^2(A) = \sigma(A)$; see [4, Theorem 2.8].

Proposition 2.2. *Let $A \in M_n$ be a unitary matrix such that $\sigma(A)$ lies in a semicircle (including the end points). Then*

$$\mathcal{X}_k^m(A) = \begin{cases} W(A) & \text{if } k = m = 1; \\ \sigma(A) & \text{if } k = 1, m > 1; \\ \sigma_k(A) & \text{if } k > 1, m > 1; \\ \Lambda_k(A) & \text{if } k > 1, m = 1. \end{cases}$$

Proof. The result for the cases $k = m = 1$, and $k > 1$ and $m = 1$, follows from Proposition 1.3((*ii*) and (*iii*)). Now, if $k = 1$ and $m > 1$, then $V^2(A) = \sigma(A)$ and hence by Proposition 1.3(*ii*), $\mathcal{X}_1^m(A) = V^m(A) = \sigma(A)$. We now assume that $k, m > 1$. By Proposition 1.3(*ii*), we have $\sigma_k(A) \subseteq \mathcal{X}_k^m(A) \subseteq V^2(A) = \sigma(A)$. So, by the hypothesis and Proposition 1.3(*vii*), we have $\mathcal{X}_k^m(A) = \mathcal{X}_k^m(A) \cap \sigma(A) \subseteq \sigma_k(A)$, and so, $\mathcal{X}_k^m(A) = \sigma_k(A)$. This completes the proof. □

It is known, see for example Proposition 1.1(*iv*), that $V^m(A)$ is not necessarily convex. In the following theorem, we will consider the convexity of $\mathcal{X}_1^m(A)$, where $m \geq 2$ and A is normal.

Theorem 2.3. *Let $2 \leq m \leq n$ be a positive integer, and $A \in M_n$ be a normal matrix. Then the following statements are equivalent:*

(*i*) *$V^m(A) = W(A)$;*

(*ii*) *$V^m(A)$ is convex;*

(*iii*) *A is a scalar matrix.*

Proof. It is enough to prove (*ii*) $\Longrightarrow$ (*iii*); because the other cases are trivial. In view of Proposition 1.3((*i*) *and* (*ii*)), and the fact that $W(A) = conv(\sigma(A))$, we see that $V^2(A) = W(A)$. So, by Proposition 1.1(*ii*), we have

$$\partial W(A) = \partial W(A) \cap V^2(A) \subseteq \sigma(A).$$

Since $\sigma(A)$ is totally disconnected and $\partial W(A)$ is a connected set, the above inclusion shows that $\sigma(A)$ must be a singleton set in $\mathbb{C}$. Hence, the normality of A implies that it is a scalar matrix. So the proof is complete. □

At the end of this section, we take our attention to matrices whose square are Hermitian. Let $A \in M_n$. Recall, e.g., see [9, Definition 1.6.2], that a point $\alpha \in \partial W(A)$ is said to be a sharp point of $W(A)$ if there are angles θ_1 and θ_2 with $0 \leq \theta_1 < \theta_2 < 2\pi$ such that for all $\theta \in (\theta_1, \theta_2)$,

$$Re(e^{\iota\theta}\alpha) = max\Big\{Re(\beta) : \beta \in W(e^{\iota\theta}A)\Big\}.$$

In [9, Theorem 1.6.3], it is also known that every sharp point of $W(A)$ is an eigenvalue of A. Now, to state our result, we need the following lemma.

Lemma 2.4. *[9, Theorem 1.6.8] Let $A \in M_n$. Then $W(A) = conv(\sigma(A))$ if and only if either A is normal or A is unitarily similar to a matrix of the form $A_1 \oplus A_2$, where A_1 is normal and $W(A_2) \subseteq W(A_1)$.*

Theorem 2.5. *Let $A \in M_n$ be such that A^2 is Hermitian. Moreover, let $2 \leq m \leq n$ be a positive integer. Then the following assertions are true:*

(*i*) *$W(A) = V^m(A)$ if and only if A is a scalar matrix;*

(*ii*) *If $W(A) = conv(V^m(A))$, then A is normal or A is unitarily similar to a matrix of the form $A_1 \oplus A_2$, where A_1 is normal and $W(A_2) \subseteq W(A_1)$.*

Proof. In view of Proposition 1.3(ii), we assume, without loss of generality, that $m = 2$. We know, e.g., see [4, Theorem 4.3], that $W(A, A^2)$ is convex. So, by Proposition 1.1(i), $\zeta \in V^2(A)$ if and only if $(\zeta, \zeta^2) \in W(A, A^2)$. Since A^2 is Hermitian, $\zeta^2 \in \mathbb{R}$. Thus, $\zeta \in \mathbb{R} \cup \imath\mathbb{R}$. This shows that

$$V^2(A) \subseteq \mathbb{R} \cup \imath\mathbb{R}. \tag{2.1}$$

Now, to prove the assertion in (i), at first we assume that $W(A) = V^2(A)$. So, by (2.1), $W(A) = V^2(A) \subseteq \mathbb{R} \cup \imath\mathbb{R}$. Hence, the convexity of $W(A)$ implies that $W(A) \subseteq \mathbb{R}$ or $W(A) \subseteq \imath\mathbb{R}$. So, A is normal, and hence, by Theorem 2.3, A is a scalar matrix. The converse of (i) is trivial.

To prove the assertion in (ii), let $W(A) = conv(V^2(A))$. Also, by (2.1), $V^2(A) \subseteq \mathbb{R} \cup \imath\mathbb{R}$. So, $W(A)(= conv(V^2(A)))$ is a line segment or a triangle, or a convex quadrilateral. Therefore, $W(A)$ has two or three, or four sharp points which are eigenvalues of A. So, $W(A) = conv(V^2(A)) = conv(\sigma(A))$, and hence, by Lemma 2.4, the result holds. So, the proof is complete. □

3. On higher rank numerical hulls of normal matrices of the form $A_1 \oplus \imath A_2$

In this section, we study the rank-k numerical hulls of matrices of the form $A_1 \oplus \imath A_2$, where A_1, A_2 are Hermitian. We characterized this set, as in Theorem 2.1(i), for the case $k > n/2$. So, the results in this section are important for the case $1 \leq k \leq \frac{n}{2}$. At first, we need the following lemma.

Lemma 3.1. *Let $A \in M_n$ be a normal matrix. Then*

$$\Lambda_k(A) \cap ext(W(A)) \subseteq \sigma_k(A),$$

where $ext(.)$ denotes the set of all extreme points.

Proof. Let $\lambda \in \Lambda_k(A) \cap ext(W(A))$, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, counting multiplicities. So, by Proposition 1.2,

$$\lambda \in \bigcap_{1 \leqslant j_1 < \cdots < j_{n-k+1} \leqslant n} conv\Big(\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}\Big).$$

Now, since $ext(conv(\{\lambda_1, \ldots, \lambda_n\}) = ext(W(A)) \subseteq \sigma(A)$, the algebraic multiplicity of λ is at least k. We know that the algebraic and geometric multiplicity of eigenvalues of normal matrices are equal. So, $\lambda \in \sigma_k(A)$, and hence, the proof is complete. □

Theorem 3.2. *Let $A = A_1 \oplus \imath A_2 \in M_n$, where A_1 and A_2 are Hermitian. Moreover, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, counting multiplicities. Then*

$$\sigma_k(A) \subseteq \mathcal{X}_k^m(A) \subseteq \bigcap_{1 \leqslant j_1 < \cdots < j_{n-k+1} \leqslant n} conv\Big(\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}\Big) \cap \sigma(A), \tag{3.1}$$

if one of the following conditions holds:
(i) A_1 and A_2 are semi-definite matrices and $m \geq 2$,
(ii) A_2 is a semi-definite matrix and $m \geq 3$.
Under the condition (i) or (ii), the set equalities in (3.1) hold if $\sigma(A) = ext(W(A))$.

Proof. If the condition (i) holds, then by Proposition 1.1(v), we have $V^2(A) = \sigma(A)$. So, by Propositions 1.3((iv) and (ii)) and 1.2, we have

$$\sigma_k(A) \subseteq \mathcal{X}_k^m(A) \subseteq V^2(A) \cap \Lambda_k(A) = \bigcap_{1\leqslant j_1<\cdots<j_{n-k+1}\leqslant n} conv\Big(\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}\Big) \cap \sigma(A),$$

and so, the result holds.

If condition (ii) holds, then using the same manner as in the proof above, the result also holds.

For the final assertion, let $\sigma(A) = ext(W(A))$. Now using Proposition 1.2 and Lemma 3.1, the set equalities hold. This completes the proof. □

Remark 3.3. Let $A = A_1 \oplus \imath A_2 \in M_n$, where A_1 and A_2 are semi-definite matrices. By Lemma 3.1 and Theorem 3.2, to study $\mathcal{X}_k^m(A)$, we have to find the members of $\Lambda_k(A) \cap \sigma(A) \cap (ext(W(A)))^c$ that belong to $\mathcal{X}_k^m(A)$. In fact, $\mathcal{X}_k^m(A)$ is the union of the set of all these elements with $\sigma_k(A)$.

In the next theorem, we find the set $\mathcal{X}_k^m(A)$ for some normal matrices $A \in M_4$ of the form $A = A_1 \oplus \imath A_2$.

Theorem 3.4. *Let $\alpha, \beta, \gamma, \theta$ are positive integers, and $A = diag(\alpha, -\beta, \imath\gamma, \imath\theta)$ and $B = diag(\alpha, -\beta, \imath\gamma, -\imath\theta)$. If $m \geq 3$ and $k, l \geq 2$, then $\mathcal{X}_k^m(A) = \emptyset = \mathcal{X}_k^l(B)$.*

Proof. We assume, without loss of generality, that $\alpha \leq \beta$ and $\gamma \leq \theta$. In view of Remark 3.3, if $\gamma = \theta$, then $\mathcal{X}_k^m(A) = \emptyset$, and for the case $\gamma < \theta$, we have

$$\mathcal{X}_k^m(A) \subseteq \{\imath\gamma\}.$$

By Proposition 1.2, $\Lambda_2(A) = \{\imath\gamma\}$, $\Lambda_2(A^2) = [-\gamma^2, \alpha^2]$ and $\Lambda_2(A^3) = \{-\imath\gamma^3\}$. So,

$$conv\Big(\Lambda_2(A, A^2, A^3)\Big) \subseteq \Lambda_2(A) \times \Lambda_2(A^2) \times \Lambda_2(A^3) = \{\imath\gamma\} \times [-\gamma^2, \alpha^2] \times \{-\imath\gamma^3\}.$$

A direct computation shows that $(\imath\gamma, -\gamma^2, -\imath\gamma^3) \notin \Lambda_2(A, A^2, A^3)$. Since $(\imath\gamma, -\gamma^2, -\imath\gamma^3)$ is a conical point of $\{\imath\gamma\} \times [-\gamma^2, \alpha^2] \times \{-\imath\gamma^3\}$,

$$\Big(\imath\gamma, -\gamma^2, -\imath\gamma^3\Big) \notin conv\Big(\Lambda_2(A, A^2, A^3)\Big).$$

So, $\imath\gamma \notin \mathcal{X}_k^m(A)$, and hence $\mathcal{X}_k^m(A) = \emptyset$. To prove the assertion for the matrix B, by Proposition 1.2, $\Lambda_2(B) = \{0\}$ and $\Lambda_2(B^2)$ is a line-segment in the positive real axis. By Proposition 1.3(iii), $\mathcal{X}_k^m(B) \subseteq \Lambda_2(B) = \{0\}$. Since $(0, 0) \notin \Lambda_2(B) \times \Lambda_2(B^2)$, $(0, 0) \notin conv(\Lambda_2(B, B^2))$, and so, $0 \notin \mathcal{X}_2^2(B)$. Hence, by Proposition 1.3((ii) and (iii)), $\mathcal{X}_k^l(B) = \emptyset$. This completes the proof. □

The final example is related to a case that we did not consider in Theorem 3.4.

Example 3.5. Let $A = diag(0, 1, 2) \oplus \imath\ diag(1)$. Obviously $\sigma_2(A) = \emptyset$. By Propositions 1.3(iii) and 1.2, $\mathcal{X}_2^2(A) \subseteq \Lambda_2(A) = \{1\}$ and $\Lambda_2(A^2) = [0, 1]$. So,

$$\Lambda_2(A, A^2) \subseteq \Lambda_2(A) \times \Lambda_2(A^2) = \{1\} \times [0, 1].$$

A direct computation shows that $(1, 1) \notin \Lambda_2(A, A^2)$, and since $(1, 1)$ is a conical point of $\Lambda_2(A) \times \Lambda_2(A^2)$, $(1, 1) \notin conv(\Lambda_2(A, A^2))$. So, $1 \notin \mathcal{X}_2^2(A)$, and hence, $\mathcal{X}_2^2(A) = \emptyset$.

4. Higher rank numerical hulls of the basic circulant matrix

Let

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ & \dots & \dots & \dots & \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix} \in M_n$$

be an arbitrary circulant matrix. Also, let $P_n = E_{12} + \cdots + E_{n-1,n} + E_{n1}$, where $E_{ij} \in M_n$ is an elementary matrix whose (i, j)-*th* entry is 1 and the other entries are 0. It is clear that

$$C = c_0 P_n^0 + c_1 P_n^1 + \cdots + c_{n-1} P_n^{n-1}.$$

By this idea, P_n is called the basic circulant matrix. It is known that P_n is unitarily similar to

$$D_n = diag(1, \omega, \dots, \omega^{n-1}),$$

where $\omega = e^{\imath 2\pi/n}$. Then, by [2, Proposition 2.3(*iii*)], $\mathcal{X}_k^m(P_n) = \mathcal{X}_k^m(D_n)$. In the next theorem, we study the higher rank numerical hulls of D_n. At first, we need the following two lemmas.

Lemma 4.1. *Let $D_n = diag(1, \omega, \dots, \omega^{n-1})$, where $\omega = e^{i2\pi/n}$. Then the following assertions are true:*

(*i*) *[4, Theorem 3.1] If $n/2 < m < n$, then $V^m(D_n) = \sigma(D_n) \cup \{0\}$. Also, if $m = n/2$, then $V^m(D_n) = \bigcup_{j=0}^{n-1} \omega^j [0, 1]$;*

(*ii*) *[4, Theorem 3.3] Let $F = (\omega^{(p-1)(q-1)}) \in M_n$, $1 \le p, q \le n$ and $3 < m < n/2$. Then $\mu \in V^m(D_n)$ if and only if there exist complex numbers $z_{m+2}, \dots, z_{n-m}$ such that $z_j = \bar{z}_{n-j+2}$ and $F^{-1}[1, \mu, \dots, \mu^m, z_{m+2}, \dots, z_{n-m}$, $\bar{\mu}^m, \dots, \bar{\mu}]^t\}$ is a nonnegative vector.*

The following lemma follows easily from [6, Corollary 2.8].

Lemma 4.2. *If $k < n/2$, then $\Lambda_k(D_n)$ is an $n-$sided convex polygon obtained by joining ω^j and ω^{j+k}, where $\omega^{j+k} = \omega^{j+k-n}$ if $j + k > n$ ($j = 1, \dots, n$).*

Theorem 4.3. *Let $D_n = diag(1, \omega, \dots, \omega^{n-1})$, where $\omega = e^{i2\pi/n}$. Moreover, let $F = (\omega^{(p-1)(q-1)}) \in M_n$; $1 \le p, q \le n$. Then the following assertions are true:*

(*i*) *If $k = 1$, then $\mathcal{X}_1^m(D_n)$ equals to*

$$\begin{cases} conv(\{1, \omega, \dots, \omega^{n-1}\}), & if\ m = 1, \\ \{\mu \in \mathbb{C} : \exists z_{m+2}, \dots, z_{n-m}\ s.t.\ z_j = \bar{z}_{n-j+2}\ and \\ F^{-1}[1, \mu, \dots, \mu^m, z_{m+2}, \dots, z_{n-m}, \bar{\mu}^m, \dots, \bar{\mu}]^t \\ \ is\ a\ nonnegative\ vector\}, & if\ 3 < m < n/2, \\ \bigcup_{j=0}^{n-1} \omega^j [0, 1], & if\ m = n/2, \\ \{0, 1, \omega, \dots, \omega^{n-1}\}, & if\ n/2 < m < n, \\ \{1, \omega, \dots, \omega^{n-1}\}, & if\ m = n; \end{cases}$$

(*ii*) *If* $1 < k < n/2$, *then*

$$\begin{cases} \mathcal{X}_k^m(D_n) = \bigcap_{0\leqslant j_1<\cdots<j_{n-k+1}\leqslant n-1} conv(\{\omega^{j_1},\ldots,\omega^{j_{n-k+1}}\}), & \text{if } m=1,\\ \mathcal{X}_k^m(D_n) \subseteq \{\mu\in\mathbb{C} : \exists z_{m+2},\ldots,z_{n-m} \text{ s.t. } z_j=\bar{z}_{n-j+2} & \\ \text{and } F^{-1}[1,\mu,\ldots,\mu^m,z_{m+2},\ldots,z_{n-m},\bar{\mu}^m,\ldots,\bar{\mu}]^t & \\ \text{is a nonnegative vector}\}, & \text{if } 3<m<n/2,\\ \mathcal{X}_k^m(D_n)\subseteq \bigcup_{j=0}^{n-1}\dfrac{\omega^k}{\omega^k+\omega-1}\omega^{j+1}[0,1], & \text{if } m=n/2,\\ \mathcal{X}_k^m(D_n)\subseteq\{0\}, & \text{if } n/2<m<n,\\ \mathcal{X}_k^m(D_n)=\emptyset, & \text{if } m=n; \end{cases}$$

(*iii*) *If* $k \geqslant n/2$, *then* $\mathcal{X}_k^m(D_n) = \emptyset$.

Proof. For $k = 1$ and $m = 1$, $\mathcal{X}_1^1(D_n) = W(D_n)$ and since D_n is a normal matrix, $\mathcal{X}_1^1(D_n) = conv(\{1,\omega,\omega^2,\ldots,\omega^{n-1}\})$.

For the case $k = 1$ and $3 < m < n/2$, by Lemma 4.1(*ii*) and Proposition 1.3(*ii*), the result holds.

By Lemma 4.1(*i*), the results in the cases $k = 1$ and $n/2 < m < n$ or $m = n/2$, also hold.

For the case $k = 1$ and $m = n$, $\mathcal{X}_1^n(D_n) = V^n(D_n) = \sigma(D_n)$.

For the case $1 < k < n/2$ and $m = 1$, by Propositions 1.3(*iii*) and 1.2, the result holds.

If $1 < k < n/2$ and $3 < m < n/2$, then the result follows from Proposition 1.3(*iv*) and Lemma 4.1(*ii*).

If $1 < k < n/2$ and $m = n/2$, then Proposition 1.3(*iv*) and Lemma 4.1(*i*) imply that $\mathcal{X}_k^m(D_n) \subseteq (\bigcup_{j=0}^{n-1}\omega^j[0,1]) \cap \Lambda_k(D_n)$. To make a better superset, using Lemma 4.2, we must find the intersection of the line segments $\omega^j[0,1]$ with $\Lambda_k(D_n)$, where $j = 0,1,\ldots,n-1$. This intersection is $\{\dfrac{\omega^{k+j+1}}{\omega^k+\omega-1}\}$. So, the above intersection equals to $\bigcup_{j=0}^{n-1}\dfrac{\omega^k}{\omega^k+\omega-1}\omega^{j+1}[0,1]$, and hence, the inclusion holds.

For the case $1 < k < n/2$ and $n/2 < m < n$, by Lemma 4.1(*i*), we have $V^m(D_n) = \sigma(D_n)\cup\{0\}$. So, by Proposition 1.3(*ii*), $\mathcal{X}_k^m(D_n) \subseteq \sigma(D_n)\cup\{0\}$.

For $k > 1$, $\sigma_k(D_n) = \emptyset$. So, by Proposition 1.3(*vii*), $\mathcal{X}_k^m(D_n)\cap\sigma(D_n) = \emptyset$. Hence, $\mathcal{X}_k^m(D_n) \subseteq \{0\}$.

For the case $1 < k < n/2$ and $m = n$, $conv(\Lambda_k(D_n, D_n^2, \ldots, D_n^n = I_n)) \subseteq \Lambda_k(D_n)\times\cdots\times\Lambda_k(D_n^{n-1})\times\{1\}$. So, if there exists a scalar $\mu \in \mathcal{X}_k^n(D_n)$, then $\mu^n = 1$. This means that there is a $0 \leq j \leq n-1$ such that $\mu = \omega^j$. By Lemma 4.2, the elements $1,\omega,\ldots,\omega^{n-2}$ and ω^{n-1} do not belong to $\Lambda_k(D_n)$ for $1 < k < n/2$. So, $\mu \notin \Lambda_k(D_n)$; a contradiction. Hence $\mathcal{X}_k^n(D_n) = \emptyset$.

By Proposition 1.2, for the case $k > n/2 + 1$, we have $\Lambda_k(D_n) = \emptyset$. Hence, $\mathcal{X}_k^m(D_n)$ is also empty.

By Proposition 1.3(*viii*), if $k = n/2$, then $\mathcal{X}_k^m(D_n) = \emptyset$. So, the result in (*iii*) holds, and hence, the proof is complete. □

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