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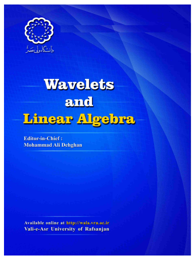

Pseudoframe multiresolution structure on locally compact abelian groups

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Abstract

Let G be a locally compact abelian group. The concept of generalized multiresolution structure (GMS) in $L^2(G)$ is discussed which is a generalization of GMS in $L^2(\mathbb{R})$. Basically, a GMS in $L^2(G)$ consists of an increasing sequence of closed subspaces of $L^2(G)$ and a pseudoframe of translation type at each level. Also, the construction of affine pseudoframes for $L^2(G)$ based on a GMS is presented.

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1. Introduction and preliminary

In 1946, Gabor introduced an interesting approach to signal decomposition [12]. In 1952, Duffin and Schaeffer abstracted Gabor's method to introduce the notion of frame in nonharmonic Fourier analysis [9]. The idea of Duffin and Schaeffer was not continued until 1986 when Daubechies et al. in [8], applied the theory of frames to wavelets and Gabor transforms. After their work, the theory of frames began to be studied widely and deeply by many authors (see [3]-[5], for example). Today, the theory of frames has been applied to signal processing, image processing, data compressing and sampling theory and so on.

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_n |\langle x, x_n\rangle|^2 \leq B\|x\|^2, \quad (x \in \mathcal{H}). \tag{1.1}$$

If the right inequality holds, then $\{x_n\}_{n\in\mathbb{Z}}$ is said to be a Bessel sequence. It is well known that for any frame $\{x_n\}_{n\in\mathbb{Z}}$ there exists another frame $\{x_n^*\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$, namely dual frame of $\{x_n\}_{n\in\mathbb{Z}}$, such that for any $x \in \mathcal{H}$

$$x = \sum_n \langle x, x_n^*\rangle x_n = \sum_n \langle x, x_n\rangle x_n^*.$$

The concept of a multiresolution analysis (MRA) was first introduced by Mallat [21] and Meyer [22]. It is a general framework for constructing orthonormal wavelet bases for $L^2(\mathbb{R})$ of the form $\{2^{j/2}\psi(2^j. - k)\}_{j,k\in\mathbb{Z}}$. The MRA-based compactly supported orthonormal wavelet systems were constructed by Daubechies [7].

Frame multiresolution analysis (FMRA) as a generalization of MRA, introduced by Benedetto and Li in [2].

As usual, we define the following operators on $L^2(\mathbb{R})$ by

$$(\tau_b f)(x) = f(x - b), \ (Df)(x) = 2^{1/2} f(2x).$$

The parameter b in the first operator can be an arbitrary real number. A frame multiresolution analysis (FMRA) for $L^2(\mathbb{R})$ consists of a sequence of closed linear subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ and a function $\phi \in V_0$ such that

1. $V_j \subseteq V_{j+1}$,
2. $\overline{\cup_j V_j} = L^2(\mathbb{R}), \cap_j V_j = \{0\}$,
3. $V_j = D^j V_0$,
4. $f \in V_0$ implies that $\tau_k f \in V_0$, for all $k \in \mathbb{Z}$,
5. $\{\tau_k \phi : k \in \mathbb{Z}\}$ is a frame for V_0.

In the above definition, if $\{\tau_k\phi : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, then $\{V_j, \phi\}_{j\in\mathbb{Z}}$ forms a multiresolution analysis (MRA) for $L^2(\mathbb{R})$. For more details on FMRA, one can see [1], [4].

Dahlke in [6], generalized the notion of MRA to locally compact abelian groups and proved an existence theorem based on generalized B-splines. For some groups G different from $\mathbb{R}^d$, multiresolution analysis were studied in [6], [10], [15] and [16]. In particular, the group analogues of the B-spline wavelet bases in $L^2(\mathbb{R})$ are defined in [6] and [10]. In [14], conditions under which a function generates a multiresolution analysis on a locally compact abelian groups were investigated.

The notion of generalized multiresolution structure (GMS) in $L^2(\mathbb{R})$ was introduced in [18]. Basically, the GMS consists of an increasing sequence of closed subspace of $L^2(\mathbb{R})$, with a pseudoframe of translates at each level. Let $\{\tau_k\phi\}_{k\in\mathbb{Z}}$ and $\{\tau_k\phi^*\}_{k\in\mathbb{Z}}$ be two sequences in $L^2(\mathbb{R})$ and $\mathcal{X}$ be a closed subspace of $L^2(\mathbb{R})$. We say $\{\tau_k\phi\}_{k\in\mathbb{Z}}$ forms a pseudoframe of translates for $\mathcal{X}$ with respect to $\{\tau_k\phi^*\}_{k\in\mathbb{Z}}$ if

$$x = \sum_k \langle x, \tau_k\phi^*\rangle \tau_k\phi, \quad (x \in \mathcal{X}).$$

In a more general case, let $\mathcal{X}$ be a closed subspace of a separable Hilbert space $\mathcal{H}$. Let $\{x_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$ be a Bessel sequence with respect to $\mathcal{X}$, and let $\{x_n^*\}_{n\in\mathbb{Z}} \subset \mathcal{H}$ be a Bessel sequence in $\mathcal{H}$. We say $\{x_n\}_{n\in\mathbb{Z}}$ is a pseudoframe for the subspace $\mathcal{X}$ (PFFS) with respect to $\{x_n^*\}_{n\in\mathbb{Z}}$ if

$$x = \sum_k \langle x, x_k\rangle x_k^*, \quad (x \in \mathcal{X}).$$

$\{x_n^*\}_{n\in\mathbb{Z}}$ is called a dual pseudoframe (or PFFS-dual) of $\{x_n\}_{n\in\mathbb{Z}}$ for the subspace $\mathcal{X}$, (see [19] and [20] for more details).

For the reader's convenience, we report a number of definitions. In this paper, we assume that G is a locally compact abelian group and Γ is a uniform lattice in G, that is Γ is a discrete subgroup that $\frac{G}{\Gamma}$ is compact. If Γ is a uniform lattice, then $\Gamma^\perp$ defined by $\{\xi \in \widehat{G} : \xi(\Gamma) = 1\}$ is a uniform lattice in $\widehat{G}$, where $\widehat{G}$ is the dual group of G [13].

Let $\pi : \Gamma \longrightarrow U(L^2(G))$ be the translation representation which is defined by $(\pi_\gamma f)(x) = f(x\gamma^{-1})$. Let σ be a (continuous) unitary operator on $L^2(G)$ with the property $\sigma^{-1}\pi_\gamma\sigma = \pi_{\alpha(\gamma)}$, where α is an injective endomorphism on Γ. Also let δ be a (continuous) topological automorphism on G such that $\delta(\Gamma) \subset \Gamma$.

The Fourier transform $\hat{}: L^1(G) \longrightarrow C_0(\hat{G}), \quad f \longmapsto \hat{f}$, is defined by $\hat{f}(\xi) = \int_G f(x)\bar{\xi}(x)dx$. The Fourier transform can be extended to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$, known as the Plancherel transform (see [11]).

In this paper, we generalize the notion of GMS for $L^2(G)$. In Section 2, a necessary and sufficient condition for existence of pseudoframes for Paley-Wiener subspaces is studied. Based on this concept, a formal definition of a GMS for a locally compact abelian group is given in Section 3. Consequently, construction methods for GMSs are also explained. Furthermore, a construction that allows us to obtain affine pseudoframes associated with such a GMS is given in Section 4.

Our conclusions are mainly generalizations of results of Li in [18].

2. Existence of pseudoframes for subspaces of $L^2(G)$

There are some examples in $L^2(G)$ such that $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ is neither generates MRA nor FMRA. Nevertheless a stable expansion for elements of a closed subspace of $L^2(G)$ exists in term of $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ (see [1, 17, 19, 20]). The following is such an example.

Example 2.1. Let $G = \mathbb{R}\times\mathbb{R}$ and $\Gamma = \mathbb{Z}\times\mathbb{Z}$. Consequently, $\hat{\Gamma} = \hat{\mathbb{Z}}\times\hat{\mathbb{Z}} = \mathbb{T}\times\mathbb{T}$. Define a function $\phi \in L^2(G)$ such that

$$\hat{\phi}(\gamma_1,\gamma_2) = \begin{cases} 1 & (\gamma_1,\gamma_2) \in [-\frac{1}{4},\frac{1}{4})^2 \\ \text{decaying to zero continuosly} & (\gamma_1,\gamma_2) \in [-\frac{1}{2},\frac{1}{2})^2 - [-\frac{1}{4},\frac{1}{4})^2 \\ 0 & \text{outside of} \quad [-\frac{1}{2},\frac{1}{2})^2 \end{cases}$$

Let $\Delta := [-\frac{1}{4},\frac{1}{4})^2$ and define $V_0 := PW_\Delta$ that is the Paley-Wiener space, the space of all band-limited functions with bandwidth in Δ (see [1]). By Shannon sampling theorem for $L^2(\mathbb{R}^d)$ [23], and for all $f \in PW_\Delta$ we have

$$f(x,y) = \textstyle\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} f(m,n)\pi_{(m,n)}\phi(x,y).$$

Since the function $\Phi(\gamma_1,\gamma_2) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} |\hat{\phi}(\gamma_1+m,\gamma_2+n)|^2$ is continuous, we are not able to find the lower frame bound for $\{\pi_{(m,n)}\phi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$. So $\{\pi_{(m,n)}\phi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ cannot be a frame for the closure of the span of $\{\pi_{(m,n)}\phi(t_1,t_2) : (m,n) \in \mathbb{Z}\times\mathbb{Z}\}$.

Also, from the fact that $\phi \notin PW_\Delta$, the sequence $\{\pi_{(m,n)}\phi\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is not a frame for PW_Δ. Whereas, if we define $V_j := PW_{2^j\Delta}$, $j \in \mathbb{Z}$, we have $V_j \subseteq V_{j+1}$, $\overline{\bigcup_j V_j} = L^2(G)$ and $\bigcap_j V_j = \{0\}$.

Example 2.1 leads us to define the concept of pseudoframes on $L^2(G)$ for a locally compact abelian group G. For a uniform lattice Γ in G, let $\pi : \Gamma \longrightarrow U(L^2(G))$ be the translation representation which is defined by $(\pi_\gamma f)(x) = f(x\gamma^{-1})$.

Definition 2.2. Let $\phi,\psi \in L^2(G)$ and $\mathcal{X}$ be a closed subspaces of $L^2(G)$. The family $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ is said to be a pseudoframe with respect to $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ for $\mathcal{X}$, if for every $x \in \mathcal{X}$,

$$x = \sum_{\gamma\in\Gamma}\langle x,\pi_\gamma\psi\rangle\pi_\gamma\phi.$$

It is important to note that $\pi_\gamma\phi$ and $\pi_\gamma\psi$ need not be contained in $\mathcal{X}$. Also they are not generally commutable, this means there exists $x \in \mathcal{X}$ such that the following is not true,

$$x = \textstyle\sum_{\gamma\in\Gamma}\langle x,\pi_\gamma\phi\rangle\pi_\gamma\psi.$$

In the following theorem, we are going to find a sufficient and necessary condition for the functions ϕ and ψ such that their translations forms a pseudoframe.

Theorem 2.3. *Let $\phi \in L^2(G)$ be such that $|\hat{\phi}| > 0$, a.e. and $\hat{\phi}$ be zero outside of $\hat{\Gamma}$. For a fixed $c > 0$, let $\Delta := \{\gamma \in \hat{G} : |\hat{\phi}(\gamma)| \geq c\}$ be closed and let $V_0 := \{f \in L^2(G) : supp\hat{f} \subseteq \Delta\}$. For a $\psi \in L^2(G)$, $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ forms a pseudoframe for V_0 with respect to $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ if and only if*

$$\hat{\phi}\overline{\hat{\psi}}\chi_\Delta = \chi_\Delta, \quad a.e.$$

Moreover, if ψ satisfies $|\hat{\psi}| > 0$ on $\hat{\Gamma}$, and the above equality holds, then $\pi_\gamma\phi$ and $\pi_\gamma\psi$ commute, in the sense that for any $x \in \mathcal{X}$,

$$x = \sum_{\gamma\in\Gamma}\langle x,\pi_\gamma\psi\rangle\pi_\gamma\phi = \sum_{\gamma\in\Gamma}\langle x,\pi_\gamma\phi\rangle\pi_\gamma\psi.$$

Proof. If $f \in V_0$, then $supp\hat{f} \subseteq \Delta$. By the assumptions, $supp\hat{\phi} \subseteq \hat{\Gamma}$. Now compactness of $\hat{\Gamma}$ implies that $supp\hat{\phi}$ is compact. On the other hand Δ is closed and so the fact that $\Delta \subseteq supp\hat{\phi}$ implies that Δ is compact. Hence $supp\hat{f}$ is compact.

By Weyl's formula we have

$$\begin{aligned}
(\sum_{\eta\in\Gamma}\langle f,\pi_\eta\psi\rangle\pi_\eta\phi\hat{)}(\gamma) &= \sum_{\eta\in\Gamma}\langle f,\pi_\eta\psi\rangle\hat{\phi}(\gamma)\overline{\gamma(\eta)}\\
&= \sum_{\eta\in\Gamma}\langle \hat{f},\widehat{\pi_\eta\psi}\rangle\hat{\phi}(\gamma)\overline{\gamma(\eta)}\\
&= \sum_{\eta\in\Gamma}\int_{\hat{G}}\hat{f}(\lambda)\overline{\hat{\psi}}(\lambda)\lambda(\eta)d\lambda\hat{\phi}(\gamma)\overline{\gamma(\eta)}\\
&= \sum_{\eta\in\Gamma}\int_{\hat{G}/\Gamma^\perp}\sum_{\xi\in\Gamma^\perp}\hat{f}(\xi\lambda)\overline{\hat{\phi}(\xi\lambda)}(\xi\lambda)(\eta)d\mu(\lambda\Gamma^\perp)\hat{\phi}(\gamma)\overline{\gamma(\eta)}\\
&= \hat{\phi}(\gamma)\sum_{\eta\in\Gamma}(\sum_{\xi\in\Gamma^\perp}\hat{f}(\xi\lambda)\overline{\hat{\psi}}(\xi\lambda))^\vee(\eta)\overline{\gamma}(\eta)\\
&= \hat{\phi}(\gamma)((\sum_{\xi\in\Gamma^\perp}\hat{f}(\xi\lambda)\overline{\hat{\psi}}(\xi\lambda))^\vee\hat{)}(\gamma)\\
&= \hat{\phi}(\gamma)\sum_{\xi\in\Gamma^\perp}\hat{f}(\xi\gamma)\overline{\hat{\psi}}(\xi\gamma).
\end{aligned}$$

The facts that $\hat{\phi}$ and $\hat{f}$ are zero outside of $\hat{\Gamma}$ imply that the only nonzero term in the last summation is $\hat{\phi}(\gamma)\hat{f}(\gamma)\overline{\hat{\psi}}(\gamma)$. So

$$\hat{\phi}\overline{\hat{\psi}}\chi_\Delta = \chi_\Delta, \quad a.e.$$

For two Bessel families $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ and $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ in $L^2(G)$, define $U, V : L^2(G) \longrightarrow l^2(\Gamma)$ by $U(f) = \{\langle f, \pi_\gamma\phi\rangle\}_{\gamma\in\Gamma}$ and $V(f) = \{\langle f, \pi_\gamma\psi\rangle\}_{\gamma\in\Gamma}$. From [19], we know that $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ forms a pseudoframe with respect to $\{\pi_\gamma\psi\}$ for $\mathcal{X}$ if and only if

$$V^*UP = P,$$

where P is the orthogonal projection on $\mathcal{X}$ and also $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ and $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ commute if and only if $V^*UP = P = PU^*V$, where V^* and U^* are the adjoints of U and V, respectively. Indeed, we have

$$\begin{aligned}
\langle f, PU^*Vg\rangle &= \langle Pf, U^*Vg\rangle\\
&= \overline{\langle U^*Vg, Pf\rangle}\\
&= \sum_{\gamma\in\Gamma}\overline{\langle g,\pi_\gamma\psi\rangle\langle\pi_\gamma\phi, Pf\rangle}\\
&= \sum_{\gamma\in\Gamma}\langle Pf,\pi_\gamma\phi\rangle\langle\pi_\gamma\psi, g\rangle\\
&= \langle Pf, \sum_{\gamma\in\Gamma}\langle g,\pi_\gamma\psi\rangle\pi_\gamma\phi\\
&= \langle Pf, g\rangle = \langle f, Pg\rangle
\end{aligned}$$

which implies that $PU^*V = P$. Thus $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ and $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ commute. □

The following result of Li (Theorem 1, [18]) is a consequence of Theorem 2.3.

Corollary 2.4. *Let $\phi \in L^2(\mathbb{R})$ be such that $|\hat{\phi}| > 0$ a.e. on a connected neighborhood of 0 in $[-\frac{1}{2}, \frac{1}{2})$ and $|\hat{\phi}| = 0$, a.e. otherwise. Define $\Omega = \{\gamma \in \hat{\mathbb{R}} : |\hat{\phi}| \geq c > 0\}$, and let $V_0 := PW_\Omega = \{f \in L^2(\mathbb{R}) : supp(\hat{f}) \subseteq \Omega\}$. Then, for a $\psi \in L^2(\mathbb{R})$ and $\{\pi_k\phi\}_{k\in\mathbb{Z}}$ is a pseudoframe of translates for V_0 with respect to $\{\pi_k\psi\}_{k\in\mathbb{Z}}$ if and only if $\hat{\phi}\overline{\hat{\psi}}.\chi_\Omega = \chi_\Omega$ a.e. Moreover, if ψ is also such that $|\hat{\psi}| > 0$ a.e. on a connected neighborhood of 0 in $[-\frac{1}{2}, \frac{1}{2})$, and $|\hat{\psi}| = 0$ a.e. otherwise, and $\hat{\phi}\overline{\hat{\psi}}.\chi_\Omega = \chi_\Omega$ a.e. holds, then $\{\pi_k\phi\}_{k\in\mathbb{Z}}$ and $\{\pi_k\psi\}_{k\in\mathbb{Z}}$ are commutative pair of pseudoframe for $\mathcal{X}$.*

3. Generalized Multiresolution Structure

In this section, by applying Theorem 2.3, we are going to construct a generalized multiresolution structure for locally compact abelian groups. First, we define the concept of generalized multiresolution structure (GMS) for $L^2(G)$, where G is a locally compact abelian group.

Let $\pi : \Gamma \longrightarrow U(L^2(G))$ be the translation representation and also, let σ be a unitary operator on $L^2(G)$ with the property $\sigma^{-1}\pi_\gamma\sigma = \pi_{\alpha(\gamma)}$, where α is an injective endomorphism on Γ. Also let δ be an automorphism on G such that $\delta(\Gamma) \subset \Gamma$.

Definition 3.1. A generalized multiresolution structure (GMS), $\{V_j, \phi, \psi\}_{j\in\mathbb{Z}}$ of $L^2(G)$ is an increasing sequence of the closed subspaces $V_j \subseteq L^2(G)$ and two elements $\phi, \psi \in L^2(G)$ such that the following conditions hold

1. $\overline{\cup_j V_j} = L^2(G), \cap_j V_j = \{0\}$,
2. $f \in V_j$ if and only if $\sigma f \in V_{j+1}$,
3. $f \in V_0$ implies that $\pi_\gamma f \in V_0$, for all $\gamma \in \Gamma$,
4. $\{\pi_\gamma\phi : \gamma \in \Gamma\}$ is a pseudoframe for V_0 with respect to $\{\pi_\gamma\psi : \gamma \in \Gamma\}$.

Remark 3.2. If $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ and $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ are dual frames, then GMS is a frame multiresolution analysis. Also, if $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ is an exact frame for V_0 and $\psi \in V_0$, then GMS is an multiresolution analysis.

Theorem 3.3. *Suppose that $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ is a pseudoframe for V_0 with respect to $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ and $V_j := \{f \in L^2(G) : \sigma^{-j}f \in V_0\}$, then $\{\sigma^j\pi_\gamma\phi\}_{\gamma\in\Gamma}$ is a pseudoframe for V_j with respect to $\{\sigma^j\pi_\gamma\psi\}_{\gamma\in\Gamma}$.*

Proof. For $f \in V_j$, we have $\sigma^{-j}f \in V_0$. So $\sigma^{-j}f = \sum_{\gamma\in\Gamma}\langle\sigma^{-j}f, \pi_\gamma\psi\rangle\pi_\gamma\phi$. We have $\sigma^* = \sigma^{-1}$, since σ is unitary. Thus

$$f = \sum_{\gamma\in\Gamma}\langle\sigma^{-j}f, \pi_\gamma\psi\rangle\sigma^j\pi_\gamma\phi = \sum_{\gamma\in\Gamma}\langle f, \sigma^j\pi_\gamma\psi\rangle\sigma^j\pi_\gamma\phi.$$

□

Corollary 3.4. *Let $\phi, \psi \in L^2(G)$ and V_0 has the properties specified in Theorem 2.3 and V_j is similar to Theorem 3.3, then $\{V_j, \phi, \psi\}_j$ forms a GMS for $L^2(G)$.*

Proof. The inclusion $V_j \subseteq V_{j+1}$ follows from the fact that V_j defined by Theorem 3.3 is equivalent to $PW_{\Delta(\delta^j)}$ and $PW_\Delta \subseteq PW_{\Delta(\delta)}$. Since $\delta(\Gamma) \subset \Gamma$ we have $\hat{\Gamma}(\delta) \subset \hat{\Gamma}$. Now let $f \notin PW_\Delta$. Then $supp\hat{f}$ is not a subset of Δ, so $supp\hat{f}$ is not a subset of $\Delta(\delta)$, consequently, $f \notin PW_{\Delta(\delta)}$. Therefore, $PW_\Delta \subseteq PW_{\Delta(\delta)}$.

Trivially the set of all band-limited functions, that their Fourier transform has a bounded support, is dense in $L^2(G)$. On the other hand, the intersection of all band-limited function is the trivial function.

For proving condition 3. of Definition 3.1, note that if $f \in V_0$, then $f = \sum_{\eta\in\Gamma}\langle f, \pi_\eta\psi\rangle\pi_\eta\phi$. Therefore,

$$\pi_\gamma f = \sum_{\eta\in\Gamma}\langle f, \pi_\eta\psi\rangle\pi_\gamma\pi_\eta\phi = \sum_{\eta\in\Gamma}\langle f, \pi_{\eta\gamma^{-1}}\psi\rangle\pi_\eta\phi = \sum_{\eta\in\Gamma}\langle \pi_\gamma f, \pi_\eta\psi\rangle\pi_\eta\phi.$$

Thus $\pi_\gamma f \in V_0$. Other conditions of Definition 3.1 are valid, obviously. □

Example 3.5. Let $G = \mathbb{R}\times\mathbb{R}$ and $\Gamma = \mathbb{Z}\times\mathbb{Z}$. Let $\phi \in L^2(G)$ be such that

$$\hat{\phi}(\gamma_1,\gamma_2) = \begin{cases} 1 & 0 \le \gamma_1^2+\gamma_2^2 \le \frac{1}{16}, \quad a.e. \\ 2-4\sqrt{\gamma_1^2+\gamma_2^2} & \frac{1}{16} \le \gamma_1^2+\gamma_2^2 \le \frac{1}{4} \quad a.e. \\ 0 & otherwise. \end{cases}$$

Put $\Delta := \{(\gamma_1,\gamma_2) \in \mathbb{R}\times\mathbb{R} : |\hat{\phi}(\gamma_1,\gamma_2)| \ge 1\} = \{(\gamma_1^2,\gamma_2^2) : 0 \le \gamma_1^2+\gamma_2^2 \le \frac{1}{16}\}$ and define $V_0 := PW_\Delta$. Now select $\psi \in L^2(G)$ such that

$$\hat{\psi}(\gamma_1,\gamma_2) = \begin{cases} 1 & 0 \le \gamma_1^2+\gamma_2^2 \le \frac{1}{16} \quad a.e. \\ 3-8\sqrt{\gamma_1^2+\gamma_2^2} & \frac{1}{16} \le \gamma_1^2+\gamma_2^2 \le \frac{9}{64} \quad a.e. \\ 0 & otherwise. \end{cases}$$

On Δ, we have $\hat{\phi}.\hat{\psi} = 1$, thus $\{V_j, \phi, \psi\}$ forms a GMS for the space of $L^2(G)$.

Note that for the Haar measure dt on G, and an endomorphism δ, the formation $d(\delta(t))$ induces a Haar measure. Thus there is a positive number $|\delta|$ such that $d(\delta(t)) = |\delta|dt$. It is obvious that $d(\delta^{-1}(t)) = |\delta|^{-1}dt$. Now we define a suitable dilation by $\sigma f(t) := |\delta|^{-\frac{1}{2}}f(\delta(t))$.

Let two complex families $\{h_0(\eta)\}_{\eta\in\Gamma}, \{h_0^*(\eta)\}_{\eta\in\Gamma}$, ($h_0, h_0^* : \Gamma \longrightarrow \mathbb{C}$ are two functions) be such that the following summations are convergent,

$$H_0(\xi) := \sum_{\eta\in\Gamma} h_0(\eta)\overline{\xi(\eta)}, H_0^*(\xi) := \sum_{\eta\in\Gamma} h_0^*(\eta)\overline{\xi(\eta)}, \quad (\xi \in \hat{G}).$$

Proposition 3.6. *Suppose H_0 and H_0^* generate ϕ and ψ, respectively, as*

$$\phi(t) = \sum_{\gamma\in\Gamma} h_0(\gamma)\sigma\pi_\gamma\phi(t), \ \psi(t) = \sum_{\gamma\in\Gamma} h_0^*(\gamma)\sigma\pi_\gamma\phi(t)$$

and $\phi, \psi \in L^2(G)$ have the properties specified in Theorem 2.3. Then $\{\pi_\gamma\phi\}_{\gamma\in\Gamma}$ forms a pseudoframe for V_0 with respect to $\{\pi_\gamma\psi\}_{\gamma\in\Gamma}$ if and only if

$$H_0.H_0^*\chi_{\Delta(\delta^{-1})} = |\delta|\chi_{\Delta(\delta^{-1})}.$$

Where $\Delta(\delta^{-1}) := \{\gamma\delta^{-1} : \gamma \in \Delta\}$.

Proof. Taking the Fourier transform of ϕ, we get

$$\begin{aligned}
\hat{\phi}(\xi) &= \sum_{\gamma\in\Gamma} h_0(\gamma)(\sigma\pi_\gamma\phi)\hat{}(\xi) \\
&= \sum_{\gamma\in\Gamma} h_0(\gamma)\int_{\hat{G}} |\delta|^{-\frac{1}{2}}\phi(\gamma^{-1}\delta(t))\overline{\xi(t)}dt \\
&= \sum_{\gamma\in\Gamma} h_0(\gamma)\int_{\hat{G}} |\delta|^{-\frac{1}{2}}\phi(t)\overline{\xi(\delta^{-1}(\gamma t)}d(\gamma^{-1}\delta(t)) \\
&= \sum_{\gamma\in\Gamma} h_0(\gamma)\int_{\hat{G}} |\delta|^{-\frac{1}{2}}\phi(t)\overline{\xi(\delta^{-1}(\gamma))\xi(\delta^{-1}(t))}|\delta|dt \\
&= |\delta|^{\frac{1}{2}}H_0(\xi\delta^{-1})\hat{\phi}(\xi\delta^{-1}).
\end{aligned}$$

So, $\hat{\phi}(\xi\delta) = |\delta|^{\frac{1}{2}}H_0(\xi)\hat{\phi}(\xi)$. Consequently $H_0.\overline{H_0^*}\chi_{\Delta(\delta^{-1})} = |\delta|\chi_{\Delta(\delta^{-1})}$. □

4. Affine pseudoframes of $L^2(G)$

We shall denote by W_0 the orthogonal complement of V_0 in V_1, as usual, in order to split a function f of V_1 into two functions in V_0 and W_0, respectively.

Definition 4.1. Let $\{V_j, \phi, \psi\}_{j\in\mathbb{Z}}$ be a given GMS and ϕ^*, ψ^* be two functions in $L^2(G)$. We say $\{\pi_\gamma\phi, \pi_\gamma\phi^*\}_{\gamma\in\Gamma}$ is an affine pseudoframe for V_1 with respect to $\{\pi_\gamma\psi, \pi_\gamma\psi^*\}_{\gamma\in\Gamma}$, if and only if

$$f = \sum_{\gamma\in\Gamma}\langle f, \pi_\gamma\psi\rangle\pi_\gamma\phi + \sum_{\gamma\in\Gamma}\langle f, \pi_\gamma\psi^*\rangle\pi_\gamma\phi^*, \quad (f \in V_0).$$

In this case $\{\pi_\gamma\psi, \pi_\gamma\psi^*\}_{\gamma\in\Gamma}$ is called a dual pseudoframe of $\{\pi_\gamma\phi, \pi_\gamma\phi^*\}_{\gamma\in\Gamma}$.

We are going to characterize conditions for which $\{\pi_\gamma\phi, \pi_\gamma\phi^*\}_{\gamma\in\Gamma}$ is an affine pseudoframe for V_1 with respect to $\{\pi_\gamma\psi, \pi_\gamma\psi^*\}_{\gamma\in\Gamma}$. First we have the following.

Proposition 4.2. *Let* $\{h_1(\gamma)\}_{\gamma\in\Gamma}$ *be such that* $H_1(\xi) = \sum_{\eta\in\Gamma} h_1(\eta)\overline{\xi(\eta)}$ *is convergent,* $H_1(0) = 0$ *and* $H_1 \in L^\infty(\hat{\Gamma})$. *Suppose that* $\phi \in L^2(G)$ *and* $\phi(t) = |\delta|^{\frac{1}{2}}\sum_{\gamma\in\Gamma} h_0(\gamma)\phi(\delta(t)\gamma^{-1})$*, for the family* $\{h_0(\eta)\}_{\eta\in\Gamma}$ *such that* $H_0(\xi) = \sum_{\eta\in\Gamma} h_0(\eta)\overline{\xi(\eta)}$ *is convergent. Then there exists* $\psi \in L^2(G)$ *such that*

$$\psi(t) = |\delta|^{\frac{1}{2}}\sum_{\gamma\in\Gamma} h_1(\gamma)\phi(\delta(t)\gamma^{-1}). \tag{4.1}$$

Proof. Define

$$\hat{\psi}(\omega) = |\delta|^{\frac{1}{2}}H_1(\omega\delta)\prod_{j=2}^{\infty}|\delta|^{\frac{1}{2}}H_0(\omega\delta^{-j}).$$

By the equality $\hat{\phi}(\xi) = H_0(\xi\delta^{-1})\hat{\phi}(\xi\delta^{-1})$, we have

$$\hat{\psi}(\omega) = |\delta|^{\frac{1}{2}} H_1(\omega\delta^{-1})\hat{\phi}(\omega\delta^{-1}) \tag{4.2}$$

Hence $\psi \in L^2(G)$, since $\phi \in L^2(G)$ and $H_1 \in L^\infty(\hat{\Gamma})$. It is now sufficient to use Parsval's Theorem and inverse Fourier transform of (4.2). to obtain (4.1). □

Suppose A is a subset of $\hat{G}$. For the next theorem we need the following $\Gamma^\perp$-periodic function

$$\Lambda_A(\gamma) = \sum_{\eta\in\Gamma^\perp} \chi_A(\gamma\eta).$$

Theorem 4.3. *Let Δ be the bandwidth of the subspace V_0 defined by Theorem 2.3. The family $\{\pi_\gamma\phi, \pi_\gamma\phi^*\}_{\gamma\in\Gamma}$ forms a pseudoframe for V_1 with respect to $\{\pi_\gamma\psi, \pi_\gamma\psi^*\}_{\gamma\in\Gamma}$ if and only if there are two functions G_0, G_1 in $L^2(\hat{\Gamma})$ such that*

$$\begin{aligned} G_0(\xi)\overline{H_0^*(\xi)}\Lambda_\Delta(\xi) + G_1(\xi)\overline{H_1^*(\xi)}\Lambda_\Delta(\xi) &= 2\Lambda_\Delta(\xi), a.e. \\ G_0(\xi)\overline{H_0^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}\Lambda_\Delta(\xi) + G_1(\xi)\overline{H_1^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}\Lambda_\Delta(\xi) &= 0, a.e. \end{aligned}$$

Proof. The fact that $\{\psi_{1,\eta}\}_{\eta\in\Gamma}$ generates the elements of V_1, implies that for any $f \in V_1$,

$$\langle f, \psi_{1,\eta}\rangle = \sum_{\gamma\in\Gamma}\langle f, \pi_\gamma\psi\rangle\langle\pi_\gamma\phi, \psi_{1,\eta}\rangle + \sum_{\gamma\in\Gamma}\langle f, \pi_\gamma\psi^*\rangle\langle\pi_\gamma\phi^*, \psi_{1,\eta}\rangle, \quad (\eta \in \Gamma).$$

Now define

$$\begin{aligned} c_0(\gamma) &= \langle f, \pi_\gamma\psi\rangle, \\ c_1(\gamma) &= \langle f, \psi_{1,\gamma}\rangle, \\ d_0(\gamma) &= \langle f, \pi_\gamma\psi^*\rangle, \end{aligned}$$

and also let

$$g_0(\eta(\delta(\gamma))^{-1}) = \langle\pi_\gamma\phi, \psi_{1,\eta}\rangle, \quad g_1(\eta(\delta(\gamma))^{-1}) = \langle\pi_\gamma\phi^*, \psi_{1,\eta}\rangle.$$

We have

$$c_1(\eta) = \sum_{\gamma\in\Gamma} c_0(\gamma)g_0(\eta(\delta(\gamma))^{-1}) + \sum_{\gamma\in\Gamma} d_0(\gamma)g_1(\eta(\delta(\gamma))^{-1})$$

By takeing the Fourier series, we have

$$\begin{aligned}
C_1(\xi) &= \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma} c_0(\gamma)g_0(\eta(\delta(\gamma))^{-1})\overline{\xi(\eta)} \\
&+ \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma} d_0(\gamma)g_1(\eta(\delta(\gamma))^{-1})\overline{\xi(\eta)} \\
&= \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma} c_0(\gamma)g_0(\eta)\overline{\xi(\eta\delta(\gamma))} \\
&+ \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma} d_0(\gamma)g_1(\eta)\overline{\xi(\eta\delta(\gamma))} \\
&= \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma} c_0(\gamma)g_0(\eta)\overline{\xi(\eta)\xi(\delta(\gamma))} \\
&+ \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma} d_0(\gamma)g_1(\eta)\overline{\xi(\eta)\xi(\delta(\gamma))} \\
&= \sum_{\gamma\in\Gamma} c_0(\gamma)\overline{\xi(\delta(\gamma))}\sum_{\eta\in\Gamma} g_0(\eta)\overline{\xi(\eta)} \\
&+ \sum_{\gamma\in\Gamma} d_0(\gamma)\overline{\xi(\delta(\gamma))}\sum_{\eta\in\Gamma} g_1(\eta)\overline{\xi(\eta)} \\
&= C_0(\xi\delta)G_0(\xi) + D_0(\xi\delta)G_1(\xi).
\end{aligned}$$

On the other hand, we know

$$\begin{aligned}
c_0(\gamma) &= \int_G f(t)\overline{\pi_\gamma\psi(t)}dt = \int_G f(t\gamma)\overline{\psi(t)}dt \\
&= \int_G f(t\gamma)\sum_{\eta\in\Gamma}|\delta|^{\frac{1}{2}}h_0^*(\eta)\overline{\psi(\eta^{-1}\delta(t))}dt \\
&= \sum_{\eta\in\Gamma}|\delta|^{\frac{1}{2}}h_0^*(\eta)\int_G f(t)\overline{\psi(\delta(t)\eta^{-1}(\delta(\gamma))^{-1})}dt \\
&= \sum_{\eta\in\Gamma}|\delta|^{\frac{1}{2}}h_0^*(\eta(\delta(\gamma))^{-1})\int_G f(t)\overline{\psi(\delta(t)\eta^{-1})}dt \\
&= \sum_{\eta\in\Gamma} h_0^*(\eta(\delta(\gamma))^{-1})c_1(\eta).
\end{aligned}$$

Similarly,

$$d_0(\gamma) = \sum_{\eta\in\Gamma} h_1^*(\eta(\delta(\gamma))^{-1})c_1(\eta).$$

Now, their Fourier series are

$$\begin{aligned}
C_0(\xi\delta) &= \sum_{\gamma\in\Gamma} c_0(\gamma)\overline{\xi(\delta(\gamma))} = \sum_{\gamma\in\Gamma}\sum_{\eta\in\Gamma} \overline{h_0^*(\eta(\delta(\gamma))^{-1})} c_1(\eta)\overline{\xi(\delta(\gamma))} \\
&= \sum_{\eta\in\Gamma}\sum_{\gamma\in\Gamma} c_1(\eta)\overline{h_0^*(\gamma)}\overline{\xi(\eta\gamma^{-1})} \\
&= \sum_{\eta\in\Gamma} c_1(\eta)\overline{\xi(\eta)} \sum_{\gamma\in\Gamma} \overline{h_0^*(\gamma)}\xi(\gamma) \\
&= \frac{1}{2}[\sum_{\eta\in\Gamma} c_1(\eta)\overline{\xi(\eta)} \sum_{\gamma\in\Gamma} \overline{h_0^*(\gamma)}\xi(\gamma) \\
&+ \sum_{\eta\in\Gamma} c_1(\eta)\overline{\xi(\eta)1_{\hat{\Gamma}}(\delta^{-1}(\eta))} \sum_{\gamma\in\Gamma} \overline{h_0^*(\gamma)}\xi(\gamma)1_{\hat{\Gamma}}(\delta^{-1}(\gamma))] \\
&= \frac{1}{2}[C_1(\xi)\overline{H_0^*(\xi)} + C_1(\xi 1_{\hat{\Gamma}}\delta^{-1})\overline{H_0^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}].
\end{aligned}$$

Similarly,

$$D_0(\xi\delta) = \frac{1}{2}[C_1(\xi)\overline{H_1^*(\xi)} + C_1(\xi 1_{\hat{\Gamma}}\delta^{-1})\overline{H_1^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}].$$

Combining the above relations, we find that

$$\begin{aligned}
2C_1(\xi) &= G_0(\xi)][C_1(\xi)\overline{H_0^*(\xi)} + C_1(\xi 1_{\hat{\Gamma}}\delta^{-1})\overline{H_0^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}] \\
&+ G_1(\xi)[C_1(\xi)\overline{H_1^*(\xi)} + C_1(\xi 1_{\hat{\Gamma}}\delta^{-1})\overline{H_1^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}] \\
&= [G_0(\xi)\overline{H_0^*(\xi)} + G_1(\xi)\overline{H_1^*(\xi)}]C_1(\xi) \\
&+ [G_0(\xi)\overline{H_0^*(\xi 1_{\hat{\Gamma}}\delta^{-1})} + G_1(\xi)\overline{H_1^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}]C_1(\xi 1_{\hat{\Gamma}}\delta^{-1}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
G_0(\xi)\overline{H_0^*(\xi)}\Lambda_\Delta(\xi) + G_1(\xi)\overline{H_1^*(\xi)}\Lambda_\Delta(\xi) &= 2\Lambda_\Delta(\xi), a.e. \\
G_0(\xi)\overline{H_0^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}\Lambda_\Delta(\xi) + G_1(\xi)\overline{H_1^*(\xi 1_{\hat{\Gamma}}\delta^{-1})}\Lambda_\Delta(\xi) &= 0, a.e.
\end{aligned}$$

□

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