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Linear combinations of wave packet frames for $L^2(\mathbb{R}^d)$

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ABSTRACT

In this paper we study necessary and sufficient conditions for some types of linear combinations of wave packet frames to be a frame for $L^2(\mathbb{R}^d)$. Further, we illustrate our results with some examples and applications.

1. Introduction

The wave packet system was introduced by Cordoba and Fefferman [6] by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. The wave packet systems have been studied by several authors, see [5, 7, 8, 11, 12, 13, 14, 15, 16, 17].

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Linear combination of frames (or redundant building blocks) is important in applied mathematics. For example, in computation processes (e.g. signal processing which depends on series expansions in terms of frames or redundant building blocks), where linear combinations of signals comes into play and carries overlapped signal to the sink (or antenna). A physical interpretation of this can be understood with the idea of transmission of a signal into the space. If we transmit a weak signal (e.g., Radio FM signal) into the space, its superposition (i.e., linear combination) is required with a signal of high energy. Aldroubi in [1] considered the following problem: given a Hilbert frame $\{f_k\}$ for a separable Hilbert space $\mathcal{H}$, define a set of functions ψ_j by taking linear combinations of the frame elements f_k. What are the conditions on the coefficients in the linear combinations, so that the new system $\{\psi_j\}$ constitutes a frame for $\mathcal{H}$? More prpecisely, Aldroubi considered a linear combination of the form

$$\psi_j = \sum_{k=1}^{\infty} \alpha_{j,k} f_k, \ (j \in \ \mathbb{N} \tag{1.1}$$

where $\alpha_{j,k}$ are scalars. Aldroubi proved sufficient conditions on $\{\alpha_{j,k}\}$ such that $\{\psi_j\}$ constitutes a frame for $\mathcal{H}$. Christensen in [3] proved sufficient conditions which are different from those proved by Aldroubi.

1.1. Plan of the Work

In this paper we consider wave packet system of functions generated by combined action of dilation, translation and modulation operators on $L^2(\mathbb{R}^d)$. More precisely, we consider a system of the form

$$\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}, \tag{1.2}$$

where $\psi \in L^2(\mathbb{R}^d)$, $\{A_j\}_{j\in\mathbb{Z}} \subset GL_d(\mathbb{R})$, $B \in GL_d(\mathbb{R})$ and $\{C_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^d$ and call it *wave packet system* in $L^2(\mathbb{R}^d)$. A frame for $L^2(\mathbb{R}^d)$ of the form $\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is called an *wave packet frame*. Sufficient conditions for the wave packet system given in (1.2) to be a frames for $L^2(\mathbb{R}^d)$ can be found in [5]. We discuss some types of linear combinations of wave packet frames in $L^2(\mathbb{R}^d)$. First we consider a linear combination of wave packet frames indexed by blocks of countable sets (see equation (3.1)). Necessary and sufficient conditions for this type of linear combinations of wave packet frames to constitutes a frame for $L^2(\mathbb{R}^d)$ are given. We present sufficient conditions on the scalars which appear in a linear combination of wave packet frames of the type given in (1.1) to constitutes a frame for $L^2(\mathbb{R}^d)$. An application of the later conditions is given.

2. Preliminaries

In this section, we recall basic notations and definitions to make the paper self-contained. Let $\mathcal{H}$ be a separable real (or complex) Hilbert space with inner product $\langle ., . \rangle$ linear in the first entry. A countable sequence $\{f_k\} \subset \mathcal{H}$ is called a frame (or Hilbert frame) for $\mathcal{H}$, if there exist numbers $0 < a_o \le b_o < \infty$ such that

$$a_o \|f\|^2 \le \|\{\langle f, f_k \rangle\}\|_{\ell^2}^2 \le b_o \|f\|^2 \text{ for all } f \in \mathcal{H}.$$

The numbers a_o and b_o are called lower and upper frame bounds, respectively. They are not unique. Associated with a frame $\{f_k\}$ for $\mathcal{H}$, there are three bounded linear operators:

$$\textit{synthesis operator} \quad V : \ell^2 \to \mathcal{H}, \quad V(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \ \{c_k\} \in \ell^2,$$

$$\textit{analysis operator} \quad V^* : \mathcal{H} \to \ell^2, \quad V^*(f) = \{\langle f, f_k \rangle\}, \ \ f \in \mathcal{H},$$

$$\textit{frame operator} \quad S = VV^* : \mathcal{H} \to \mathcal{H}, \quad S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \ \ f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k \quad \left(= \sum_{k=1}^{\infty} \langle f, S^{-1}f_k \rangle f_k \right). \tag{2.1}$$

The scalars $\{\langle S^{-1}f, f_k \rangle\}$ are called *frame coefficients* of the vector $f \in \mathcal{H}$. The representation of f in the reconstruction formula (2.1) need not be unique. This reflects one of the important properties of frames in applied mathematics. For utility of frames in different directions in applied mathematics, see [2, 4, 9, 10].
As usual $L^2(\mathbb{R}^d)$ denote the Hilbert space of complex-valued Lebesgue square integrable functions with the inner product given by

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f \, \overline{g} \, dt,$$

where $\overline{g}$ denote the complex conjugate of g.
By $GL_d(\mathbb{R})$ we denote the set of all invertible d by d matrices over $\mathbb{R}$. Let $a, b \in \mathbb{R}^d$ and let C be a real $d \times d$ matrix. We consider operators $T_a, \ E_b, D_C : L^2(\mathbb{R}^d) \to \ L^2(\mathbb{R}^d)$ given by

Translation by $a \leftrightarrow \ T_a f(t) = f(t - a)$

Modulation by $b \leftrightarrow \ E_b f(t) = e^{2\pi i b \cdot t} f(t)$, where $b \cdot t$ denotes the inner product of b and t in $\mathbb{R}^d$.

Dilation by $C \leftrightarrow D_C f(t) = |detC|^{\frac{1}{2}} f(Ct)$.

For $f \in L^1(\mathbb{R}^d) \bigcap L^2(\mathbb{R}^d)$, the *Fourier transform* $\hat{f}$ is defined by

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \cdot \gamma} dx, \ \gamma \in \mathbb{R}^d.$$

3. Wave Packet Frames for $L^2(\mathbb{R}^d)$

Let $\psi \in L^2(\mathbb{R}^d)$, $\{A_j\}_{j \in \mathbb{Z}} \subset GL_d(\mathbb{R})$, $B \in GL_d(\mathbb{R})$ and $\{C_m\}_{m \in \mathbb{Z}} \subset \mathbb{R}^d$ and $\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m \in \mathbb{Z}, k \in \mathbb{Z}^d}$ be a given wave packet frame in $L^2(\mathbb{R}^d)$.
Consider a sequence of vectors $\{\Psi_{r,s,t}\}_{r,t \in \mathbb{Z}, s \in \mathbb{Z}^d}$ defined by

$$\Psi_{r,s,t} = \sum_{(j,k,m) \in \mathbb{I}_{r,s,t}} \alpha_{j,k,m} D_{A_j} T_{Bk} E_{C_m} \psi, \ (r, t \in \mathbb{Z}, s \in \mathbb{Z}^d), \tag{3.1}$$

where $\bigcup_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \mathbb{I}_{r,s,t} = \mathbb{Z}\times\mathbb{Z}^d\times\mathbb{Z}$, $\mathbb{I}_{r,s,t} \bigcap \mathbb{I}_{r',s',t'} = \emptyset$, $(r,s,t) \neq (r',s',t')$ for all $r,t,r',t' \in \mathbb{Z}$, $s,s' \in \mathbb{Z}^d$ and $\alpha_{j,k,m}$ are scalars. The system $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is not a frame for $L^2(\mathbb{R}^d)$, in general.

Regarding the existence of wave packet frames for $L^2(\mathbb{R}^d)$ of the type $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$, we have following example.

Example 3.1. Let $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a wave packet frame for $L^2(\mathbb{R}^d)$ and let $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ be the sequence of vectors given in (3.1).

(i) Choose $\alpha_{j,k,m} = 1$, $j,m \in \mathbb{Z}, k \in \mathbb{Z}^d$ and $\mathbb{I}_{r,s,t} = \{(r,s,t)\}$, $r,t \in \mathbb{Z}, s \in \mathbb{Z}^d$. Then

$$\begin{aligned}\Psi_{r,s,t} &= \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} \alpha_{j,k,m} D_{A_j}T_{Bk}E_{C_m}\psi \\ &= D_{A_r}T_{Bs}E_{C_t}\psi.\end{aligned}$$

Therefore, by hypothesis the system $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$.

(ii) Let $E_{C_m} = I_{L^2(\mathbb{R}^d)}$ (the identity operator on $L^2(\mathbb{R}^d)$) for all $m \in \mathbb{Z}$, A be any expansive $d\times d$ matrix (i.e., every eigenvalue ζ of A satisfies $|\zeta| > 1$) and let $A_j = A^j$ for all $j \in \mathbb{Z}$. Then, there exist $\psi \in L^2(\mathbb{R}^d)$ such that $\hat{\psi} = \chi_E$, where E is a compact subset of $\mathbb{R}^d$. Therefore

$$\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} = \{D_{A^j}T_{Bk}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$ (see Theorem 12.3 in [10] p. 357).
Choose $\alpha_{j,k,m} = j + \|k\|_{\mathbb{Z}^d} + m$, $j,m \in \mathbb{Z}, k \in \mathbb{Z}^d$ and $\mathbb{I}_{r,s,t} = \{(r,s,t)\}$, where $r,t \in \mathbb{Z}, s \in \mathbb{Z}^d$. Then

$$\begin{aligned}\Psi_{r,s,t} &= \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} \alpha_{j,k,m} D_{A_j}T_{Bk}E_{C_m}\psi \\ &= (r + \|s\|_{\mathbb{Z}^d} + t) D_{A_r}T_{Bs}E_{C_t}\psi.\end{aligned}$$

The sequence $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is not a frame for $L^2(\mathbb{R}^d)$. Indeed, let b_o be an upper frame bound for $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$. Then, we can find a positive integer n_o such that $b_o < n_o$.
Fix $k_o \in \mathbb{Z}^d$ and choose $f_o = D_{A_{n_o}}T_{Bk_o}E_{C_{n_o}}\psi \in L^2(\mathbb{R}^d)$ (note that $\|f_o\| = 1$).
Then

$$\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle f_o, \Psi_{r,s,t}\rangle|^2 = \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |(r + \|s\|_{\mathbb{Z}^d} + t)\langle D_{A_{n_o}}T_{Bk_o}E_{C_{n_o}}\psi, D_{A_r}T_{Bs}E_{C_t}\psi\rangle|^2. \tag{3.2}$$

By using (3.2) and orthonormality of the system $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$, we have

$$\begin{aligned}|2n_o + \|k_o\|_{\mathbb{Z}^d}|^2 = |n_o + \|k_o\|_{\mathbb{Z}^d} + n_o|^2 &= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle f_o, \Psi_{r,s,t}\rangle|^2 \\ &\leq b_o\|f_o\|^2 \\ &= b_o,\end{aligned}$$

which contradicts the fact that $b_o < n_o$. Hence $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is not a frame for $L^2(\mathbb{R}^d)$.

The following theorem provides sufficient conditions on the scalars $\{\alpha_{j,k,m}\}$ such that the sequence of vectors $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ constitutes a frame for $L^2(\mathbb{R}^d)$.

Theorem 3.2. *Let $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a wave packet frame for $L^2(\mathbb{R}^d)$ and let $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \subset L^2(\mathbb{R}^d)$ be a sequence of vectors (defined in (3.1)). Assume that $\sup\limits_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\mathbb{I}_{r,s,t}| < \infty$, $\{\alpha_{j,k,m}\}$ is positively confined sequence (i.e., $\sup\limits_{j,k,m}|\alpha_{j,k,m}| < \infty$) and*

$$\inf_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}\left\{\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}}|\alpha_{j,k,m}|^2 - \sum_{\substack{(j,k,m),(j',k',m')\in\mathbb{I}_{r,s,t}\\(j,k,m)\neq(j',k',m')}}|\alpha_{j,k,m}|\,|\alpha_{j',k',m'}|\right\} > 0.$$

Then, $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$.

Proof. Let a_o, b_o be frame bounds for the frame $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$.

Choose $K = \sup\limits_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\mathbb{I}_{r,s,t}| < \infty$.

We compute

$$\begin{aligned}
\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}\left|\langle f,\Psi_{r,s,t}\rangle\right|^2 &= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}\left|\left\langle f,\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}}\alpha_{j,k,m}D_{A_j}T_{Bk}E_{C_m}\psi\right\rangle\right|^2\\
&= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}\left|\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}}\overline{\alpha_{j,k,m}}\langle f,D_{A_j}T_{Bk}E_{C_m}\psi\rangle\right|^2\\
&\leq K\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}\ \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}}|\overline{\alpha_{j,k,m}}|^2|\langle f,D_{A_j}T_{Bk}E_{C_m}\psi\rangle|^2\\
&\leq K\sup_{j,k,m}|\alpha_{j,k,m}|^2\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}\ \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}}|\langle f,D_{A_j}T_{Bk}E_{C_m}\psi\rangle|^2\\
&\leq \left(K\sup_{j,k,m}|\alpha_{j,k,m}|^2 b_o\right)\|f\|^2,\ f\in L^2(\mathbb{R}^d).
\end{aligned}$$

Hence upper frame condition for $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is satisfied.

For the lower frame condition, we compute

$$
\begin{aligned}
&\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle f,\Psi_{r,s,t}\rangle|^2 \\
&= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \left|\left\langle f, \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} \alpha_{j,k,m} D_{A_j} T_{Bk} E_{C_m}\psi\right\rangle\right|^2 \\
&= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \left|\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} \overline{\alpha_{j,k,m}} \langle f, D_{A_j} T_{Bk} E_{C_m}\psi\rangle\right|^2 \\
&= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} [\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} |\overline{\alpha_{j,k,m}}|^2 |\langle f, D_{A_j} T_{Bk} E_{C_m}\psi\rangle|^2 \\
&+ \sum_{\substack{(j,k,m),(j',k',m')\in\mathbb{I}_{r,s,t}\\(j,k,m)\neq(j',k',m')}} \alpha_{j,k,m}\overline{\alpha_{j',k',m'}}\,\overline{\langle f, D_{A_j} T_{Bk} E_{C_m}\psi\rangle}\langle f, D_{A_{j'}} T_{Bk'} E_{C_{m'}}\psi\rangle] \\
&\geq \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} [\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} |\alpha_{j,k,m}|^2 |\langle f, D_{A_j} T_{Bk} E_{C_m}\psi\rangle|^2 \\
&- \sum_{\substack{(j,k,m),(j',k',m')\in\mathbb{I}_{r,s,t}\\(j,k,m)\neq(j',k',m')}} |\alpha_{j,k,m}|\,|\alpha_{j',k',m'}|\,|\overline{\langle f, D_{A_j} T_{Bk} E_{C_m}\psi\rangle}|\,|\langle f, D_{A_{j'}} T_{Bk'} E_{C_{m'}}\psi\rangle|] \\
&\geq \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle f, D_{A_r} T_{Bs} E_{C_t}\psi\rangle|^2 \left[\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} |\alpha_{j,k,m}|^2 - \sum_{\substack{(j,k,m),(j',k',m')\in\mathbb{I}_{r,s,t}\\(j,k,m)\neq(j',k',m')}} |\alpha_{j,k,m}|\,|\alpha_{j',k',m'}|\right] \\
&\geq a_o \inf_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \left[\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} |\alpha_{j,k,m}|^2 - \sum_{\substack{(j,k,m),(j',k',m')\in\mathbb{I}_{r,s,t}\\(j,k,m)\neq(j',k',m')}} |\alpha_{j,k,m}|\,|\alpha_{j',k',m'}|\right] \|f\|^2,
\end{aligned}
$$

for all $f \in L^2(\mathbb{R}^d)$. Hence $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$. □

Next, we give a necessary condition for the sequence of vectors $\{\Psi_{r,s,t}\}$ defined in (3.1) to be a frame for $L^2(\mathbb{R}^d)$.

Proposition 3.3. *Let $\{D_{A_j} T_{Bk} E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a wave packet frame for $L^2(\mathbb{R}^d)$ and let $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \subset L^2(\mathbb{R}^d)$ be a sequence given in (3.1). If $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$, then there exists a constant $\lambda > 0$ such that*

$$\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle \Psi_{r,s,t}, f\rangle|^2 \leq \lambda \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m}\psi, f\rangle|^2 \text{ for all } f \in L^2(\mathbb{R}^d). \tag{3.3}$$

Proof. Assume that $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with frame bounds a', b'. Then,

$$\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle \Psi_{r,s,t}, f\rangle|^2 \leq b'\|f\|^2, \ f \in L^2(\mathbb{R}^d). \tag{3.4}$$

Let a_o be a lower frame bound of the frame $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$.
Then

$$a_o\|f\|^2 \leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle f, D_{A_j}T_{Bk}E_{C_m}\psi\rangle|^2,\ f\in L^2(\mathbb{R}^d).$$

That is,

$$\|f\|^2 \leq \frac{1}{a_o}\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle f, D_{A_j}T_{Bk}E_{C_m}\psi\rangle|^2,\ f\in L^2(\mathbb{R}^d). \tag{3.5}$$

Choose $\lambda = \frac{b'}{a_o} > 0$. Then, by using (3.4) and (3.5), we have

$$\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle \Psi_{r,s,t}, f\rangle|^2 \leq \lambda \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle f, D_{A_j}T_{Bk}E_{C_m}\psi\rangle|^2 \text{ for all } f\in L^2(\mathbb{R}^d).$$

The proposition is proved. □

We now demonstrate by a concrete example that the condition given in Proposition 3.3 is not sufficient.

Example 3.4. Let $E_{C_m} = I_{L^2(\mathbb{R}^d)}$ (the identity operator on $L^2(\mathbb{R}^d)$) for all $m\in\mathbb{Z}$, let A be any expansive $d\times d$ matrix and $A_j = A^j$ for all $j\in\mathbb{Z}$. Choose $\alpha_{j,k,m} = 1$, $j,m\in\mathbb{Z}, k\in\mathbb{Z}^d$. Then, there exist $\psi\in L^2(\mathbb{R}^d)$ such that $\widehat{\psi} = \chi_E$, where E is a compact subset of $\mathbb{R}^d$ and hence, $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} = \{D_{A^j}T_{Bk}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is an orthonormal system in $L^2(\mathbb{R}^d)$.
Let $k_o\in\mathbb{Z}^d$ be arbitrary but fixed.

Define

$$\begin{aligned}
\mathbb{I}_{1,k_o,t} &= \{(1,k_o,t),(2,k_o,t)\},\ t\in\mathbb{Z}.\\
\mathbb{I}_{r,k_o,t} &= \{(r+1,k_o,t)\}, r\geq 2,\ t\in\mathbb{Z}.\\
\mathbb{I}'_{r,k_o,t} &= \{(r,k_o,t)\}, r\leq 0,\ r,\ t\in\mathbb{Z}.\\
\mathbb{I}_{r,s,t} &= \{(r,s,t)\}, s\neq k_o,\ r,t\in\mathbb{Z},\ s\in\mathbb{Z}^d
\end{aligned}$$

Choose $f_0 = D_{A^1}T_{Bk_o}E_{C_0}\psi - D_{A^2}T_{Bk_o}E_{C_0}\psi \quad (= D_{A^1}T_{Bk_o}\psi - D_{A^2}T_{Bk_o}\psi)$. Then, f_0 is a nonzero vector in $L^2(\mathbb{R}^d)$.
By using orthonormality of the system $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$, we compute

$$\begin{aligned}
\Big\langle f_o, \sum_{\substack{j,k,m\in\mathbb{I}_{1,k_o,t}\\ t\in\mathbb{Z}}} D_{A_j}T_{Bk}E_{C_m}\psi\Big\rangle &= \Big\langle D_{A^1}T_{Bk_o}E_0\psi - D_{A^2}T_{Bk_o}E_0\psi, \sum_{\substack{(j,k,m)\in\mathbb{I}_{1,k_o,t}\\ t\in\mathbb{Z}}} D_{A_j}T_{Bk}E_{C_m}\psi\Big\rangle\\
&= \langle D_{A^1}T_{Bk_o}\psi, D_{A^1}T_{Bk_o}\psi\rangle + \langle D_{A^1}T_{Bk_o}\psi, D_{A^2}T_{Bk_o}\psi\rangle\\
&- \langle D_{A^2}T_{Bk_o}\psi, D_{A^1}T_{Bk_o}\psi\rangle - \langle D_{A^2}T_{Bk_o}\psi, D_{A^2}T_{Bk_o}\psi\rangle\\
&= 1+0-0-1\\
&= 0.
\end{aligned}$$

Again by using orthonormality of the system $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$, we have

$$\begin{aligned}\Bigg\langle f_0, \sum_{\substack{(j,k,m)\in\mathbb{I}_{r,k_o,t}\\ r\geq 2, r,t\in\mathbb{Z}}} D_{A_j}T_{Bk}E_{C_m}\psi\Bigg\rangle &= \Bigg\langle D_{A^1}T_{Bk_o}E_{C_o}\psi - D_{A^2}T_{Bk_o}E_{C_0}\psi, \sum_{\substack{(j,k,m)\in\mathbb{I}_{r,k_o,t}\\ r\geq 2, r,t\in\mathbb{Z}}} D_{A_j}T_{Bk}E_{C_m}\psi\Bigg\rangle\\ &= 0,\end{aligned}$$

$$\begin{aligned}\Bigg\langle f_0, \sum_{\substack{(j,k,m)\in\mathbb{I}'_{r,k_o,t}\\ r\leq 0, r,t\in\mathbb{Z}}} D_{A_j}T_{Bk}E_{C_m}\psi\Bigg\rangle &= \Bigg\langle D_{A^1}T_{Bk_o}E_{C_o}\psi - D_{A^2}T_{Bk_o}E_{C_0}\psi, \sum_{\substack{(j,k,m)\in\mathbb{I}'_{r,k_o,t}\\ r\leq 0, r,t\in\mathbb{Z}}} D_{A_j}T_{Bk}E_{C_m}\psi\Bigg\rangle\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\Bigg\langle f_0, \sum_{\substack{(j,k,m)\in\mathbb{I}_{r,s,t}\\ r,t\in\mathbb{Z}, s\in\mathbb{Z}^d\setminus\{k_o\}}} D_{A_j}T_{Bk}E_{C_m}\psi\Bigg\rangle &= \Bigg\langle D_{A^1}T_{Bk_o}E_{C_o}\psi - D_{A^2}T_{Bk_o}E_{C_0}\psi, \sum_{\substack{(j,k,m)\in\mathbb{I}_{r,s,t}\\ r,t\in\mathbb{Z}, s\in\mathbb{Z}^d\setminus\{k_o\}}} D_{A_j}T_{Bk}E_{C_m}\psi\Bigg\rangle\\ &= 0.\end{aligned}$$

Therefore, $\langle f_0, \Psi_{r,s,t}\rangle = 0$ for all $r,t\in\mathbb{Z}, s\in\mathbb{Z}^d$. That is, the system $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is not complete in $L^2(\mathbb{R}^d)$. Hence $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is not a frame for $L^2(\mathbb{R}^d)$.

To show $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ satisfies the inequality (3.3), we compute

$$\begin{aligned}&\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle\Psi_{r,s,t}, f\rangle|^2\\ &= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}\Bigg|\Bigg\langle\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}}\alpha_{j,k,m}D_{A_j}T_{Bk}E_{C_m}\psi, f\Bigg\rangle\Bigg|^2\\ &= \sum_{t\in\mathbb{Z}}\Bigg|\Bigg\langle\sum_{(j,k,m)\in\mathbb{I}_{1,k_o,t}} D_{A_j}T_{Bk}E_{C_m}\psi, f\Bigg\rangle\Bigg|^2 + \sum_{r\geq 2, r,t\in\mathbb{Z}}\Bigg|\Bigg\langle\sum_{(j,k,m)\in\mathbb{I}_{r,k_o,t}} D_{A_j}T_{Bk}E_{C_m}\psi, f\Bigg\rangle\Bigg|^2\\ &\qquad + \sum_{r\leq 0, r,t\in\mathbb{Z}}\Bigg|\Bigg\langle\sum_{(j,k,m)\in\mathbb{I}'_{r,k_o,t}} D_{A_j}T_{Bk}E_{C_m}\psi, f\Bigg\rangle\Bigg|^2\\ &\qquad\qquad + \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d\setminus\{k_o\}}\Bigg|\Bigg\langle\sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} D_{A_j}T_{Bk}E_{C_m}\psi, f\Bigg\rangle\Bigg|^2\\ &\leq 2\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}|\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2.\end{aligned}$$

Therefore, the system $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ satisfies the condition (3.3) given in the Proposition 3.3 for $\lambda = 2$.

The following theorem give a necessary condition for the sequence of vectors $\{\Psi_{r,s,t}\}$ defined in (3.1) to be a frame for $L^2(\mathbb{R}^d)$ in terms of scalars $\alpha_{j,k,m}$.

Theorem 3.5. *Let $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a Bessel sequence for $L^2(\mathbb{R}^d)$ with bounds b_o, and let $\{\alpha_{j,k,m}\}$ is positively confined sequence. If $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with lower bounds γ_o, then*

$$\gamma_o \leq K \sup_{j,k,m} |\alpha_{j,k,m}|^2 b_o,$$

where $K = \sup_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\mathbb{I}_{r,s,t}| < \infty.$

Proof. We compute

$$\begin{aligned}\gamma_o\|f\|^2 \leq \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \left|\langle f, \Psi_{r,s,t}\rangle\right|^2 &= \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \left|\left\langle f, \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} \alpha_{j,k,m} D_{A_j}T_{Bk}E_{C_m}\psi\right\rangle\right|^2 \\ &\leq K \sup_{j,k,m} |\alpha_{j,k,m}|^2 \sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \sum_{(j,k,m)\in\mathbb{I}_{r,s,t}} |\langle f, D_{A_j}T_{Bk}E_{C_m}\psi\rangle|^2 \\ &\leq \left(K \sup_{j,k,m} |\alpha_{j,k,m}|^2 b_o\right)\|f\|^2, \ f \in L^2(\mathbb{R}^d),\end{aligned}$$

where $K = \sup_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\mathbb{I}_{r,s,t}| < \infty.$
This completes the proof. □

If coefficients (in the sense of Fourier) associated with the system $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ are images of coefficients associated with the wave packet frame for $L^2(\mathbb{R}^d)$ under a certain bounded linear operator, then situation is different than those given in the Proposition 3.3. In this direction, the following result characterize the system $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ as a frame for $L^2(\mathbb{R}^d)$.

Theorem 3.6. *Let $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a wave packet frame for $L^2(\mathbb{R}^d)$ and let $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} \subset L^2(\mathbb{R}^d)$ be a sequence given in (3.1). Assume that $T : \ell^2(\mathbb{Z}^{d+2}) \to \ell^2(\mathbb{Z}^{(d+2)})$ be a bounded linear operator such that*

$$T(\{\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}) = \{\langle \Psi_{r,s,t}, f\rangle\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}, \ f \in L^2(\mathbb{R}^d).$$

Then, $\{\Psi_{r,s,t}\}_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ if and only if there exists a constant $\mu > 0$ such that

$$\sum_{r,t\in\mathbb{Z},s\in\mathbb{Z}^d} |\langle \Psi_{r,s,t}, f\rangle|^2 \geq \mu \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2 \text{ for all } f \in L^2(\mathbb{R}^d).$$

Proof. For the proof of this theorem, we refer [1, 17]. □

Now, we consider linear combinations of wave packet frame of the form

$$\Phi_s = \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \alpha_{s,j,k,m} D_{A_j}T_{Bk}E_{C_m}\psi \ \ (s \in \mathbb{Z}).$$

The following theorem gives sufficient conditions on the scalars $\{\alpha_{s,j,k,m}\}_{s,j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ such that the vectors $\{\Phi_s\}_{s\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R}^d)$.

Theorem 3.7. *Suppose that* $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a wave packet frame for* $L^2(\mathbb{R}^d)$ *with bounds* a_o, b_o *and let* $\{\alpha_{s,j,k,m}\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \subseteq \ell^2(\mathbb{Z}^{d+2})$, *for each* $s \in \mathbb{Z}$. *If*

$$\mu := \inf_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\left[\sum_{s\in\mathbb{Z}}|\alpha_{s,j,k,m}|^2 - \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}}\left|\sum_{s\in\mathbb{Z}}\alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right|\right] > 0,$$

$$\nu := \sup_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\sum_{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d}\left|\sum_{s\in\mathbb{Z}}\alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| < \infty.$$

Then, $\{\Phi_s\}_{s\in\mathbb{Z}} \equiv \left\{\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\alpha_{s,j,k,m}D_{A_j}T_{Bk}E_{C_m}\psi\right\}_{s\in\mathbb{Z}}$ *is a frame for* $L^2(\mathbb{R}^d)$ *with frame bounds* $\mu a_o,\ \nu b_o$.

Proof. We compute

$$\begin{aligned}
\sum_{s\in\mathbb{Z}}|\langle\Phi_s, f\rangle|^2 &= \sum_{s\in\mathbb{Z}}\left|\left\langle\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\alpha_{s,j,k,m}D_{A_j}T_{Bk}E_{C_m}\psi, f\right\rangle\right|^2\\
&= \sum_{s\in\mathbb{Z}}\left|\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\alpha_{s,j,k,m}\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle\right|^2\\
&= \sum_{s\in\mathbb{Z}}\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}|\alpha_{s,j,k,m}|^2|\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2\\
&\quad+\sum_{s\in\mathbb{Z}}\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}}\alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle\langle f, D_{A_{j'}}T_{Bk'}E_{C_{m'}}\psi\rangle\\
&= (\maltese) + (\maltese\,\maltese).
\end{aligned}\tag{3.6}$$

By Cauchy-Schwartz's inequality, we have

$$\begin{aligned}
|(\maltese\,\maltese)| &\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}}|\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle\langle f, D_{A_{j'}}T_{Bk'}E_{C_{m'}}\psi\rangle|\left|\sum_{s\in\mathbb{Z}}\alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right|\\
&\leq \left(\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}}|\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2\left|\sum_{s\in\mathbb{Z}}\alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right|\right)^{\frac{1}{2}}\\
&\quad\times\left(\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}}|\langle f, D_{A_{j'}}T_{Bk'}E_{C_{m'}}\psi\rangle|^2\left|\sum_{s\in\mathbb{Z}}\alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right|\right)^{\frac{1}{2}}.
\end{aligned}$$

We observe that the two terms in the last product are identical. Indeed, for all $f \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d \\ (j',k',m')\neq(j,k,m)}} |\langle f, D_{A_{j'}} T_{Bk'} E_{C_{m'}} \psi\rangle|^2 \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \\
&= \sum_{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d} \sum_{\substack{j,m\in\mathbb{Z},k\in\mathbb{Z}^d \\ (j,k,m)\neq(j',k',m')}} |\langle f, D_{A_{j'}} T_{Bk'} E_{C_{m'}} \psi\rangle|^2 \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d \\ (j,k,m)\neq(j',k',m')}} |\langle D_{A_j} T_{Bk} E_{C_m} \psi, f\rangle|^2 \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right|.
\end{aligned}$$

Therefore, for all $f \in L^2(\mathbb{R}^d)$, we have

$$|(\maltese\ \maltese)| \leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d \\ (j,k,m)\neq(j',k',m')}} |\langle D_{A_j} T_{Bk} E_{C_m} \psi, f\rangle|^2 \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right|. \tag{3.7}$$

Thus, by using (3.6) and (3.7), we have

$$\begin{aligned}
&\sum_{s\in\mathbb{Z}} |\langle \Phi_s, f\rangle|^2 \\
&= \sum_{s\in\mathbb{Z}} \left|\left\langle \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \alpha_{s,j,k,m} D_{A_j} T_{Bk} E_{C_m} \psi, f\right\rangle\right|^2 \\
&\geq \sum_{s\in\mathbb{Z}} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\alpha_{s,j,k,m}|^2 |\langle D_{A_j} T_{Bk} E_{C_m} \psi, f\rangle|^2 \\
&\qquad - \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d \\ (j,k,m)\neq(j',k',m')}} |\langle D_{A_j} T_{Bk} E_{C_m} \psi, f\rangle|^2 \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m} \psi, f\rangle|^2 \left[\sum_{s\in\mathbb{Z}} |\alpha_{s,j,k,m}|^2 - \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d \\ (j,k,m)\neq(j',k',m')}} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right|\right] \\
&\geq \mu \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m} \psi, f\rangle|^2 \\
&\geq \mu a_o \|f\|^2 \text{ for all } f \in L^2(\mathbb{R}^d).
\end{aligned}$$

Hence lower frame condition is satisfied.
For upper frame condition, by using (3.6) and (3.7), we compute

$$
\begin{aligned}
&\sum_{s\in\mathbb{Z}} |\langle \Phi_s, f\rangle|^2 \\
&\leq \sum_{s\in\mathbb{Z}} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\alpha_{s,j,k,m}|^2 |\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2 \\
&\qquad + \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d \\ (j,k,m)\neq(j',k',m')}} |\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2 \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2 \, [\, \sum_{s\in\mathbb{Z}} |\alpha_{s,j,k,m}|^2 \\
&\qquad + \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d \\ (j,k,m)\neq(j',k',m')}} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \,] \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2 \left[\sum_{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \right] \\
&\leq \nu \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j}T_{Bk}E_{C_m}\psi, f\rangle|^2 \\
&\leq \nu b_o \|f\|^2 \text{ for all } f \in L^2(\mathbb{R}^d).
\end{aligned}
$$

Hence $\{\Phi_s\}_{s\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R}^d)$ with desired frame bounds. □

3.1. Application:

The following example gives an application of Theorem 3.7.

Example 3.8. Let $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a wave packet frame for $L^2(\mathbb{R}^d)$. The set $\mathbb{Z}\times\mathbb{Z}^d\times\mathbb{Z}$ is a countable set, being a finite product of countable sets. Therefore, there exits a bijection $\Omega : \mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}^d\times\mathbb{Z}$. For $s \in \mathbb{Z}$, define $\{\alpha_{s,j,k,m}\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \subset \ell^2(\mathbb{Z}^{d+2})$ as follows:

$$
\alpha_{\substack{s,j,k,m \\ s\neq 0}} = \begin{cases} 1 + 1/s, & (j,k,m) = \Omega(s) \\ 0 & (j,k,m) \neq \Omega(s) \end{cases}
$$

and

$$
\alpha_{0,j,k,m} = \begin{cases} 1, & (j,k,m) = \Omega(0) \\ 0 & (j,k,m) \neq \Omega(0). \end{cases}
$$

By definition of $\Phi_s \left(= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \alpha_{s,j,k,m} D_{A_j}T_{Bk}E_{C_m}\psi\right)$, we have

$$
\Phi_s = \begin{cases} (1 + 1/s) D_{A_j}T_{Bk}E_{C_m}\psi, & (j,k,m) = \Omega(s), s \neq 0 \\ D_{A_j}T_{Bk}E_{C_m}\psi, & (j,k,m) = \Omega(0). \end{cases}
$$

For any $j, m \in \mathbb{Z}, k \in \mathbb{Z}^d$, we compute

$$\begin{aligned}
&\sum_{s\in\mathbb{Z}} |\alpha_{s,j,k,m}|^2 - \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \\
&= \begin{cases} |1+1/s|^2 - 0, & (j,k,m) = \Omega(s), s \neq 0 \\ 1 - 0 & (j,k,m) = \Omega(0) \end{cases} \\
&= \begin{cases} |1+1/s|^2, & (j,k,m) = \Omega(s), s \neq 0 \\ 1 & (j,k,m) = \Omega(0). \end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}
\mu &= \inf_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left[\sum_{s\in\mathbb{Z}} |\alpha_{s,j,k,m}|^2 - \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \right] \\
&= \inf_{s\in\mathbb{Z}\setminus\{0\}} \{|1+1/s|^2, 1\} \\
&= 1.
\end{aligned}$$

To find ν, we compute

$$\begin{aligned}
&\sum_{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \\
&= \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j,k,m}}\right| + \sum_{\substack{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d\\(j,k,m)\neq(j',k',m')}} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \\
&= \begin{cases} |1+1/s|^2 + 0, & (j,k,m) = \Omega(s), s \neq 0 \\ 1 + 0 & (j,k,m) = \Omega(0) \end{cases} \\
&= \begin{cases} |1+1/s|^2, & (j,k,m) = \Omega(s), s \neq 0 \\ 1 & (j,k,m) = \Omega(0). \end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}
\nu &= \sup_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left[\sum_{j',m'\in\mathbb{Z},k'\in\mathbb{Z}^d} \left|\sum_{s\in\mathbb{Z}} \alpha_{s,j,k,m}\overline{\alpha_{s,j',k',m'}}\right| \right] \\
&= \sup_{s\in\mathbb{Z}\setminus\{0\}} \{|1+1/s|^2, 1\} \\
&= 4.
\end{aligned}$$

Thus, we have shown that $\mu > 0$ and $\nu < \infty$. Hence by Theorem 3.7, $\{\Phi_s\}_{s\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R}^d)$.

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