Dilation of a family of $g$-frames

Mohammad Reza Abdollahpour$^{a,*}$

$^a$Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, Ardabil, Islamic Republic of Iran

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ABSTRACT

In this paper, we first discuss about canonical dual of $g$-frame
\[ \Lambda P = \{ \Lambda_i P \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \}, \]
where $\Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ is a $g$-frame for a Hilbert space $\mathcal{H}$ and $P$ is the orthogonal projection from $\mathcal{H}$ onto a closed subspace $M$. Next, we prove that, if $\Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \}$ and $\Theta = \{ \Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I \}$ be respective $g$-frames for non zero Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and $\Lambda$ and $\Theta$ are unitarily equivalent (similar), then $\Lambda$ and $\Theta$ can not be weakly disjoint. On the other hand, we study dilation property for $g$-frames and we show that two $g$-frames for a Hilbert space have dilation property, if they are disjoint, or they are similar, or one of them is similar to a dual $g$-frame of another one. We also prove that a family of $g$-frames for a Hilbert space has dilation property, if all the members in that family have the same deficiency.

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1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $F = \{ f_i \}_{i \in I}$ is called a frame for $\mathcal{H}$, if there exist two positive constants $A, B$ such that
\[
A \| f \|^2 \leq \sum_{i \in I} |(f, f_i)|^2 \leq B \| f \|^2, \quad f \in \mathcal{H}.
\]  

*Corresponding author

Email address: m.abdollah@uma.ac.ir (Mohammad Reza Abdollahpour)

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If \( A = B = 1 \) in (1.1), then we say that \( F = \{ f_i \}_{i \in I} \) is a Parseval frame for \( \mathcal{H} \). Let \( F = \{ f_i \}_{i \in I} \) be a frame for \( \mathcal{H} \). In this case,

\[
T_F : l_2(I) \to \mathcal{H}, \quad T_F(\{c_i\}_{i \in I}) = \sum_{i \in I} c_i f_i
\]
is a bounded and onto operator and its adjoint is \( T_F^*(f) = \{ \langle f, f_i \rangle \}_{i \in I} \), for all \( f \in \mathcal{H} \) \([6]\). The operators \( T_F, T_F^* \) and \( S_F = T_F T_F^* \) are called the synthesis, analysis and frame operator of \( F = \{ f_i \}_{i \in I} \), respectively. If \( F = \{ f_i \}_{i \in I} \) is a frame for \( \mathcal{H} \), then \( S_F \) is an invertible positive operator and we have

\[
f = \sum_{i \in I} \langle f, S_F^{-1} f_i \rangle f_i, \quad f \in \mathcal{H}.
\]  

(1.2)

A sequence \( F = \{ f_i \}_{i \in I} \) is called a Riesz basis for \( \mathcal{H} \), if \( \text{span}\{ f_i \}_{i \in I} = \mathcal{H} \) and there exist two positive constants \( A, B \) such that for any finite scalar sequence \( \{ c_i \} \) we have

\[
A \sum_i |c_i|^2 \leq \sum_i \|c_i f_i\|^2 \leq B \sum_i |c_i|^2.
\]

Let \( F = \{ f_i \}_{i \in I} \) and \( G = \{ g_i \}_{i \in I} \) be two frames for a Hilbert space \( \mathcal{H} \). We say that \( G \) is a dual frame for \( F \), if

\[
f = \sum_{i \in I} \langle f, g_i \rangle f_i, \quad f \in \mathcal{H}.
\]

From (1.2), we conclude that \( \tilde{F} = \{ S_F^{-1} f_i \}_{i \in I} \) is a dual frame of \( F \), which is called the canonical dual of \( F \). It is proved in \([6]\), each Riesz basis for \( \mathcal{H} \) is a frame and has only one dual frame.

The concepts of disjoint frames and strongly disjoint frames introduced by Han and Larson \([7]\), and these notions generalized to frames in Banach spaces by Casazza, Han and Larson \([5]\). In 2006, more general extension of frames, the so-called \( g \)-frames, introduced by Sun \([9]\). Some properties of \( g \)-frames have been investigated in papers \([2, 3, 4]\).

Throughout this paper, \( \mathcal{H} \) and \( \mathcal{K} \) are separable Hilbert spaces and \( \{ \mathcal{H}_i \}_{i \in I} \) is a sequence of separable Hilbert spaces.

**Definition 1.1.** We call a sequence \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) a \( g \)-frame for \( \mathcal{H} \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \), if there exist two positive constants \( A \) and \( B \) such that

\[
A \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B \|f\|^2, \quad f \in \mathcal{H}.
\]

A and \( B \) are called the lower and upper \( g \)-frame bounds, respectively.

We call \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) a tight \( g \)-frame if \( A = B \) and Parseval \( g \)-frame if \( A = B = 1 \).

If there is no confusion, we use \( g \)-frame (\( g \)-frame for \( \mathcal{H} \)) instead of \( g \)-frame for \( \mathcal{H} \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \).

Let \( \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) \) be given for all \( i \in I \). Let us define the set

\[
\widetilde{\mathcal{H}} = \{ \{ f_i \}_{i \in I} : f_i \in \mathcal{H}_i, \sum_{i \in I} \|f_i\|^2 < \infty \}
\]
with the inner product given by \( \langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle \). It is easy to show that \( \hat{\mathcal{H}} \) is a Hilbert space with respect to the pointwise operations. It is proved in [8], if \( \Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) is a \( g \)-Bessel sequence for \( \mathcal{H} \), then the operator

\[
T_\Lambda : \mathcal{H} \to \mathcal{H}, \quad T_\Lambda(\{f_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^*(f_i)
\]

is well defined and bounded and its adjoint is \( T_\Lambda^* f = \{\Lambda_i^*f\}_{i \in I} \) for all \( f \in \mathcal{H} \). Also, a sequence \( \Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) is a \( g \)-frame for \( \mathcal{H} \) if and only if the operator \( T_\Lambda \) defined in (1.3) is a bounded and onto operator. We call operators \( T_\Lambda \) and \( T_\Lambda^* \), the synthesis and analysis operators of \( \Lambda \), respectively. If \( \Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) is a \( g \)-frame for \( \mathcal{H} \), then

\[
S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_{i \in I} \Lambda_i^* \Lambda_i f
\]

is a bounded invertible positive operator [9], and every \( f \in \mathcal{H} \) has the following representation

\[
f = \sum_{i \in I} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f = \sum_{i \in I} \Lambda_i^* \Lambda_i S_{\Lambda_i}^{-1} f.
\]

(1.4)

\( S_\Lambda \) is called the \( g \)-frame operator of \( \Lambda \). Let \( \Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) be a \( g \)-frame for \( \mathcal{H} \) with \( g \)-frame bounds \( A, B \) and let \( \tilde{\Lambda}_i = \Lambda_i S_{\Lambda_i}^{-1} \), for all \( i \in I \). Then \( \tilde{\Lambda} = \{\tilde{\Lambda}_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) is a \( g \)-frame for \( \mathcal{H} \) with bounds \( \frac{1}{B} \) and \( \frac{1}{A} \) [9].

**Definition 1.2.** Let \( \Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) and \( \Theta = \{\Theta_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) be two \( g \)-frames for \( \mathcal{H} \) such that

\[
f = \sum_{i \in I} \Theta_i^* \Lambda_i f, \quad f \in \mathcal{H},
\]

then \( \Theta \) is called a dual \( g \)-frame of \( \Lambda \).

By (1.4), \( \tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i \in I} \) is a dual \( g \)-frame of \( \{\Lambda_i\}_{i \in I} \), which is called the canonical dual of \( \Lambda = \{\Lambda_i\}_{i \in I} \).

**Definition 1.3.** A sequence \( \Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) is called

1. a \( g \)-Riesz basis for \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \), if there exist two positive constants \( A \) and \( B \) such that for any finite subset \( F \subseteq I \) we have

\[
A \sum_{i \in F} \|g_i\|^2 \leq \sum_{i \in F} \Lambda_i^* g_i \|^2 \leq B \sum_{i \in F} \|g_i\|^2, \quad g_i \in \mathcal{H}_i,
\]

and \( \Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in I\} \) is \( g \)-complete, i.e.,

\[
\{f : \Lambda_i f = 0, \forall i \in I\} = \{0\}.
\]

2. a \( g \)-orthonormal basis for \( \mathcal{H} \) with respect to \( \{\mathcal{H}_i\}_{i \in I} \), if for all \( f \in \mathcal{H} \), \( \sum_{i \in I} \|\Lambda_i f\|^2 = \|f\|^2 \), and

\[
\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \quad g_i \in \mathcal{H}_i, \quad g_j \in \mathcal{H}_j, \quad i, j \in I.
\]
2. Dilation of g-frames

The concepts of disjoint g-frames and strongly disjoint g-frames were introduced in [1]. In this section, we investigate dilation of g-frames and we show that disjoint g-frames for a Hilbert space have dilation property.

**Definition 2.1.** Let \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) and \( \Theta = \{ \Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I \} \) be g-frames for Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), respectively. Then \( \Lambda \) and \( \Theta \) are called

1. disjoint, if \( \text{Range}T_{\Lambda}^* \cap \text{Range}T_{\Theta}^* = \{0\} \) and \( \text{Range}T_{\Lambda}^* + \text{Range}T_{\Theta}^* \) is a closed subspace of \( \mathcal{H} \).
2. complementary pair, if \( \text{Range}T_{\Lambda}^* \cap \text{Range}T_{\Theta}^* = \{0\} \) and
   \[
   \text{Range}T_{\Lambda}^* + \text{Range}T_{\Theta}^* = \mathcal{H}.
   \]
3. weakly disjoint if \( \text{Range}T_{\Lambda}^* \cap \text{Range}T_{\Theta}^* = \{0\} \).

**Proposition 2.2 ([1]).** Two g-frames \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) and \( \Theta = \{ \Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I \} \) are disjoint if and only if \( \{ \Gamma_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i) : i \in I \} \) is a g-frame for \( \mathcal{H} \oplus \mathcal{K} \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \), where
   \[
   \Gamma_i : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}_i, \quad \Gamma_i(f \oplus g) = \Lambda_i f + \Theta_i g,
   \]
   for all \( i \in I \).

**Proposition 2.3 ([1]).** Two g-frames \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) and \( \Theta = \{ \Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I \} \) are complementary pair if and only if \( \{ \Gamma_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}_i) : i \in I \} \) is a g-Riesz basis for \( \mathcal{H} \oplus \mathcal{K} \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \), where \( \Gamma_i \) is defined by (2.1), for all \( i \in I \).

**Proposition 2.4 ([1]).** Two g-frames \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) and \( \Theta = \{ \Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I \} \) are weakly disjoint if and only if
   \[
   \{f \oplus g : \Gamma_i(f \oplus g) = 0, \forall i \in I\} = \{0\},
   \]
   where \( \Gamma_i \) is defined by (2.1), for all \( i \in I \).

**Proposition 2.5.** Let \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) and \( \Theta = \{ \Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I \} \) be Parseval g-frames for \( \mathcal{H} \) and \( \mathcal{K} \), respectively. Then \( \text{Range}T_{\Lambda}^* \oplus \text{Range}T_{\Theta}^* = \mathcal{H} \) if and only if \( \{ \Gamma_i \}_{i \in I} \) is a g-orthonormal basis for \( \mathcal{H} \oplus \mathcal{K} \), where \( \Gamma_i \) is defined by (2.1), for all \( i \in I \).

**Proof.** If \( \{ \Gamma_i \}_{i \in I} \) is a g-orthonormal basis for \( \mathcal{H} \oplus \mathcal{K} \) then
   \[
   \|f\|^2 + \|g\|^2 = \sum_{i \in I} \|f_i(f \oplus g)|^2
   = \sum_{i \in I} \|\Lambda_i f\|^2 + \sum_{i \in I} \|\Theta_i g\|^2 + 2 \text{Re} \sum_{i \in I} \langle \Lambda_i f, \Theta_i g \rangle,
   \]
   and
   \[
   \text{Re} \sum_{i \in I} \langle \Lambda_i f, \Theta_i g \rangle = 0, \quad f \in \mathcal{H}, \; g \in \mathcal{K}.
   \]
If we replace $g$ by $ig$ in (2.2), then

$$Im \sum_{i \in I} \langle \Lambda_i f, \Theta_i g \rangle = 0, \quad f \in \mathcal{H}, \ g \in \mathcal{K}.$$ 

Therefore $RangeT^*_\Lambda \perp RangeT^*_\Theta$. Since $\Gamma = \{ \Gamma_i \in B(\mathcal{H} \oplus \mathcal{K}, \mathcal{H}) : i \in I \}$ is a $g$-orthonormal basis, $T^*_\Gamma$ is onto. But $RangeT^*_\Lambda + RangeT^*_\Theta = RangeT^*_\Gamma$, hence $RangeT^*_\Lambda + RangeT^*_\Theta = \hat{\mathcal{H}}$.

For the converse implication, we have

$$\sum_{i \in I} ||\Gamma_i(f \oplus g)||^2 = \sum_{i \in I} ||\Lambda_i f + \Theta_i g||^2 = \sum_{i \in I} ||\Lambda_i f||^2 + \sum_{i \in I} ||\Theta_i g||^2$$

$$= ||f||^2 + ||g||^2 = ||f \oplus g||^2,$$

for all $f \oplus g \in \mathcal{H} \oplus \mathcal{K}$. If $\{g_i\}_{i \in I} \in \hat{\mathcal{H}}$, then $\{g_i\}_{i \in I} = \{\Lambda_i f\}_{i \in I} + \{\Theta_i g\}_{i \in I}$ for some $f \in \mathcal{H}$ and for some $g \in \mathcal{K}$. Therefore, $g_i = \Lambda_i f + \Theta_i g_i$, for all $i \in I$. We have

$$\left\| \sum_{i \in I} \Gamma_i g_i \right\|^2 = \left\| \sum_{i \in I} (\Lambda_i g_i + \Theta_i g_i) \right\|^2 = \left\| \sum_{i \in I} \Lambda_i g_i \right\|^2 + \left\| \sum_{i \in I} \Theta_i g_i \right\|^2$$

$$= \left\| \sum_{i \in I} \Lambda_i (\Lambda_i f + \Theta_i g) \right\|^2 + \left\| \sum_{i \in I} \Theta_i (\Lambda_i f + \Theta_i g) \right\|^2$$

$$= \left\| f + \sum_{i \in I} \Lambda_i \Theta_i g \right\|^2 + \left\| g + \sum_{i \in I} \Theta_i \Lambda_i f \right\|^2.$$

Since $\sum_{i \in I} \Lambda_i \Theta_i g = 0$ and $\sum_{i \in J} \Theta_i \Lambda_i f = 0$,

$$\left\| \sum_{i \in I} \Gamma_i g_i \right\|^2 = \left\| f \right\|^2 + \left\| g \right\|^2 = \sum_{i \in I} \left\| \Lambda_i f \right\|^2 + \sum_{i \in I} \left\| \Theta_i g \right\|^2$$

$$= \sum_{i \in I} \left\| \Lambda_i f + \Theta_i g \right\|^2 = \sum_{i \in I} \left\| g \right\|^2.$$

So

$$\left\| \sum_{i \in I} \Gamma_i g_i \right\|^2 = \sum_{i \in I} \left\| g \right\|^2, \quad \{g_i\}_{i \in I} \in \hat{\mathcal{H}}. \quad (2.3)$$

By (2.3) we have

$$\left\| \Gamma^*_i g_i \right\|^2 = \left\| g_i \right\|^2; \quad i \in I, \ g_i \in \mathcal{H}_i. \quad (2.4)$$

Again, (2.3) implies that

$$\left\| \Gamma^*_i g_i + \Gamma^*_j g_j \right\|^2 = \left\| g_i \right\|^2 + \left\| g_j \right\|^2; \quad i, j \in I, \ g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j,$$

or

$$\left\| \Gamma^*_i g_i \right\|^2 + \left\| \Gamma^*_j g_j \right\|^2 + 2Re(\Gamma^*_i g_i, \Gamma^*_j g_j) = \left\| g_i \right\|^2 + \left\| g_j \right\|^2; \quad g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j,$$

for all $i, j \in I$. Therefore, by (2.4)

$$\langle \Gamma^*_i g_i, \Gamma^*_j g_j \rangle = \delta_{ij} \langle g_i, g_j \rangle, \quad g_i \in \mathcal{H}_i, g_j \in \mathcal{H}_j,$$

for all $i, j \in I$. \qed
Let $F = \{f_i\}_{i \in I}$ be a Riesz basis for a Hilbert space $\mathcal{H}$ with unique dual frame $\tilde{F} = \{\tilde{f}_i\}_{i \in I}$. If $M \subset \mathcal{H}$ is a closed subspace of $\mathcal{H}$ and $P$ is the orthogonal projection form $\mathcal{H}$ onto $M$, then $PF = \{PF_i\}_{i \in I}$ is a frame for $M$ with dual frame $P\tilde{F} = \{P\tilde{f}_i\}_{i \in I}$. In general, $P\tilde{F} = \{P\tilde{f}_i\}_{i \in I}$ is not the canonical dual of $PF = \{PF_i\}_{i \in I}$. But, if $P$ commutes with the frame operator $SF$, then $P\tilde{F} = \{P\tilde{f}_i\}_{i \in I}$ is the canonical dual of $PF = \{PF_i\}_{i \in I}$ (see [7]). Here, we generalize this result to $g$-frames.

**Proposition 2.6.** Let $P$ be an orthogonal projection from $\mathcal{H}$ onto a closed subspace $M$ and let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$. Then $\Lambda P = \{\Lambda_i P \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a $g$-frame for $M$ with respect to $\{\mathcal{H}_i\}_{i \in I}$ and

$$\forall i \in I, \quad \Lambda_i P = \Lambda_i \Rightarrow \Lambda_i P = S^{-1}_\Lambda P,$$

where $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Lambda_i P = \{\Lambda_i P \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ are canonical dual of $\Lambda$ and $\Lambda P$, respectively.

**Proof.** Let $f \in M$ and $A, B$ be the $g$-frame bounds for $\Lambda$, then

$$A\|f\|^2 = A\|P f\|^2 \leq \sum_{i \in I}\|\Lambda_i P f\|^2 \leq B\|P f\|^2 = B\|f\|^2.$$ 

If $\Lambda_i P = \Lambda_i P$, for all $i \in I$, then $\Lambda_i P S^{-1}_\Lambda P = S^{-1}_\Lambda P$, for all $i \in I$. Therefore, we have $PS^{-1}_\Lambda P = S^{-1}_\Lambda P$, and so $PS^{-1}_\Lambda P = PS^{-1}_\Lambda P$, which implies that $S^{-1}_\Lambda P = PS^{-1}_\Lambda P$. By taking adjoint we get $PS^{-1}_\Lambda P = PS^{-1}_\Lambda P$, and hence $PS^{-1}_\Lambda P = S^{-1}_\Lambda P$.

Now we assume that $PS^{-1}_\Lambda P = S^{-1}_\Lambda P$ and $f \in M$, then

$$f = \sum_{i \in I} (\Lambda_i P)^*(\Lambda_i P)f = \sum_{i \in I} P\Lambda_i^* P S^{-1}_\Lambda P f. \quad (2.5)$$

Since $f \in M \subset \mathcal{H}$, we can write $f = \sum_{i \in I} \Lambda_i^* S^{-1}_\Lambda f$ or

$$f = P f = \sum_{i \in I} P\Lambda_i^* S^{-1}_\Lambda P f.$$

Now, (2.5) and our assumption imply that

$$0 = \sum_{i \in I} P\Lambda_i^* \Lambda_i (PS^{-1}_\Lambda - S^{-1}_\Lambda) P f = \sum_{i \in I} P\Lambda_i^* \Lambda_i (PS^{-1}_\Lambda - S^{-1}_\Lambda) P f = S^{-1}_\Lambda (PS^{-1}_\Lambda f - S^{-1}_\Lambda P f),$$

for all $f \in M$. Therefore $PS^{-1}_\Lambda P = S^{-1}_\Lambda P$, and so $\Lambda_i P = \Lambda_i P$, for all $i \in I$. \hfill $\square$

Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{K}, \mathcal{H}_i) : i \in I\}$ be $g$-frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. We recall that $\Lambda$ and $\Theta$ are unitarily equivalent (similar), if there exists a unitary (an invertible) operator $U \in B(\mathcal{H}, \mathcal{K})$ such that

$$\Lambda_i = \Theta_i U, \quad i \in I.$$
Proposition 2.7. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(K, \mathcal{H}_i) : i \in I\}$ be $g$-frames for non zero Hilbert spaces $\mathcal{H}$ and $K$, respectively. If $\Lambda$ and $\Theta$ are unitarily equivalent (similar), then
\[
\overline{\text{span}}[\Gamma_i(\mathcal{H}_i)]_{i \in I} \neq \mathcal{H} \oplus K,
\]
where $\Gamma_i$ is defined by (2.1), for all $i \in I$.

**Proof.** Let $U \in B(\mathcal{H}, K)$ be a unitary (an invertible) operator such that $\Lambda_i = \Theta_i U$ for any $i \in I$. If $0 \neq g \in K$, then there exists $f \in \mathcal{H}$ and $Uf = -g$. Then $\Theta_i(Uf + g) = 0$, for all $i \in I$. Hence
\[
\{f \oplus g : \Gamma_i(f \oplus g) = 0, \ i \in I\} \neq \{0\},
\]
consequently $\overline{\text{span}}[\Gamma_i(\mathcal{H}_i)]_{i \in I} \neq \mathcal{H} \oplus K$, (see [8]).

Corollary 2.8. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(K, \mathcal{H}_i) : i \in I\}$ be respective $g$-frames for non zero Hilbert spaces $\mathcal{H}$ and $K$. If $\Lambda$ and $\Theta$ are unitarily equivalent (similar), then $\Lambda$ and $\Theta$ can not be weakly disjoint. Moreover, if $\Lambda$ and $\Theta$ are unitarily equivalent (similar), then $\Gamma = \{\Gamma_i \in B(\mathcal{H} \oplus K, \mathcal{H}_i) : i \in I\}$ is not a $g$-frame for $\mathcal{H} \oplus K$, where $\Gamma_i$ is defined by (2.1), for all $i \in I$.

Let $\{e_{ij}\}_{j \in J_i}$ be an orthonormal basis for $\mathcal{H}_i$, for every $i \in I$. It is proved in [8], $\{E_{ij}\}_{i \in I, j \in J_i}$ is an orthonormal basis for $\mathcal{H}_i$, where
\[
(E_{ij})_k = \begin{cases} e_{ij}, & i = k \\ 0, & i \neq k \end{cases}. \tag{2.6}
\]

We use the above fact in the rest of this paper.

Proposition 2.9. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be a $g$-frame for Hilbert space $\mathcal{H}$ with respect to $\mathcal{H}_i$, Then there exist a Hilbert space $\mathcal{H} \subset K$ and a g-Riesz basis $\Delta = \{\Delta_i \in B(K, \mathcal{H}_i) : i \in I\}$ for $K$ with respect to $\mathcal{H}_i$, such that $\Lambda_i = \Delta_i P_{\mathcal{H}}$ for all $i \in I$, where $P_{\mathcal{H}}$ is the orthogonal projection from $K$ onto $\mathcal{H}$.

**Proof.** Let $\Theta_i = \Lambda_i S^{-1}_{\Lambda}$, for all $i \in I$. Then $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a Parseval $g$-frames for $\mathcal{H}$ and $\text{Range} T^*_{\Theta_i} = \text{Range} T^*_{\Lambda_i}$. Let $P$ be the orthogonal projection from $\mathcal{H}$ onto $\text{Range} T^*_{\Theta_i}$. We define the operators
\[
\varphi_i : P^\perp \mathcal{H} \to \mathcal{H}_i, \quad \varphi_i(g) = \sum_{j \in J_i} \langle g, P^\perp E_{ij} \rangle e_{ij}, \tag{2.7}
\]
for all $i \in I$, where $E_{ij}$ is defined by (2.6). Then $\varphi = \{\varphi_i \in B(P^\perp \mathcal{H}, \mathcal{H}_i) : i \in I\}$ is a Parseval $g$-frame for $P^\perp \mathcal{H}$. In fact
\[
\sum_{i \in I} \|\varphi_i g\|^2 = \sum_{i \in I} \left\| \sum_{j \in J_i} \langle g, P^\perp E_{ij} \rangle e_{ij} \right\|^2 = \sum_{i \in I} \sum_{j \in J_i} |\langle g, P^\perp E_{ij} \rangle|^2 = \|g\|^2,
\]
We define the operator $F$.

According to the Proposition 2.5, (2.8) and (2.9) imply that consequently so, $g$ for all $f \in \mathcal{H}$ and $g \in P^2 \hat{\mathcal{H}}$. So, 

$$\text{Range} T^*_\phi \perp \text{Range} T^*_\psi.$$ (2.8)

On the other hand, if $g = \{g_i\}_{i \in I} \in P^2 \hat{\mathcal{H}}$ then we have 

$$\varphi_i g = \sum_{j \in J_i} \langle g, P^2 E_{ij} \rangle e_{ij} = \sum_{j \in J_i} \langle \{g_i\}_{i \in I}, E_{ij} \rangle e_{ij}$$

$$= \sum_{j \in J_i} \langle g_i, e_{ij} \rangle e_{ij} = g_i,$$

so, $g = \{\varphi_i g\}_{i \in I}$. Thus 

$$P^2 g = \{\varphi_i(P^2 g)\}_{i \in I}; \quad g = P g + T^*_\phi(P^2 g), \quad g \in \hat{\mathcal{H}}.$$ 

consequently 

$$\hat{\mathcal{H}} = \text{Range} T^*_\phi + \text{Range} T^*_\psi.$$ (2.9)

According to the Proposition 2.5, (2.8) and (2.9) imply that $\{\Gamma_i\}_{i \in I}$ is a $g$-orthonormal basis for $\mathcal{H} \oplus P^2 \hat{\mathcal{H}}$, where 

$$\Gamma_i : \mathcal{H} \oplus P^2 \hat{\mathcal{H}} \rightarrow \mathcal{H}, \quad \Gamma_i(f \oplus g) = \Theta_i f + \varphi_i g.$$ (2.10)

We define the operator $F \in B(\mathcal{H} \oplus P^2 \hat{\mathcal{H}})$ by $F(f \oplus g) = S^2 f \oplus g$, then $F$ is invertible. Let $\Delta_i = \Gamma_i F$, for all $i \in I$. In this case, $\{\Delta_i\}_{i \in I}$ is a $g$-Riesz basis for $K = \mathcal{H} \oplus P^2 \hat{\mathcal{H}}$ (see [2]). Clearly, $\Delta_i P_{\mathcal{H}} = \Lambda_i$, for all $i \in I$.

**Definition 2.10.** Let $\mathcal{F}$ be a family of $g$-frames for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$. We say that $\mathcal{F}$ has dilation property, if there is a larger Hilbert space $\mathcal{K} \subset K$ such that for every $\Lambda = \{\Lambda_i\}_{i \in I} \in \mathcal{F}$, there exists a $g$-Riesz basis $\Gamma = \{\Gamma_i\}_{i \in I}$ for $K$ such that $\Lambda_i = \Gamma_i P_{\mathcal{H}}$, for all $i \in I$, where $P_{\mathcal{H}}$ is orthogonal projection from $K$ onto $\mathcal{H}$.

In the next proposition we provide some sufficient conditions, under which a family of $g$-frames with two members has dilation property.

**Proposition 2.11.** Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I\}$ be two $g$-frames for Hilbert spaces $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i \in I}$. If one of the following conditions holds, then $\mathcal{F} = \{\Lambda, \Theta\}$ has the dilation property.
(1) \( \Lambda \) and \( \Theta \) are similar.

(2) \( \Lambda \) and \( \Theta \) are disjoint.

(3) \( \Theta \) is similar to a dual \( g \)-frame of \( \Lambda \).

Proof. (1) Let \( T \in B(\mathcal{H}) \) be an invertible operator and \( \Theta_i = \Lambda_i T \), for all \( i \in I \). By Proposition 2.9, then there exist a Hilbert space \( \mathcal{H} \subset K \) \((K = \mathcal{H} \oplus P^1 \mathcal{H})\), where \( P_H \) is the orthogonal projection from \( \hat{\mathcal{H}} \) onto \( \text{Range}T^\Lambda_i \) and a \( g \)-Riesz basis \( \Gamma = \{ \Gamma_i \in B(K, \mathcal{H}_i) : i \in I \} \) for \( K \) with \( \Lambda_i = \Gamma_i P_H \) for all \( i \in I \). Let us define \( \Delta_i \in B(K, \mathcal{H}_i) \) by \( \Delta_i = \Gamma_i (T \oplus I) \), where

\[
T \oplus I : K \to K, \quad (T \oplus I)(f \oplus g) = Tf \oplus g.
\]

Since \( T \oplus I \) is invertible and \( \Gamma = \{ \Gamma_i \}_{i \in I} \) is a \( g \)-Riesz basis for \( K \), then \( \Delta = \{ \Delta_i \}_{i \in I} \) is a \( g \)-Riesz basis for \( K \) and \( \Theta_i = \Delta_i P_H \) for all \( i \in I \).

(2) Since \( \Lambda = \{ \Lambda_i \}_{i \in I} \) and \( \Theta = \{ \Theta_i \}_{i \in I} \) are disjoint, by Proposition 2.2, \( \{ \psi_i \}_{i \in I} \) and \( \{ \varphi_i \}_{i \in I} \) are \( g \)-frames for \( \mathcal{H} \oplus \mathcal{H} \), where for all \( i \in I \), \( \psi_i, \varphi_i : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}_i \) are defined by

\[
\psi_i(f \oplus g) = \Lambda_i f + \Theta_i g, \quad \varphi_i(f \oplus g) = \Theta_i f + \Lambda_i g, \quad f, g \in \mathcal{H}.
\]

From the other hand, \( \{ \psi_i \}_{i \in I} \) and \( \{ \varphi_i \}_{i \in I} \) are similar. Hence by (1), there exist a Hilbert space \( \mathcal{H} \oplus \mathcal{H} \subset K \), and two \( g \)-Riesz basis \( \Gamma = \{ \Gamma_i \}_{i \in I} \) and \( \Delta = \{ \Delta_i \}_{i \in I} \) for \( K \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \), such that \( \psi_i = \Gamma_i P_{\mathcal{H} \oplus \mathcal{H}} \) and \( \varphi_i = \Delta_i P_{\mathcal{H} \oplus \mathcal{H}} \) for all \( i \in I \), where \( P_{\mathcal{H} \oplus \mathcal{H}} \) is the orthogonal projection from \( K \) onto \( \mathcal{H} \oplus \mathcal{H} \). If we identify \( \mathcal{H} \) by \( \mathcal{H} \oplus 0 \oplus 0 \) and consider \( P_H \) is the orthogonal projection from \( K \) onto \( \mathcal{H} \oplus 0 \oplus 0 \), then \( \Lambda_i = \Gamma_i P_H \) and \( \Theta_i = \Delta_i P_H \) for all \( i \in I \).

(3) Let \( \phi = \{ \phi_i \}_{i \in I} \) be a dual \( g \)-frame for \( \Lambda = \{ \Lambda_i \}_{i \in I} \) and \( T \in B(\mathcal{H}) \) be an invertible operator so that \( \Theta_i = \phi_i T \), for all \( i \in I \). By Theorem 2.9 of [1], there exists a Hilbert space \( \mathcal{H} \subset K \) and two \( g \)-Riesz basis \( \Gamma = \{ \Gamma_i \}_{i \in I} \) and \( \Delta = \{ \Delta_i \}_{i \in I} \) for \( K \) with \( \Lambda_i = \Gamma_i P_H \) and \( \phi_i = \Delta_i P_H \) for all \( i \in I \), where \( P_H \) is the orthogonal projection from \( K \) onto \( \mathcal{H} \). Let us define

\[
W_i : K \to \mathcal{H}_i, \quad W_i = \Delta_i (T \oplus I), \quad i \in I.
\]

Then \( W = \{ W_i \}_{i \in I} \) is a \( g \)-Riesz basis for \( K \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \), and \( \Theta_i = W_i P_H \), for all \( i \in I \). \( \square \)

Definition 2.12. Let \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{H}_i) : i \in I \} \) be a \( g \)-frame for \( \mathcal{H} \). We define the deficiency of \( \Lambda \) to be \( \dim(\text{Range}T^\Lambda_i)^\perp \).

In the following theorem we provide a sufficient condition for a family of \( g \)-frame \( \mathcal{F} \) such that \( \mathcal{F} \) has the dilation property.

Theorem 2.13. Let \( \mathcal{F} \) be a family of \( g \)-frames for \( \mathcal{H} \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \). Then \( \mathcal{F} \) has the dilation property if all members of \( \mathcal{F} \) have the equal deficiency.

Proof. Fix a \( g \)-frame \( \Lambda = \{ \Lambda_i \}_{i \in I} \) in \( \mathcal{F} \) and let \( \Theta = \{ \Theta_i \}_{i \in I} \) be any \( g \)-frame in \( \mathcal{F} \). Let \( K = \mathcal{H} \oplus P^1 \hat{\mathcal{H}} \) and \( M = \mathcal{H} \oplus Q^1 \hat{\mathcal{H}} \), where \( P \) and \( Q \) are the orthogonal projection from \( \hat{\mathcal{H}} \) onto \( \text{Range}T^\Lambda_i \) and \( \text{Range}T^\Theta_i \), respectively. We define

\[
\varphi_i : P^1 \hat{\mathcal{H}} \to \mathcal{H}_i, \quad \varphi_i(g) = \sum_{j \in J_i} (g, P^1 E_{ij}) e_{ij},
\]
and
\[ \psi_i : Q^+ \hat{H} \to \mathcal{H}, \quad \psi_i(h) = \sum_{j \in J} \langle h, Q^+ E_{ij} \rangle e_{ij}, \]
for all \( i \in I \), where \( E_{ij} \) is defined by 2.6. Then \( \varphi = \{ \varphi_i \}_{i \in I} \) and \( \psi = \{ \psi_i \}_{i \in I} \) are respective \( g \)-frames for \( P^+ \hat{H} \) and \( Q^+ \hat{H} \). Now, we consider bounded operators
\[ \Gamma_i : K \to \mathcal{H}, \quad \Gamma_i(f \oplus g) = \Lambda_i f + \varphi_i g, \]
(2.11) and
\[ \Phi_i : M \to \mathcal{H}, \quad \Phi_i(f \oplus h) = \Theta_i f + \psi_i h. \]
(2.12)
A argument similar to the proof of Proposition 2.9 shows that
\[ \hat{H} = \text{Range}T^*_A + \text{Range}T^*_\psi, \quad \text{Range}T^*_A \perp \text{Range}T^*_\psi. \]
So by Proposition 2.3, \( \Gamma = \{ \Gamma_i \}_{i \in I} \) is a \( g \)-Riesz basis for \( K \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \). Similarly, \( \Phi = \{ \Phi_i \}_{i \in I} \) is a \( g \)-Riesz basis for \( M \) with respect to \( \{ \mathcal{H}_i \}_{i \in I} \). Since \( \dim(\text{Range}T^*_A)^+ = \dim(\text{Range}T^*_\psi)^+ \), there is a unitary operator \( W \) from \( (\text{Range}T^*_A)^+ \) onto \( (\text{Range}T^*_\psi)^+ \). In fact, if \( \{ x_i \}_{i \in J} \) and \( \{ y_i \}_{i \in J} \) are orthonormal bases for \( (\text{Range}T^*_A)^+ \) and \( (\text{Range}T^*_\psi)^+ \), respectively, then we may consider
\[ W : (\text{Range}T^*_A)^+ \to (\text{Range}T^*_\psi)^+, \quad Wf = \sum_{i \in J} \langle f, x_i \rangle y_i. \]
It is easy to show that \( W \) is a unitary operator. Let us define
\[ \Delta_i : K \to \mathcal{H}, \quad \Phi_i(f \oplus g) = \Theta_i f + \psi_i W g, \quad i \in I. \]
Since \( \Delta_i = \Phi_i F \), for all \( i \in I \) and the operator
\[ F : K \to M, \quad F(f \oplus g) = f \oplus W g \]
is invertible, \( \Delta = \{ \Delta_i \}_{i \in I} \) is a \( g \)-Riesz basis for \( K \). Clearly, \( \Gamma_i P_H = \Lambda_i \) and \( \Delta_i P_H = \Theta_i \) for ever \( i \in I \), therefore \( \mathcal{F} \) has the dilation property. \( \square \)

References