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# On some special classes of Sonnenschein matrices

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## Abstract

In this paper we consider the special classes of Sonnenschein matrices, namely the Karamata matrices $K[\alpha,\beta] = (a_{n,k})$ with the entries

$$a_{n,k} = \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v},$$

and calculate their row and column sums and give some applications of these sums.



## 1. Introduction

Let $f(z)$ be an analytic function in $D_f = \{z \in \mathbb{C} : |z| < r, r \geq 1\}$ with $f(1) = 1$. The matrix $S = S_f = (a_{n,k})$, where $(a_{n,k})$ are defined by $[f(z)]^n = \sum_{k=0}^{\infty} a_{n,k} z^k$ is called a Sonnenschein matrix [7, 9]. The special choice

$$f(z) = \frac{\alpha + (1-\alpha-\beta)\, z}{1-\beta z}, \; z \in \mathbb{C} \backslash \{\frac{1}{\beta}\},$$

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where $\alpha$ and $\beta$ are complex numbers, gives the Karamata matrix $K[\alpha,\beta]$ and its coefficients are given by [3, 4]

$$a_{n,k} = \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v}.$$

In particular, $K[1-\alpha, 0]$ and $K[0, 1-\alpha]$, give the Euler [1] matrix $E^\alpha$, and the Taylor [10] matrix $T^\alpha$, respectively.

In this paper we are going to compute the row and column sums of the Karamata matrix $K[\alpha,\beta]$ for the cases $\alpha,\beta \in (0,1)$ and to give some applications for these sums. Our results are based on the following binomial coefficients identities:

(i) $\frac{1}{(1-x)^{k+1}} = \frac{1}{k!} \sum_{n=0}^{\infty} (n+k) \dots (n+1)\, x^n, \qquad |x| < 1.$

(ii) $\binom{t}{v} = 0, \quad (v > t \text{ or } v < 0)\,.$

We begin with the following lemma which is essential in the text.

**Lemma 1.1.** *Let $n,k,v \in \mathbb{N} \cup \{0\}$ and $\alpha < 1$. Then*

$$\sum_{n=v}^{\infty} \binom{n}{v} \binom{n+k-v-1}{k-v} \alpha^{n-v} = \frac{\alpha \binom{k}{v} + (1-\alpha)\binom{k-1}{v-1}}{(1-\alpha)^{k+1}}.$$

*Proof.* Using the identity (i), we have

$$\begin{aligned}
\sum_{n=v}^{\infty} \binom{n}{v} \binom{n+k-v-1}{k-v} \alpha^{n-v} &= \sum_{n=0}^{\infty} \binom{n+v}{v} \binom{n+k-1}{k-v} \alpha^{n} \\
&= \frac{1}{v!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+v)(n+k-1)!}{n!} \alpha^n \\
&= \frac{1}{v!(k-v)!} \sum_{n=1}^{\infty} \frac{(n+k-1)!}{(n-1)!} \alpha^n + \frac{1}{(v-1)!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+k-1)!}{n!} \alpha^n \\
&= \frac{\alpha}{v!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} \alpha^n + \frac{1}{(v-1)!(k-v)!} \sum_{n=0}^{\infty} \frac{(n+k-1)!}{n!} \alpha^n \\
&= \frac{k!}{v!(k-v)!} \frac{\alpha}{(1-\alpha)^{k+1}} + \frac{(k-1)!}{(v-1)!(k-v)!} \frac{1}{(1-\alpha)^{k}} \\
&= \frac{\alpha\binom{k}{v} + (1-\alpha)\binom{k-1}{v-1}}{(1-\alpha)^{k+1}},
\end{aligned}$$

as desired. □

**Theorem 1.2.** *For the Karamata matrix $K[\alpha,\beta]$, the sum of the first column is $\frac{1}{1-\alpha}$, the sum of all other columns are $\frac{1-\beta}{1-\alpha}$, and the sum of all rows are 1.*

*Proof.* For $k = 0$, obviously $\sum_{n=0}^{\infty} a_{n,0} = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$. If $k \geq 1$, applying Lemma 1.1 together with the identity (i), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_{n,k} &= \sum_{n=0}^{\infty}\sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\
&= \sum_{v=0}^{k}\sum_{n=0}^{\infty} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\
&= \sum_{v=0}^{k} (1-\alpha-\beta)^v \beta^{k-v} \sum_{n=v}^{\infty} \binom{n}{v} \binom{n+k-v-1}{k-v} \alpha^{n-v} \\
&= \sum_{v=0}^{k} (1-\alpha-\beta)^v \beta^{k-v} \frac{\alpha\binom{k}{v} + (1-\alpha)\binom{k-1}{v-1}}{(1-\alpha)^{k+1}} \\
&= \frac{\alpha}{(1-\alpha)^{k+1}} \sum_{v=0}^{k} \binom{k}{v} (1-\alpha-\beta)^v \beta^{k-v} \\
&\qquad + \frac{1-\alpha}{(1-\alpha)^{k+1}} \sum_{v=1}^{k} \binom{k-1}{v-1} (1-\alpha-\beta)^v \beta^{k-v} \\
&= \frac{\alpha(1-\alpha)^k}{(1-\alpha)^{k+1}} + \frac{1-\alpha}{(1-\alpha)^{k+1}} (1-\alpha-\beta) \sum_{v=1}^{k} \binom{k-1}{v-1} (1-\alpha-\beta)^v \beta^{k-v} \\
&= \frac{\alpha(1-\alpha)^k + (1-\alpha)(1-\alpha-\beta)(1-\alpha)^{k-1}}{(1-\alpha)^{k+1}} \\
&= \frac{1-\beta}{1-\alpha}.
\end{aligned}$$

For the row sums of the Karamata matrix, fix $n \in \mathbb{N} \cup \{0\}$, using identities (i) and (ii), we have

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{n,k} &= \sum_{k=0}^{\infty}\sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\
&= \sum_{k=0}^{\infty}\sum_{v=0}^{n} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\
&= \sum_{v=0}^{n} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \sum_{k=v}^{\infty} \binom{n+k-v-1}{k-v} \beta^{k-v} \\
&= \sum_{v=0}^{n} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \beta^{k} \\
&= (1-\beta)^n \frac{1}{(1-\beta)^n} = 1.
\end{aligned}$$

□

## 2. Applications

For $0 < p < \infty$, the Karamata sequence space is defined by

$$\mathcal{K}_p^{\alpha,\beta} = \left\{ (x_n) \in \mathbb{C} : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p < \infty \right\}.$$

More precisely, $\mathcal{K}_p^{\alpha,\beta}$ is the set of all sequences such that $K[\alpha,\beta]-$transforms of them are in the space $\ell_p$. In particular, $\mathcal{K}_p^{0,0} = \ell_p$ and $\mathcal{K}_p^{1-\alpha,0} = e_p^{\alpha}$, the Euler sequence space of order $\alpha$ [2], which is defined as below:

$$e_p^{\alpha} = \left\{ (x_n) \in \mathbb{C} : \sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} \binom{n}{k} (1-\alpha)^{n-k} \alpha^k x_k \right|^p < \infty \right\}.$$

**Lemma 2.1.** *The set $\mathcal{K}_p^{\alpha,\beta}$ becomes a linear space with the coordinatewise addition and scalar multiplication, which is the semi normed space with the semi norm*

$$\|x\|_{\mathcal{K}_p^{\alpha,\beta}} := \left( \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p \right)^{\frac{1}{p}},$$

*for $1 \le p < \infty$. Further, it is a $p-$semi normed space with the $p-$semi norm $|||x||| = \|x\|_{\mathcal{K}_p^{\alpha,\beta}}^p$, whenever $0 < p < 1$.*

An easy calculation shows that the absolute property does not hold on the space $\mathcal{K}_p^{\alpha,\beta}$, that is, $\|x\|_{\mathcal{K}_p^{\alpha,\beta}} \neq \||x|\|_{\mathcal{K}_p^{\alpha,\beta}}$ for at least one sequence in the space $\mathcal{K}_p^{\alpha,\beta}$, and this says us that $\mathcal{K}_p^{\alpha,\beta}$ is a sequence space of non-absolute type, where $|x| = (|x_k|)$. It is immediate by the well-known inclusion $\ell_p \subset \ell_q$ that the inclusion $\mathcal{K}_p^{\alpha,\beta} \subset \mathcal{K}_q^{\alpha,\beta}$ holds whenever $p \le q$.

**Theorem 2.2.** *The inclusion $\ell_p \subseteq \mathcal{K}_p^{\alpha,\beta}$ holds for $1 \le p < \infty$.*

*Proof.* Let $x \in \ell_p$. Applying Theorem 1.2 with Hölder's inequality for $1 < p < \infty$, we obtain

$$\begin{aligned} &\left| \sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p \\ &\quad \le \left( \sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} |x_k|^p \right) \\ &\qquad \times \left( \underbrace{\sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v}}_{=1} \right)^{p-1} \end{aligned}$$

$$= \sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} |x_k|^p. \tag{2.1}$$

Applying (2.1) with Theorem 1.2, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} x_k \right|^p \\
&\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} |x_k|^p \\
&\leq \sum_{k=0}^{\infty} |x_k|^p \sum_{n=0}^{\infty} \sum_{v=0}^{k} \binom{n}{v} (1-\alpha-\beta)^v \alpha^{n-v} \binom{n+k-v-1}{k-v} \beta^{k-v} \\
&= |x_0|^p \left(\frac{1}{1-\alpha}\right) + |x_1|^p \left(\frac{1-\beta}{1-\alpha}\right) + |x_2|^p \left(\frac{1-\beta}{1-\alpha}\right) + |x_3|^p \left(\frac{1-\beta}{1-\alpha}\right) + \cdots \\
&\leq \left(\frac{1}{1-\alpha}\right) \sum_{k=0}^{\infty} |x_k|^p,
\end{aligned}$$

which yields us that

$$\|x\|_{\mathcal{K}_p^{\alpha,\beta}} \leq \left(\frac{1}{1-\alpha}\right)^{1/p} \|x\|_{\ell_p}. \tag{2.2}$$

Therefore, $x \in \mathcal{K}_p^{\alpha,\beta}$, i.e. $\ell_p \subseteq \mathcal{K}_p^{\alpha,\beta}$. By similar discussions, it may be easily proved that the inequality (2.2) holds in the case $p = 1$ and so we omit the detail. This completes the proof. □

**Theorem 2.3.** *Let* $0 < p < 1$, *and* $x = (x_n)$ *be a non-negative sequence in* $\mathcal{K}_p^{\alpha,\beta}$. *Then* $x$ *belongs to the space* $\ell_p$, *and we have*

$$\|x\|_{\ell_p} \leq \left(\frac{1-\beta}{1-\alpha}\right)^{-1/p} \|x\|_{\mathcal{K}_p^{\alpha,\beta}}. \tag{2.3}$$

*Proof.* The proof is similar to that of Theorem 2.2. The only difference is due to the first part in which the Hölder's inequality for $0 < p < 1$, should be applied and therefore the inequality sign must be reversed. □

We refer the readers to [5, 6, 8] to give more information about sequence spaces.

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