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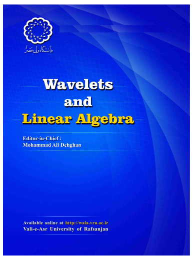

On I-biflat and I-biprojective Banach algebras

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Abstract

In this paper, we introduce new notions of I-biflatness and I-biprojectivity, for a Banach algebra A, where I is a closed ideal of A. We show that $M(G)$ is $L^1(G)$-biprojective (I-biflat) if and only if G is a compact group (an amenable group), respectively. Also we show that, for a non-zero ideal I, if the Fourier algebra $A(G)$ is I-biprojective, then G is a discrete group. Some examples are given to show the differences between these new notions and the classical ones.

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1. Introduction

B. E. Johnson in [7] extended the concept of amenability from locally compact groups to the Banach algebras. In fact a Banach algebra A is amenable (super amenable) Banach algebra if every continuous derivation $D : A \to X^*(X)$ is inner, for every Banach A-bimodule X, respectively. Also Johnson showed that, A is amenable if and only if A has a bounded net $(m_\alpha)_\alpha \subseteq A \otimes_p A$ such that $\pi_A(m_\alpha)a \to a$ and $a \cdot m_\alpha - m_\alpha \cdot a \to 0$, where $\pi_A : A \otimes_p A \to A$ is given by $\pi_A(a \otimes b) = ab$, for every $a, b \in A$. In the other word, he found out that, there is a direct relation between the Banach algebra $A \otimes_p A$ and amenability of A. There exists another approach to study the structure of Banach algebra through the homology of Banach algebras. Some concepts like biflatness and biprojectivity were introduced, which they have direct relation with supper amenability and amenability of Banach algebras, for more details reader referred to [6] or [10]. Indeed A is biflat (biprojective), if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}(A \otimes_p A)$ such that

$$\pi_A^{**} \circ \rho(a) = a(\pi_A \circ \rho(a) = a), \quad (a \in A),$$

respectively. We should recall that A is amenable if and only if A is biflat and posses a bounded approximate identity, see [10].

The content of this paper is as follows, after recalling some background notations and definitions, we investigate some properties of these new notions. We show that if the Fourier algebra $A(G)$ is I-biprojective then G must be a discrete group and the measure algebra $M(G)$ is $L^1(G)$-biprojective if and only if G is a compact group. We show that $M(G)$ is $L^1(G)$-biflat if and only if G is an amenable group.

We recall that if X is a Banach A-bimodule, then with the following actions X^* is also a Banach A-bimodule:

$$a \cdot f(x) = f(x \cdot a), \quad f \cdot a(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

The projective tensor product of A with A is denoted by $A \otimes_p A$. The Banach algebra $A \otimes_p A$ is a Banach A-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Throughout this paper, $\Delta(A)$ denotes the character space of A, that is, all non-zero multiplicative linear functionals on A. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension on A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**}.

2. I-biprojectivity

Definition 2.1. Let A be a Banach algebra and I be a closed ideal of A. Then A is called I-biprojective if there exists a bounded "A-bimodule morphism" $\rho : I \to A \otimes_p A$ such that $\pi_A \circ \rho(i) = i$ for every $i \in I$.

Theorem 2.2. *Let A be a Banach algebra and I be a closed ideal of A. Suppose that I posses a left and right approximate identity. Then A is I-biprojective if and only if I is biprojective.*

Proof. Let A be I-biprojective. Then there exists a bounded A-bimodule morphism $\rho : I \to A \otimes_p A$ such that $\pi_A \circ \rho(i) = i$ for all $i \in I$. Let (e_α) and (e_β) be left and right approximate identity of I, respectively. Suppose that $i \in I$ is an arbitrary element of I. Then

$$\rho(i) = \rho(\lim_\alpha e_\alpha i) = \lim_\alpha e_\alpha \cdot \rho(i) = \lim_\alpha e_\alpha \cdot \rho(\lim_\beta ie_\beta) = \lim_\alpha \lim_\beta e_\alpha \cdot \rho(i) \cdot e_\beta.$$

It follows that $\rho(i) \in I \otimes_p I$. Thus ρ is a bounded I-bimodule morphism such that $\pi_I \circ \rho(i) = i$, for all $i \in I$. So I is biprojective.
For converse, suppose that I is biprojective. Then there exists a bounded I-bimodule morphism $\rho : I \to I \otimes_p I$ such that $\pi_I \circ \rho(i) = i$, for all $i \in I$. Since I has a left approximate identity, $\overline{I^2} = I$. Let $i \in I$. Then there exist nets (a_α) and (b_α) in I such that $i = \lim_\alpha a_\alpha b_\alpha$. Thus

$$x \cdot \rho(i) = x \cdot \rho(\lim a_\alpha b_\alpha) = x \cdot \lim a_\alpha \rho(b_\alpha) = \lim x a_\alpha \rho(b_\alpha) = \rho(xi), \quad (x \in A, i \in I)$$

□

Let A be a Banach algebra and $\phi \in \Delta(A)$. We recall that a Banach algebra A is ϕ-biprojective, if there exists a bounded A-bimodule morphism $\rho : A \to A \otimes_p A$ such that $\phi \circ \pi_A \circ \rho(a) = \phi(a)$, for all $a \in A$. For more details about this concept the reader referred to [11].

Proposition 2.3. *Let G be a locally compact group and $A(G)$ be the Fourier algebra over G. Let I be a non-zero closed ideal of $A(G)$. If $A(G)$ is I-biprojective, then G is discrete.*

Proof. Let $\rho : I \to A(G) \otimes_p A(G)$ be a bounded $A(G)$-bimodule morphism such that $\pi_{A(G)} \circ \rho(i) = i$ for each $i \in I$. It is known that $\Delta(A(G)) = \{\phi_t | t \in G\}$, where $\phi_t(f) = f(t)$ for $f \in A(G)$. Clearly $\cap_{t \in G} \ker \phi_t = \{0\}$. Then there exists a $t_0 \in G$ such that $\phi_{t_0}|_I \neq 0$. Pick $i_0 \in I$ such that $\phi_{t_0}(i_0) = 1$. Define $\eta_{i_0} : A(G) \to I$ by $\eta_{i_0}(a) = ai_0$, for each $a \in A(G)$. We show that $\rho \circ \eta_{i_0} : A(G) \to A(G) \otimes_p A(G)$ is a right $A(G)$-module morphism. Since ρ is a bounded $A(G)$-bimodule morphism and $A(G)$ is a commutative Banach algebra with respect to the pointwise multiplication, we have

$$\rho \circ \eta_{i_0}(a) \cdot b = \rho(ai_0) \cdot b = \rho(ai_0 b) = \rho(abi_0) = \rho \circ \eta_{i_0}(ab),$$

for each $a, b \in A(G)$. Similarly one can show that $\rho \circ \eta_{i_0}$ is a bounded left $A(G)$-module morphism. Now consider

$$\phi_{t_0} \circ \pi_{A(G)} \circ \rho \circ \eta_{i_0}(a) = \phi_{t_0} \circ \pi_A \circ \rho(ai_0) = \phi_{t_0}(ai_0) = \phi_{t_0}(a).$$

Then $A(G)$ is ϕ_{t_0}-biprojective. So by [11, Corollary 2] G is a discrete group. □

Let A be a Banach algebra and $\phi \in \Delta(A)$. We recall that A is left ϕ-contractible, if there exists an element $m \in A$ such that $am = \phi(a)m$ and $\phi(m) = 1$. For more details reader referred to [9]. The character ϕ on $L^1(G)$ is called augmentation character, if it specified by $\phi(f) = \int_G f(x)dx$, where dx is denoted for the left Haar measure and $f \in L^1(G)$.

Proposition 2.4. *Let G be a locally compact group. Then the measure algebra $M(G)$ is $L^1(G)$-biprojective if and only if G is compact.*

Proof. It is well-known that $L^1(G)$ has a bounded approximate identity. Then $M(G)$ is $L^1(G)$ biprojective if and only if $L^1(G)$ is biprojective. It is known that $L^1(G)$ is biprojective if and only if G is compact. □

Proposition 2.5. *Let A be a Banach algebra and I be its closed ideal. Also let B be a Banach algebra and J be its closed ideal. If A and B are I and J-biprojective, respectively, then $A \otimes_p B$ is $I \otimes_p J$-biprojective.*

Proof. First it is easy to see that $I \otimes_p J$ is a closed ideal of $A \otimes_p B$. Suppose that $\rho : I \to A \otimes_p A$ and $\eta : J \to B \otimes_p B$ are bounded A and B-bimodule morphisms, respectively. Let $\Upsilon : (A \otimes_p A) \otimes_p (B \otimes_p B) \to (A \otimes_p B) \otimes_p (A \otimes_p B)$ defined by $\Upsilon((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)$, for each $a_1, a_2 \in A$ and $b_1, b_2 \in B$. One can see that $\Upsilon \circ (\rho \otimes \eta) : I \otimes_p J \to (A \otimes_p B) \otimes_p (A \otimes_p B)$ is $A \otimes_p B$-bimodule morphism. On the other hand, suppose that $i \in I$ and $j \in J$. Let $\rho(i) = \sum_{k=1}^{\infty} x_k^i \otimes y_k^i$ and $\eta(j) = \sum_{s=1}^{\infty} a_s^j \otimes b_s^j$, where $(x_k^i)_k$ and $(y_k^i)_k$ are some nets in A and also $(a_s^j)_s$ and $(b_s^j)_s$ are some nets in B. Consider

$$\begin{aligned}
\pi_{A\otimes_p B} \circ \Upsilon \circ \rho \otimes \eta(i \otimes j) &= \pi_{A\otimes_p B} \circ \Upsilon(\rho(i) \otimes \rho(j)) \\
&= \pi_{A\otimes_p B} \circ \Upsilon((\sum_{k=1}^{\infty} x_k^i \otimes y_k^i) \otimes (\sum_{s=1}^{\infty} a_s^j \otimes b_s^j)) \\
&= \pi_{A\otimes_p B}(\sum_{s,k=1}^{\infty} (x_k^i \otimes a_s^j) \otimes (y_k^i \otimes b_s^j)) \\
&= \sum_{k=1}^{\infty} x_k^i y_k^i \otimes \sum_{s=1}^{\infty} a_s^j b_s^j \\
&= \pi_A \circ \rho(i) \otimes \pi_B \circ \eta(j) = i \otimes j.
\end{aligned}$$

Then the proof is complete. □

Lemma 2.6. *Let A and B be Banach algebras. Also let I and J be closed ideal for A and B, respectively. Suppose that I has an identity and J has a nonzero idempotent. If $A \otimes_p B$ is $I \otimes_p J$-biprojective, then I is super amenable.*

Proof. Since I has an identity, using [5, Theorem VII.1.63] it is enough to show that I is biprojective and by Theorem 2.2, we only show that A is I-biprojective. By hypothesis $A \otimes_p B$ is $I \otimes_p J$-biprojective, then there exists a bounded $A \otimes_p B$-bimodule morphism $\rho_1 : I \otimes_p J \to (A \otimes_p B) \otimes_p (A \otimes_p B)$ such that $\pi_{A\otimes_p B} \circ \rho_1(i \otimes j) = i \otimes j$, where $i \in I$ and $j \in J$. Let regard $I \otimes_p J$ as a Banach A-bimodule via the following actions

$$a \cdot (i \otimes j) = ai \otimes j \quad (i \otimes j) \cdot a = ia \otimes j,$$

where $a \in A, i \in I$ and $j \in J$. Suppose that e_I is an identity of I and x_0 is a nonzero idempotent for J. We also regard the projective tensor product $(A \otimes_p B) \otimes_p (A \otimes_p B)$ as a Banach A-bimodule with the following module actions:

$$a' \bullet \Big((a \otimes b) \otimes (c \otimes d)\Big) = (a'a \otimes x_0 b) \otimes (c \otimes d),$$

and

$$\Big((a\otimes b)\otimes(c\otimes d)\Big)\bullet a' = (a\otimes b)\otimes(ca'\otimes dx_0),$$

for all $a, c, a' \in A$ and $b, d \in B$. By the above considerations we have:

$$\begin{aligned}\rho_1(ai\otimes x_0) &= \rho_1((a\otimes x_0)(i\otimes x_0))\\ &= (a\otimes x_0)\rho_1(i\otimes x_0)\\ &= a\bullet\rho_1(i\otimes x_0)\end{aligned}$$

Similarly one can show that $\rho_1(ia\otimes x_0) = \rho_1(i\otimes x_0)\bullet a$, where $a \in A, i \in I$. Choose $\psi \in B^*$ such that $\psi(x_0) = 1$. Define

$$T : (A\otimes_p B)\otimes_p(A\otimes_p B)\to A\otimes_p A$$

by

$$T((a\otimes b)\otimes(c\otimes d)) = \psi(bd)a\otimes c,$$

for each $a, c \in A$ and $b, d \in B$. Clearly T is a bounded A-bimodule morphism. Define $\rho : I \to A\otimes_p A$ via $\rho(i) = T\circ\rho_1(i\otimes x_0)$. It is easy to see that ρ is a bounded A-bimodule morphism and

$$\pi_A\circ T = (id_A\otimes\psi)\circ\pi_{A\otimes_p B},$$

where $id_A\otimes\psi(a\otimes b) = \psi(b)a$ for $a \in A$ and $b \in B$. Hence

$$\pi_A\circ\rho(i) = \pi_A\circ T\circ\rho_1(i\otimes x_0) = (id_A\otimes\psi)\circ\pi_{A\otimes_p B}\circ\rho_1(i\otimes x_0) = i,$$

the last equality holds because $\psi(x_0) = 1$. □

3. I-biflat Banach algebras

Definition 3.1. Let A be a Banach algebra and I be a closed ideal of A. We say that A is I-biflat if there exists a bounded "A-bimodule morphism" $\rho : I \to (A\otimes_p A)^{**}$ such that $\pi_A^{**}\circ\rho(i) = i$ for every $i \in I$.

We recall that for Banach A-bimodules X and Y, a net $(\rho_\alpha)_\alpha$ of bounded linear operators from X into Y is called approximate A-bimodule morphism, if for each $a \in A$ and $x \in X$ we have

$$\rho_\alpha(a\cdot x) - a\cdot\rho_\alpha(x)\xrightarrow{\|\cdot\|}0 \qquad \rho_\alpha(x\cdot a) - \rho_\alpha(x)\cdot a\xrightarrow{\|\cdot\|}0.$$

We remind that for Banach algebras X and Y the *weak* operator topology* (W^*OT) on $B(X, Y^*)$ (the set of all bounded linear operators from X into Y^*) is a topology determined by seminorms $\{p_{x,f} : x \in X, f \in Y\}$ that $p_{x,f}(T) = |T(x)(f)|$, where $T \in B(X, Y^*)$. In the other word $T_\alpha \xrightarrow{W^*OT} T$ if and only if for every $x \in X$; $T_\alpha(x)\xrightarrow{w^*}T(x)$. Note that, since $B(X, Y^*) \cong (X\otimes_p Y)^*$, every bounded set in $B(X, Y^*)$ has a w^*-limit point, with respect to w^*-topology on $(X\otimes_p Y)^*$.

Lemma 3.2. *Let A be a Banach algebra and I be a closed ideal of A. Suppose that there exists a net of bounded approximate A-bimodule morphisms $(\rho_\alpha)_\alpha$, which $\rho_\alpha : I \to (A\otimes_p A)^{**}$ and $\lim\pi_A^{**}\circ\rho_\alpha(x) = x$, where $x \in I$. Then A is I-biflat.*

Proof. Since $(\rho_\alpha)_\alpha$ is a bounded net in $B(I, (A \otimes_p A)^{**}) \cong (I \otimes_p (A \otimes_p A)^*)^*$, by Alaglou's theorem $(\rho_\alpha)_\alpha$ has w^*-limit point say ρ, which one can show that ρ is a bounded A-bimodule morphism. On the other hand since π_A^{**} is a w^*-continuous map, $\rho_\alpha(x) \xrightarrow{w^*} \rho(x)$ implies that $\pi_A^{**} \circ \rho_\alpha(x) \xrightarrow{w^*} \pi_A^{**} \circ \rho(x)$. Also using $\pi_A^{**} \circ \rho_\alpha(x) \xrightarrow{\|\cdot\|} x$, follows that $\pi_A^{**} \circ \rho(x) = x$, where $x \in I$. □

Theorem 3.3. *Let A be a Banach algebra and I be a closed ideal of A and let I has a bounded approximate identity. Then I is a biflat Banach algebra if and only if A is I-biflat.*

Proof. Similar to the proof of Theorem 2.2. □

Consider the semigroup $\mathbb{N}$ with the semigroup operation $m \wedge n = \min\{m, n\}$, where m and n are in $\mathbb{N}$. The semigroup algebra with respect to $\mathbb{N}_\wedge$ is denoted by $\ell^1(\mathbb{N}_\wedge)$. This Banach algebra has a bounded approximate identity $(\delta_n)_n$. For more details about this algebra the reader referred to [1].

Corollary 3.4. *Let $\ell^1(\mathbb{N}_\wedge)^\sharp$ be the unitization of $\ell^1(\mathbb{N}_\wedge)$. Then $\ell^1(\mathbb{N}_\wedge)^\sharp$ is not $\ell^1(\mathbb{N}_\wedge)$-biflat.*

Proof. Suppose that $\ell^1(\mathbb{N}_\wedge)^\sharp$ is $\ell^1(\mathbb{N}_\wedge)$ biflat. Then since this algebra has a bounded approximate identity, by above Theorem $\ell^1(\mathbb{N}_\wedge)$ is biflat, then by [5, Theorem VII.1.63] $\ell^1(\mathbb{N}_\wedge)$ is an amenable Banach algebra. So by [2, Theorem 2] the set of idempotents of $\mathbb{N}_\wedge$ is finite which is impossible. □

Corollary 3.5. *Let G be a locally compact group. Then $M(G)$ is $L^1(G)$-biflat if and only if G is an amenable group.*

Proof. It is well-known that $L^1(G)$ is biflat if and only if G is amenable. Since $L^1(G)$ posses a bounded approximate identity Theorem 3.3 finishes the proof. □

Corollary 3.6. *There exists a Banach algebra A and a closed ideal I of it such that A is I-biflat but A is not I-biprojective.*

Proof. Let G be a non-compact, amenable group (for example the abelian group $\mathbb{R}$). So $L^1(G)$ is amenable. Then by above Corollary $M(G)$ is $L^1(G)$-biflat but by Proposition 2.4 $M(G)$ is not $L^1(G)$-biprojective. □

A Banach algebra A is called ϕ-biflat if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$ for all $a \in A$, [11].

Lemma 3.7. *Let I be a closed ideal of a commutative Banach algebra A and also let $I \nsubseteq \cap_{\phi \in \Delta(A)} \ker \phi$. If A is I-biflat, then A is ϕ-biflat, for some $\phi \in \Delta(A)$.*

Proof. Let $i_0 \in I$ such that for some $\phi \in \Delta(A)$, $\phi(i_0) = 1$. Let $\rho : I \to (A \otimes_p A)^{**}$ be a bounded A-bimodule morphism that $\pi_A^{**} \circ \rho(i) = i$ for every $i \in I$. Suppose that $\theta_{i_0} : A \to I$ specified by $\theta_{i_0}(a) = ai_0$. Consider $\eta_{i_0} = \rho \circ \theta_{i_0} : A \to (A \otimes_p A)^{**}$. It is easy to see that η_{i_0} is a bounded A-bimodule morphism (use commutativity here). Moreover

$$\tilde{\phi} \circ \pi_A^{**} \circ \eta_{i_0}(a) = \tilde{\phi} \circ \pi_A^{**} \circ \rho \circ \theta_{i_0}(a) = \tilde{\phi} \circ \pi_A^{**} \circ \rho(ai_0) = \phi(ai_0) = \phi(a)\phi(i_0) = \phi(a).$$

Then the proof is complete. □

Lemma 3.8. *Let A be a Banach algebra which is a closed ideal of A^{**}. If A^{**} is A-biprojective, then A is a biflat Banach algebra.*

Proof. Let $\rho : A \to A^{**} \otimes_p A^{**}$ be a bounded A^{**}-bimodule morphism such that $\pi_{A^{**}} \circ \rho(a) = a$, for every $a \in A$. Then clearly ρ is a bounded A-bimodule morphism. On the other hand, there exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, satisfies the following;

(i) $\psi(a \otimes b) = a \otimes b$,

(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,

(iii) $\pi_A^{**}(\psi(m)) = \pi_{A^{**}}(m)$,

see [4, Lemma 1.7]. Let define $\eta = \psi \circ \rho : A \to (A \otimes_p A)^{**}$. Then it is easy to see that η is a bounded A-bimodule morphism such that

$$\pi_A^{**} \circ \eta(a) = \pi_A^{**} \circ \psi \circ \rho(a) = \pi_{A^{**}} \circ \rho(a) = a.$$

Then the proof is complete. □

A Banach algebra A is called ϕ-Johnson amenable if and only if there exists a bounded net (m_α) in $(A \otimes_p A)^{**}$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0, \quad \phi \circ \pi_A(m_\alpha) \to 1,$$

see [11].
For any locally compact group G, it is well-known that $L^1(G)^{**}$ is a closed ideal of $M(G)^{**}$ [3, Proposition 1.3].

Proposition 3.9. *Let G be a locally compact group. If $M(G)^{**}$ is $L^1(G)^{**}$-biflat, then G is amenable.*

Proof. Assume that $M(G)^{**}$ is $L^1(G)^{**}$-biflat. Then there exists a bounded $M(G)^{**}$-bimodule morphism $\rho : L^1(G)^{**} \to (M(G)^{**} \otimes_p M(G)^{**})^{**}$ such that $\pi_{M(G)^{**}}^{**} \circ \rho(a) = a$ for every $a \in L^1(G)^{**}$. Suppose that $\phi \in \Delta(L^1(G))$ and pick $i_0 \in L^1(G)$ such that $\phi(i_0) = 1$. We denote $\tilde{\phi}$ and $\tilde{\tilde{\phi}}$ for unique extension ϕ to $L^1(G)^{**}$ and $L^1(G)^{****}$. Let R_{i_0} and L_{i_0} denoted for the map of right and left multiplication by i_0, respectively. Since $L^1(G)^{**}$ is a closed ideal in $M(G)^{**}$, the map $R_{i_0} \otimes L_{i_0} : M(G)^{**} \otimes_p M(G)^{**} \to L^1(G)^{**} \otimes_p L^1(G)^{**}$ is a bounded $M(G)^{**}$-bimodule morphism. Also one can easily see that $(R_{i_0} \otimes L_{i_0})^{**}$ is a bounded $M(G)^{**}$-bimodule morphism. On the other hand, there exists

$$\psi : L^1(G)^{**} \otimes_p L^1(G)^{**} \to (L^1(G) \otimes_p L^1(G))^{**}$$

such that for $a, b \in L^1(G)$ and $m \in L^1(G)^{**} \otimes_p L^1(G)^{**}$, the following holds;

(i) $\psi(a \otimes b) = a \otimes b$,

(ii) $\psi(m) \cdot a = \psi(m \cdot a), \qquad a \cdot \psi(m) = \psi(a \cdot m)$,

(iii) $\pi^{**}_{L^1(G)}(\psi(m)) = \pi_{L^1(G)^{**}}(m)$,

see [4, Lemma 1.7]. Define $\eta : \psi^{**} \circ (R_{i_0} \otimes L_{i_0})^{**} \circ \rho|_{L^1(G)} : L^1(G) \to (L^1(G) \otimes_p L^1(G))^{****}$, which is a bounded $L^1(G)$-bimodule morphism. Let $(e_\alpha)_{\alpha \in J}$ be a bounded approximate identity for $L^1(G)$. Then for every $a \in L^1(G)$, we have $a \cdot \eta(e_\alpha) - \eta(e_\alpha) \cdot a \to 0$. Also we have

$$\tilde{\tilde{\phi}} \circ \pi^{****}_{L^1(G)} \circ \psi^{**} \circ (R_{i_0} \otimes L_{i_0})^{**} \circ \rho(e_\alpha) \to 1,$$

to see these, consider

$$\begin{aligned}
\tilde{\tilde{\phi}} \circ \pi^{****}_{L^1(G)} \circ \psi^{**} \circ (R_{i_0} \otimes L_{i_0})^{**} \circ \rho(a) &= \tilde{\tilde{\phi}} \circ (\pi^{**}_{L^1(G)} \circ \psi \circ (R_{i_0} \otimes L_{i_0}))^{**} \circ \rho(a)\\
&= \tilde{\tilde{\phi}} \circ (\pi_{L^1(G)^{**}} \circ (R_{i_0} \otimes L_{i_0}))^{**} \circ \rho(a)\\
&= \tilde{\tilde{\phi}} \circ \pi^{**}_{M(G)^{**}} \circ \rho(a)\\
&= \phi(a),
\end{aligned}$$

where $a \in L^1(G)$. Let $\epsilon > 0$. Pick finite sets $F \subseteq L^1(G)$ and $\Phi \subseteq (L^1(G) \otimes_p L^1(G))^{***}$. Let

$$K = \{a \cdot \xi | a \in F, \xi \in \Phi\} \cup \{\xi \cdot a | a \in F, \xi \in \Phi\}.$$

Hence there exists $\nu = \nu(\epsilon, F, \Phi) \in J$ such that for $a \in F$,

$$||a \cdot \eta(e_\nu) - \eta(e_\nu) \cdot a|| \leq \frac{\epsilon}{3K_0}, \qquad |\tilde{\tilde{\phi}} \circ \pi^{****}_{L^1(G)} \circ \eta(e_\nu) - 1| < \epsilon,$$

where $K_0 = \max\{||\xi||, \xi \in \Phi\}$. Goldstine's theorem implies that, there exists a net $(b_\lambda)_\lambda$ in $(L^1(G) \otimes_p L^1(G))^{**}$ such that $\widehat{b_\lambda} \xrightarrow{w^*} \eta(e_\nu)$. On the other hand since $\pi^{****}_{L^1(G)}$ is a w^*-continuous map, $\pi^{**}_{L^1(G)}(b_\lambda) = \pi^{****}_{L^1(G)}(\widehat{b_\lambda}) \xrightarrow{w^*} \pi^{****}_{L^1(G)} \circ \eta(e_\nu)$. Hence there exists a $\lambda_0 = \lambda_0(\epsilon, F, \Phi)$ such that $|\vartheta(b_{\lambda_0}) - \eta(e_\nu)(\vartheta)| < \frac{\epsilon}{3}$ and $|\tilde{\phi} \circ \pi^{**}_{L^1(G)}(b_{\lambda_0}) - \tilde{\tilde{\phi}} \circ \pi^{****}_{L^1(G)} \circ \eta(e_\nu)| < \epsilon$, where $\vartheta \in K$. Consider

$$|\tilde{\phi} \circ \pi^{**}_{L^1(G)}(b_{\lambda_0}) - 1| = |\tilde{\phi} \circ \pi^{**}_{L^1(G)}(b_{\lambda_0}) - \tilde{\tilde{\phi}} \circ \pi^{**}_{L^1(G)} \circ \eta(e_\nu) + \tilde{\tilde{\phi}} \circ \pi^{**}_{L^1(G)} \circ \eta(e_\nu) - 1| < c\epsilon,$$

for some $c \in \mathbb{C}$. Since $|\vartheta(b_{\lambda_0}) - \eta(e_\nu)(\vartheta)| < \frac{\epsilon}{3}$, we have

$$\begin{aligned}
&|\xi(a \cdot b_{\lambda_0} - b_{\lambda_0} \cdot a)| \leq\\
&|\xi(a \cdot b_{\lambda_0}) - a \cdot \eta(e_\nu)(\xi)| + |a \cdot \eta(e_\nu)(\xi) - \eta(e_\nu) \cdot a(\xi)| + |\eta(e_\nu) \cdot a(\xi) - \xi(b_{\lambda_0} \cdot a)|\\
&< \epsilon.
\end{aligned}$$

Therefore $a \cdot b_\lambda - b_\lambda \cdot a \xrightarrow{w} 0$, so by Mazur's lemma we can assume that the limitation happens in the norm topology. Then there exists a bounded net $(b_\lambda)_\lambda \subseteq (L^1(G) \otimes_p L^1(G))^{**}$ (since $(\eta(e_\nu))_{\nu \in J}$ is a bounded net) such that

$$a \cdot b_\lambda - b_\lambda \cdot a \xrightarrow{||\cdot||} 0, \qquad \tilde{\phi} \circ \pi^{**}_{L^1(G)}(b_\lambda) \to 1.$$

One can proceed as above consider $(b_\lambda)_\lambda \subseteq L^1(G) \otimes_p L^1(G)$, hence $L^1(G)$ is ϕ-Johnson amenable. By [11, Lemma 3.1], $L^1(G)$ is ϕ-biflat, now apply [11, Lemma 4.1] to see that G is amenable. □

4. Examples

We establish some examples which show the differences of our new notion with classical ones.

Example 4.1. We give a Banach algebra which is not biprojective but it is I-biprojective for some closed ideal of it.
Let A be a Banach space which $\dim A > 1$ and let $\phi \in A^* - \{0\}$. Let define a multiplication on A via $ab = \phi(a)b$ for every a and b in A. Then clearly with this multiplication A becomes a Banach algebra. Let $A^\sharp$ be the unitization of A. It is known that A is a closed ideal of $A^\sharp$. We claim that $A^\sharp$ is not biprojective but is A-biprojective. Suppose conversely that $A^\sharp$ is biprojective. Applying [10, Corollary 2.3.11] A is amenable. Using [10, Proposition 2.2.1] A has a bounded approximate identity say (e_α). Then

$$a = \lim_\alpha ae_\alpha = \phi(a)\lim_\alpha e_\alpha.$$

Let a_0 be an element of A such that $\phi(a_0) = 1$. It follows that

$$a_0 = \lim_\alpha a_0e_\alpha = \phi(a_0)\lim_\alpha e_\alpha = \lim_\alpha e_\alpha.$$

Hence

$$a = a\lim_\alpha e_\alpha = \lim_\alpha ae_\alpha = \phi(a)\lim_\alpha e_\alpha = \phi(a)a_0.$$

So $\dim A = 1$ which is a contradiction.
We claim that $A^\sharp$ is A-biprojective. Suppose that x_0 is an element in A such that $\phi(x_0) = 1$. Define $\rho_{x_0} : A \to A^\sharp \otimes_p A^\sharp$ by $\rho_{x_0}(a) = x_0 \otimes a$. Since

$$b \cdot \rho_{x_0}(a) = bx_0 \otimes a = \phi(b)x_0 \otimes a = x_0 \otimes \phi(b)a = x_0 \otimes ba = \rho_{x_0}(ba),$$

for every $b \in A$. Then ρ_{x_0} is a left A-module morphism. If b is the unit of $A^\sharp$, then clearly we have $b\cdot\rho_{x_0}(a) = \rho_{x_0}(ba)$, where $a \in A$. Then ρ_{x_0} is a bounded left $A^\sharp$-module morphism. It is easy to see that ρ_{x_0} is a bounded right $A^\sharp$-module morphism. Moreover

$$\pi_{A^\sharp} \circ \rho_{x_0}(a) = \pi_{A^\sharp}(x_0 \otimes a) = x_0a = \phi(x_0)a = a,$$

for every $a \in A$. Then $A^\sharp$ is an A-biprojective Banach algebra.

Example 4.2. We give another example of Banach algebras which is not biprojective but for some closed ideal I, is I-biprojective. Let $A = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{C}\}$. As one of the main results of [8] this algebra is not biprojective. Let $I = \{\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} | x, y \in \mathbb{C}\}$. It is easy to see that I is a closed ideal of A. Define $\rho : I \to A \otimes_p A$ by $\rho(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. For an arbitrary element

$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ of A and $\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$ of I, we have

$$\begin{aligned}
\rho(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}) \cdot \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \\
&= \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & c \\ 0 & c \end{pmatrix} \\
&= \begin{pmatrix} 0 & xc \\ 0 & yc \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\
&= \rho(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}).
\end{aligned}$$

Hence ρ is a bounded right A-module morphism. One can easily see that ρ is a bounded left A-module morphism. Also we have

$$\begin{aligned}
\pi_A \circ \rho(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}) &= \pi_A(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}) \\
&= \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}.
\end{aligned}$$

Then A is not biprojective but it is I-biprojective.

It is easy to see that every biprojective Banach algebras are I-biprojective for every closed ideal I of A.

Example 4.3. There exists a closed ideal I in a Banach algebra A such that I is not biprojective but A is I-biprojective. Let $A = \{\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} | x, y \in \mathbb{C}\}$ and $I = \{\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} | z \in \mathbb{C}\}$. Clearly $I^2 = \{0\}$ and $AI = \{0\}$ and also $IA = I$. We go toward a contradiction and suppose that I is biprojective. Then I is biflat, hence by [10, Exercise 4.3.14] $\overline{I^2} = I$ which is impossible. Since A is a biprojective Banach algebra (Define $\rho : A \to A \otimes_p A$ by $\rho(\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and see the arguments at the above Example), it is easy to see that A is $I-$biprojective.

We give a non-biprojective Banach algebra A which posses a non-biprojective closed ideal I. But A is I-biprojective.

Example 4.4. Let $A = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{C}\}$ and $I = \{\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} | z \in \mathbb{C}\}$ be its closed ideal. Note that A is not biprojective see [8]. Also by previous Example I is not biprojective. Now use the similar arguments of Example 4.2 to show that A is I-biprojective.

Consider the semigroup $\mathbb{N}_\vee$, with the semigroup operation $m \vee n = \max\{m, n\}$, where m and n are in $\mathbb{N}$. It is easy to see that $\ell^1(\mathbb{N}_\vee)$ has an unit δ_1. For more details about this semigroup algebra reader referred to [1].
Let A be a Banach algebra and $\phi \in \Delta(A)$. Then A is called a ϕ-biflat Banach algebra, if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$ for every $a \in A$. See [11] for more information about this concept.

Example 4.5. We give a Banach algebra A and a closed ideal I of it which is not I-biflat but A and I are ϕ-biflat. Let $I = \ell^1(\mathbb{N}_\vee)$ and $A = \ell^1(\mathbb{N}_\vee)^\sharp$ be its unitization. We know that $\ell^1(\mathbb{N}_\vee)$ is ϕ-biflat for every $\phi \in \Delta(\ell^1(\mathbb{N}_\vee))$ see [11, Example 5.5]. The Banach algebra $\ell^1(\mathbb{N}_\vee)$ is not biflat. Otherwise if $\ell^1(\mathbb{N}_\vee)$ is biflat, then since it has unit, by [10, Exercise 4.3.14] it must be amenable, then by [2, Theorem 2] the set of idempotents of $\mathbb{N}_\vee$ is finite which is impossible. So by Theorem 3.3 A is not I-biflat but A and I are ϕ-biflat for every $\phi \in \Delta(A)$.

Example 4.6. There exists a Banach algebra A with a closed ideal I such that A is I-biflat but I is not biflat. Again let A and I be as in the Example 4.3. Since A is I-biprojective, A is I-biflat. On the other hand since $I^2 = \{0\}$ which is not dense in I, then by [10, Example 4.3.14] I is not biflat. Let A and I be as in the Example 4.4, with the same arguments as in the Example 4.4. Note that I is not biflat. By the main result of [8] A is not biflat but by the same arguments as in the Example 4.2, one can see that A is I-biflat.

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