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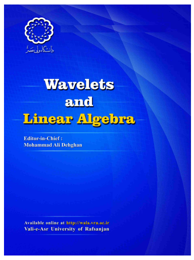

Littlewood Subordination Theorem and Composition Operators on Function Spaces with Variable Exponents

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Abstract

This study concerns a detailed analysis of composition operators C_φ on the classical Bergman spaces, as well as on the Hardy and Bergman spaces with variable exponents. Here, φ is an analytic self-map of the open unit disk in the complex plane. Accordingly, conditions for the boundedness of these operators are obtained. It is worth mentioning that the Littlewood subordination theorem plays a fundamental role in proving the stated theorems in which we use the Rubio de Francia extrapolation theorem.

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1. Introduction

The study of composition operators as part of operator theory on analytic function spaces dates back to the work of E. Nordgren in the mid-1960's. As a result, there appears several papers in which researchers studied the composition operators acting on function spaces in the unit disk, as well as in the unit ball of $\mathbb{C}^n$. It should be emphasized that most of these papers generally discuss the boundedness and the compactness of composition operators on analytic function spaces. In particular, Stessin and Zhu proved that the composition operators are bounded on the Hardy and Bergman spaces defined on the unit disk as well as on the polydisk and provided necessary and sufficient conditions for the compactness of these operators on the indicated spaces (see [22] and [23]). It should be underlined that, variable exponent Lebesgue spaces, as generalization of Lebesgue spaces, were first introduced by Orlicz [19]. After that, O. Kováčik, and J. Rákonsník [14] developed the fundamental properties of these spaces. More significantly, in 2004, the correct regularity condition on the exponents to guarantee the boundedness of maximal functions was found by L. Diening [7] for bounded domains and subsequently by Cruz-Uribe et al. [6], and independently by A. Nekvinda [18], for unbounded domains.

Very recently, the study of variable exponent Banach spaces of analytic functions, in particular, the study of variable exponent Hardy, Bergman, and Fock spaces have attracted attention. The following references would offer some good examples of the current studies on these topics [1, 2, 3, 8, 12].

It should be emphasized that, little is known about the variable exponent Hardy spaces on the open unit disk. The first approach to tackle this issue was taken in [12] and [13]. Next, G. R. Chacón and G. A. Chacón introduced the variable exponent version of the Hardy space on the upper-half plane and studied some properties of these spaces [1].

In this paper, we first review the basic concepts of the theory of variable exponent Hardy and Bergman spaces. We then discuss some previous works regarding the boundedness of composition operators both in constant exponenet spaces and in variable exponent spaces. We give sufficient conditions for the boundedness of the composition operators on the classical Bergman space, as well as on the variable exponent Bergman space and on the variable exponent Hardy space. The main point of our results is the use of Littlewood subordination theorem in the proofs of the theorems presented here. It is worth mentioning that in proving the boundedness of composition operators on analytic function spaces, the concept of Carleson measure is usually used (see [22, 23]), however we will focus on the use of the Littlewood subordination theorem.

2. Preliminaries

Due to a slight difference in the definition of concepts related to variable exponent Hardy and Bergman spaces, we present these concepts in two separate subsections.

2.1. Variable Exponent Hardy Space

To achieve our goals, we first, review the basic concepts of variable exponent Hardy and Bergman spaces as defined by G. R. Chacón, G. A. Chacón and H. Rafeiro [1, 2]. We also recall the theorems needed in the proof of our main theorems.

Throughout this paper, we use the symbol $\mathbb{D}$ for the open unit disk in the complex plane. A measurable and essentially bounded function $p : [0, 2\pi] \to [1, \infty)$ such that $p(0) = p(2\pi)$, is called an *exponent function*. The variable exponent Lebesgue space, $L^{p(\cdot)}(\mathbb{T})$, is defined as the space of all measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\rho_{p(\cdot)}(f) = \int_0^{2\pi} |f(e^{i\theta})|^{p(\theta)} d\theta < \infty$$

where $\mathbb{T}$ stands for the boundary of the unit disk $\mathbb{D}$. The norm of a function f in this space is defined by

$$\|f\|_{p(\cdot)} = \inf_{\lambda} \left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}.$$

This space is a Banach space with respect to this norm [5].

A function $p : [0, 2\pi] \to [1, \infty)$ is said to be *log-Hölder continuous* or satisfy to the *Dini-Lipschitz condition* on $[0, 2\pi]$, if there exists a positive constant C_{log} such that

$$|p(x) - p(y)| \leq \frac{C_{log}}{\log(\frac{1}{|x-y|})},$$

for all $x, y \in [0, 2\pi]$. The set all of log-Hölder continuous functions in $[0, 2\pi]$ for which $1 < p_- \leq p_+ < \infty$ is denoted by $\mathcal{P}^{\log}([0, 2\pi])$ where

$$p_+ = \operatorname*{ess\ sup}_{x\in[0,2\pi]} p(x) \qquad and \qquad p_- = \operatorname*{ess\ inf}_{x\in[0,2\pi]} p(x).$$

Let $f : \mathbb{D} \to \mathbb{C}$ and $0 < r < 1$. We define the dilation function $f_r : \mathbb{T} \to \mathbb{C}$ by $f_r(\zeta) = f(r\zeta)$. Given $p : [0, 2\pi] \to [1, \infty)$, the *variable exponent harmonic Hardy space* $h^{p(\cdot)}(\mathbb{D})$ is defined as the space of harmonic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{h^{p(\cdot)}(\mathbb{D})} = \sup_{0\leq r<1} \|f_r\|_{L^{p(\cdot)}(\mathbb{T})} < \infty.$$

We now recall the definition of the variable exponent Hardy space $H^{p(\cdot)}(\mathbb{D})$ as defined in reference [1]. Suppose that $p : [0, 2\pi] \to [1, \infty)$ is a $2\pi-$periodic function. The *variable exponent Hardy space* $H^{p(\cdot)}(\mathbb{D})$ is defined as the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that $f \in h^{p(\cdot)}(\mathbb{D})$.

In an analogous manner as in the classical setting, it is shown that $H^{p(\cdot)}(\mathbb{D})$ can be identified as the subspace of functions in $L^{p(\cdot)}(\mathbb{T})$ whose negative Fourier coefficients are zero and as such, $H^{p(\cdot)}(\mathbb{D})$ is a Banach space (see [9] for more details).

Now, we recall an important tool for handling these sorts of problems. Given a function $f \in L^1_{loc}(\mathbb{D})$ and $z \in \mathbb{D}$, the *Hardy-Littlewood maximal function* is defined by

$$Mf(z) = \sup_{r>0} \frac{1}{|B(z, r)|} \int_{B(z,r)} |f(w)| d\mu(w)$$

where μ is the Lebesgue measure on $\mathbb{D}$. L. Diening has shown that on bounded domains, log-Hölder continuity is a sufficient condition for the boundedness of the maximal operator on $L^{p(\cdot)}(\mathbb{D})$ (see [7]).

Lemma 2.1. *([7]) Let $p \in \mathcal{P}^{\log}([0, 2\pi])$. Then the Hardy-Littlewood maximal function is bounded on $L^{p(\cdot)}(\mathbb{D})$, i.e., there exists $C > 0$ such that*

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{D})} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{D})}.$$

In classical analysis, the Muckenhoupt weight class A_p consists of those weight functions ω for which the Hardy-Littlewood maximal operator is bounded on $L^p(d\omega)$. Let $p > 1$, then a measurable function ω on $\mathbb{D}$ such that $0 < \omega(x) < \infty$, almost everywhere, is called a *weight of class A_p* and is written $\omega \in A_p$ if the following condition holds,

$$[\omega]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(z)dA(z)\right)\left(\frac{1}{|Q|}\int_Q \omega(z)^{1-p'}dA(z)\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{D}$ with sides parallel to the coordinate axes and $|Q|$ stands for the Lebesgue measure of Q.
In special case when $p = 1$, a measurable function ω on $\mathbb{D}$ such that $0 < \omega(z) < \infty$, almost everywhere, is called a weight of *class* A_1 and is written $\omega \in A_1$ if we have

$$[\omega]_{A_1} := \operatorname{ess}\sup_{z\in\mathbb{D}} \frac{M\omega(z)}{\omega(z)} < \infty,$$

where M is the Hardy-Littlewood maximal operator. Note that, if $\omega \in A_1$, then the following inequality holds:

$$M\omega(z) \leq [\omega]_{A_1}\omega(z), \quad a.e.$$

Moreover, if $\omega \in A_1$, the following relations are valid

$$[\omega]_{A_1}\omega(z) \geq M\omega(z) \geq \omega(\mathbb{D}), \tag{2.1}$$

where

$$\omega(B) = \int_B \omega(z)dA(z), \quad B \subset \mathbb{D}.$$

This implies that

$$1 \leq \frac{[\omega]_{A_1}\omega(z)}{\omega(\mathbb{D})}. \tag{2.2}$$

For more details on Muckenhoupt weights, see [4] and [5].

Theorem 2.2. *([5]) Let $\Omega \subset \mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies $1 < p_- \leq p_+ < \infty$. Then the Hardy-Littlewood maximal function M is bounded on $L^{p(\cdot)}(\Omega)$ if and only if M is bounded on the dual space $L^{p'(\cdot)}(\Omega)$.*

Theorem 2.3. *([5]) (Rubio de Francia extrapolation theorem) Given $\Omega \subset \mathbb{R}^n$, let for some $p_0 \geq 1$, the family $\mathcal{F}$ consisting all non-negative measurable functions on Ω is such that for all $\omega \in A_1$, there exists $C_0 > 0$ such that*

$$\int_\Omega F(x)^{p_0}\omega(x)dx \leq C_0 \int_\Omega G(x)^{p_0}\omega(x)dx; \quad F, G \in \mathcal{F}. \tag{2.3}$$

If $p(\cdot) \in \mathcal{P}(\Omega)$ with $p_0 \leq p_- \leq p_+ < \infty$ and if the maximal operator is bounded on the dual space $L^{\left(\frac{p(\cdot)}{p_0}\right)'}(\Omega)$, then there exists $C_{p(\cdot)} > 0$ such that

$$\|F\|_{L^{p(\cdot)}} \leq C_{p(\cdot)}\|G\|_{L^{p(\cdot)}}.$$

By Proposition 3.3 of [5], we know that for every $f : \mathbb{D} \to \mathbb{C}$, the following inequality holds

$$|f(z)| \leq Mf(z), \quad z \in \mathbb{D}, \tag{2.4}$$

this is of course a consequence of the Lebesgue differentiation theorem (see Section 2.9 of [6]). Finally, we recall the Littlewood subordination theorem as follows.

Theorem 2.4. *([4])(Littlewood Subordination Theorem) Let φ be an analytic map of $\mathbb{D}$ into itself such that $\varphi(0) = 0$. If G is a subharmonic function in $\mathbb{D}$, then for $0 < r < 1$, the following inequality is valid*

$$\int_0^{2\pi} G(\varphi(re^{i\theta}))d\theta \leq \int_0^{2\pi} G(re^{i\theta})d\theta.$$

Theorem 2.5. *([4]) If φ is an analytic function of the disk into itself and $p \geq 1$, then the composition operator $C_\varphi : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is bounded and*

$$\left(\frac{1}{1 - |\varphi(0)|^2}\right)^{\frac{1}{p}} \leq \|C_\varphi\| \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)}\right)^{\frac{1}{p}}.$$

2.2. Variable Exponent Bergman Space

First, we recall the definition of Bergman spaces with constant exponent. Let $dA(z) = dx\,dy/\pi$ be the normalized Lebesgue area measure on $\mathbb{D}$. For $1 \leq p < \infty$, the *Bergman space* $A^p(\mathbb{D})$ is defined as the space of all analytic functions on $\mathbb{D}$ that satisfy the following relation

$$\|f\|^p_{A^p} := \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty.$$

This space equipped with the following norm is a Banach space (see [11]):

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z)\right)^{1/p}.$$

G. R. Chacón and H. Rafeiro in 2014 introduced the concept of the variable exponent Bergman spaces [2]. A measurable function $p : \mathbb{D} \to [1, \infty)$ is called a *variable exponent*. Let $\mathcal{P}(\mathbb{D})$ be the set of all variable exponents $p(\cdot)$ for which $p_+ < \infty$, where

$$p_+ = p_{\mathbb{D}}^+ := \operatorname*{ess\,sup}_{z\in\mathbb{D}} \; p(z),$$
$$p_- = p_{\mathbb{D}}^- := \operatorname*{ess\,inf}_{z\in\mathbb{D}} \; p(z).$$

For a complex-valued measurable function $f : \mathbb{D} \to \mathbb{C}$, the modular $\rho_{p(\cdot)}$ is defined by

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{D}} |f(z)|^{p(z)} \, d\mu(z)$$

where μ is the Lebesgue measure on $\mathbb{D}$. The *Luxemburg-Nakano* norm induced by this modular is given by

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}. \tag{2.5}$$

The *variable exponent Lebesgue space*, $L^{p(\cdot)}(\mathbb{D})$, consists of all complex-valued functions $f : \mathbb{D} \to \mathbb{C}$ for which $\rho_{p(\cdot)}(f) < \infty$. In this case, we write $p(\cdot) \in \mathcal{P}(\mathbb{D})$. It is well-known that if $p(\cdot) \in \mathcal{P}(\mathbb{D})$, then the $L^{p(\cdot)}(\mathbb{D})$ equipped with the given norm is a Banach space (see [5] or [8]). Moreover, the dual of $L^{p(\cdot)}(\mathbb{D})$ is denoted by $L^{p'(\cdot)}(\mathbb{D})$ where the conjugate exponent $p'(\cdot)$ satisfies the following relation for every $z \in \mathbb{D}$

$$\frac{1}{p(z)} + \frac{1}{p'(z)} = 1.$$

So far, we assumed that the variable exponent p enjoys the boundedness property $p_+ < \infty$. To get more efficient results, we need to impose some more restrictions. A function $p : \mathbb{D} \to \mathbb{R}$ is said to be *log-Hölder continuous* if there exists a positive constant C such that for all $z, w \in \mathbb{D}$ with $|z - w| < \frac{1}{2}$, the following relation holds

$$|p(z) - p(w)| \leq \frac{C}{\log(\frac{1}{|z-w|})}. \tag{2.6}$$

We denote by $\mathcal{P}^{\log}(\mathbb{D})$, the set of all log-Hölder continuous functions in $\mathbb{D}$ for which $1 < p_- \leq p_+ < \infty$. Now, we recall the definition of variable exponent Bergman spaces on the unit disk of the complex plane. Given $p \in \mathcal{P}(\mathbb{D})$, the *variable exponent Bergman space* $A^{p(\cdot)}(\mathbb{D})$ consists of all analytic functions in in the unit disk for which

$$\int_{\mathbb{D}} |f(z)|^{p(z)} dA(z) < \infty.$$

In other words, $A^{p(\cdot)}(\mathbb{D})$ consists of all analytic functions in the open unit disk that are simultaneously elements of $L^{p(\cdot)}(\mathbb{D})$ with respect to the normalized Lebesgue area measure in $\mathbb{D}$. These spaces were studied in detail by Chaćon and Rafeiro [2].

Now, we turn our attention to Carleson measure. For constant exponent Bergman spaces, a finite positive Borel measure μ on $\mathbb{D}$ is said to be a *Carleson measure* for $A^p(\mathbb{D})$ if there exists $C > 0$ such that for every $f \in A^p(\mathbb{D})$ we have the following relation

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \leq C \int_{\mathbb{D}} |f(z)|^p dA(z).$$

In other words, μ is a Carleson measure for $A^p(\mathbb{D}, dA)$ if $A^p(\mathbb{D}, dA)$ is continuously embedded in $L^p(\mathbb{D}, d\mu)$.

For more details and characterizations of the Carleson measures for the Bergman spaces, see [10] and [11]. Recently, Chaćon, Rafeiro and Vallejo in [3] have defined the Carleson measure for variable exponent Bergman spaces.

Given a finite positive Borel measure μ on the unit disk $\mathbb{D}$, we call μ a *Carleson measure for the variable exponent Bergman space* $A^{p(\cdot)}(\mathbb{D}, dA)$ if there exists a positive constant C such that for every $f \in A^{p(\cdot)}(\mathbb{D}, dA)$ the following relation holds [3]

$$||f||_{L^{p(\cdot)}(\mathbb{D},\mu)} \leq C||f||_{A^{p(\cdot)}(\mathbb{D},dA)}.$$

It is well-known that in Bergman spaces, being a Carleson measure is independent of p; in other words, μ is a Carleson measure for $A^p(\mathbb{D})$ for some $p > 0$ if and only if μ is a Carleson measure for $A^p(\mathbb{D})$ for every $p > 0$ (see [10]).

Let φ be an analytic self-map on $\mathbb{D}$. Then φ induces an operator C_φ on every space of analytic functions on the unit disk; this operator is given by $C_\varphi(f) = f \circ \varphi$ and is called the *composition operator*. We, also recall for $\alpha > -1$, the *weighted Bergman space* $A^p_\alpha(\mathbb{D})$, is defined as the space of analytic functions f for which the following property holds

$$\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) < +\infty,$$

where

$$dA_\alpha(z) = \pi^{-1}(\alpha + 1)(1 - |z|^2)^\alpha dx\, dy.$$

Suppose $\alpha > -1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic. We define a positive Borel measure $\mu_{\varphi,\alpha}$ on $\mathbb{D}$ as follows: If E is a Borel subset of $\mathbb{D}$, we define

$$\mu_{\varphi,\alpha}(E) := A_\alpha\left(\varphi^{-1}(E)\right) = (\alpha + 1) \int_{\varphi^{-1}(E)} \left(1 - |z|^2\right)^\alpha dA(z).$$

Finally, similar to Lemma 2.1, we have

Lemma 2.6. *([5]) Let $p \in \mathcal{P}^{\log}(\mathbb{D})$. Then the Hardy-Littlewood maximal function is bounded on $L^{p(\cdot)}(\mathbb{D})$ i.e. there exists positive constant $C_{p(\cdot)}$ such that*

$$||Mf||_{L^{p(\cdot)}(\mathbb{D})} \leq C_{p(\cdot)}||f||_{L^{p(\cdot)}(\mathbb{D})}.$$

3. Main Result

We now prove the main results of this paper. First, we present the following lemma which allows us to use the Littlewood subordination theorem to prove the boundedness of composition operators on Hardy and Bergman spaces.

Lemma 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function such that $\varphi(0) = 0$, $0 < p < \infty$ and $0 \neq f \in Hol(\mathbb{D})$. Then $|f|^p$ is a subharmonic function and the following inequality holds*

$$\int_0^{2\pi} |f\left(\varphi(re^{i\theta})\right)|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

Proof. Let $0 \neq f \in Hol(\mathbb{D})$. The function $\eta(t) = e^{pt}$ is increasing and convex. On the other hand, since f is holomorphic, $u = \log|f|$ is a subharmonic function, therefore the function

$$\eta \circ u = e^{p \log |f|} = |f|^p$$

is subharmonic too (see [20]).

Now, applying the Littlewood subordination theorem to the subharmonic function $|f|^p$, we have

$$\int_0^{2\pi} |f\left(\varphi(re^{i\theta})\right)|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

□

Theorem 3.2. *Let $p > 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then the composition operator $C_\varphi : A^p(\mathbb{D}) \to A^p(\mathbb{D})$ is bounded.*

Proof. See [4]. □

Theorem 3.3. *Let $p(\cdot) \in \mathcal{P}^{log}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $f \circ \varphi \in A^{p(\cdot)}(\mathbb{D})$ for every $f \in A^{p(\cdot)}(\mathbb{D})$, and let $p(\cdot)$ and φ satisfy the condition*

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\frac{2}{p(z)}}}{(1 - |\varphi(z)|^2)^{\frac{2}{p(\varphi(z))}}} < \infty.$$

Then $C_\varphi : A^{p(\cdot)}(\mathbb{D}) \to A^{p(\cdot)}(\mathbb{D})$ is bounded.

Proof. The first thing we should comment is the supremum condition that φ and $p(\cdot)$ must satisfy. As we shall see, this is indeed a necessary condition for the boundedness of the composition operator. Let us assume that C_φ is bounded. Recall the evaluation functional

$$\gamma_z(f) = f(z), \quad f \in A^{p(\cdot)}(\mathbb{D}),$$

which is known to be bounded and satisfies the following estimate (see [3, Theorem 3.4])

$$|\gamma_z(f)| \leq \frac{C\|f\|_{A^{p(\cdot)}(\mathbb{D})}}{(1 - |z|^2)^{\frac{2}{p(z)}}}$$

where C is some constant. Now, let

$$k_a(z) = \frac{1}{(1-\overline{a}z)^2}, \quad a, z \in \mathbb{D}.$$

It follows from Proposition 3.3 of [3] that

$$\|k_a^{2/p(a)}\| = \|\frac{1}{(1-\overline{a}z)^{4/p(a)}}\| \leq \frac{C}{(1-|a|^2)^{2/p(a)}},$$

from which it follows that

$$\|(1-|a|^2)^{2/p(a)}k_a^{2/p(a)}\| \leq C.$$

We now consider the function

$$g_a(z) := (1-|a|^2)^{2/p(a)}k_a^{2/p(a)}(z)$$

which has a norm less than or equal to C. Therefore, if C_φ is bounded, we then have

$$\begin{aligned} |(C_\varphi g_a)(z)| &= |g_a(\varphi(z))| \\ &= |\gamma_z(g_a \circ \varphi)| \\ &\leq \frac{C\|g_a \circ \varphi\|_{A^{p(\cdot)}(\mathbb{D})}}{(1-|z|^2)^{\frac{2}{p(z)}}} \\ &\leq \frac{C\|C_\varphi\|\|g_a\|_{A^{p(\cdot)}(\mathbb{D})}}{(1-|z|^2)^{\frac{2}{p(z)}}} \\ &\leq \frac{M}{(1-|z|^2)^{\frac{2}{p(z)}}}. \end{aligned}$$

This implies that there is $M > 0$ such that

$$|g_a(\varphi(z))|\,(1-|z|^2)^{\frac{2}{p(z)}} \leq M, \quad z, a \in \mathbb{D}.$$

In particular, put $a = \varphi(z)$ to get

$$\sup_{z\in\mathbb{D}} \left(1-|\varphi(z)|^2\right)^{\frac{2}{p(\varphi(z))}} k_{\varphi(z)}^{\frac{2}{p(\varphi(z))}}(\varphi(z))(1-|z|^2)^{\frac{2}{p(z)}} \leq M,$$

or

$$\sup_{z\in\mathbb{D}} \left(1-|\varphi(z)|^2\right)^{\frac{2}{p(\varphi(z))}} \frac{1}{(1-|\varphi(z)|^2)^{\frac{4}{p(\varphi(z))}}}(1-|z|^2)^{\frac{2}{p(z)}} \leq M,$$

and finally,

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\frac{2}{p(z)}}}{(1-|\varphi(z)|^2)^{\frac{2}{p(\varphi(z))}}} \leq M,$$

which proves the necessary condition. So far we have justified the assumption made on the symbol and on the exponent. To prove the boundedness of C_φ, let $\omega \in A_1$ and $1 < p_0 \leq p_-$. From (2.4) and the fact that $0 < \omega(\eta) < \infty$, we have

$$\int_{\mathbb{D}} |f(\varphi(\eta))|^{p_0} \omega(\eta) dA(\eta) \leq \int_{\mathbb{D}} |f(\varphi(\eta))|^{p_0} M\omega(\eta) dA(\eta). \tag{3.1}$$

Since, $0 < \omega(\eta) < \infty$, it follows that the integrals in the definition of A_p class weights are positive, therefore, both of them must be finite. This implies that ω and $\omega^{1-p'}$ are locally integrable. Now since ω is locally integrable in $\mathbb{R}^n$, there exists $C_0 > 0$ such that for every $\eta \in \mathbb{D}$, we have

$$M\omega(\eta) \leq [\omega]_{A_1} \omega(\eta) \leq [\omega]_{A_1} \omega(\mathbb{D}) < C_0. \tag{3.2}$$

Therefore, we can write

$$\int_{\mathbb{D}} |f(\varphi(\eta))|^{p_0} M\omega(\eta) dA(\eta) \leq C_0 \int_{\mathbb{D}} |f(\varphi(\eta))|^{p_0} dA(\eta).$$

Now, let $\alpha > -1$ and

$$d\mu(\eta) = dA_\alpha(\eta) = (\alpha + 1)(1 - |\eta|^2)^\alpha dA(\eta)$$

be a measure on $\mathbb{D}$, therefore, for every $E \subseteq \mathbb{D}$, we define the push forward measure by

$$\mu(E) := A_\alpha(\varphi^{-1}(E)) = (\alpha + 1) \int_{\varphi^{-1}(E)} (1 - |\eta|^2)^\alpha dA(\eta).$$

Therefore, we have the following relation [23]

$$\int_{\mathbb{D}} |f(\varphi(\eta))|^2 dA_\alpha(\eta) = \int_{\mathbb{D}} |f(\eta)|^2 d\mu(\eta). \tag{3.3}$$

Let $a = \varphi(0)$ and $\psi(\eta) = (\varphi_a \circ \varphi)(\eta)$ that $\varphi_a(\eta) = \frac{a-\eta}{1-\bar{a}\eta}$. Note that ψ is an analytic function on $\mathbb{D}$ satisfying $\psi(0) = \varphi_a(\varphi(0)) = \varphi_a(a) = 0$. By applying Littlewood subordination theorem, for every analytic function $f : \mathbb{D} \to \mathbb{C}$,

$$\int_{\mathbb{D}} |f(\psi(\eta))|^2 dA_\alpha(\eta) \leq \int_{\mathbb{D}} |f(\eta)|^2 dA_\alpha(\eta). \tag{3.4}$$

Now, replacing $f \circ \varphi_a$ with f in (3.4), we get

$$\int_{\mathbb{D}} |(f \circ \varphi_a)(\psi(\eta))|^2 dA_\alpha(\eta) \leq \int_{\mathbb{D}} |f \circ \varphi_a(\eta)|^2 dA_\alpha(\eta), \tag{3.5}$$

from which, using $\varphi_a \circ \psi = \varphi_a(\varphi_a \circ \varphi) = \varphi$, we obtain

$$\int_{\mathbb{D}} |(f \circ \varphi)(\eta)|^2 dA_\alpha(\eta) \leq \int_{\mathbb{D}} |f \circ \varphi_a(\eta)|^2 dA_\alpha(\eta). \tag{3.6}$$

By using the relations (3.3) and (3.6) we have

$$\begin{aligned}
\int_{\mathbb{D}} |f(\eta)|^2 d\mu(\eta) &= \int_{\mathbb{D}} |f(\varphi(\eta))|^2 dA_\alpha(\eta) \\
&\leq \int_{\mathbb{D}} |f \circ \varphi_a(\eta)|^2 dA_\alpha(\eta) \\
&= \int_{\mathbb{D}} |f(\eta)|^2 dA_\alpha(\eta) \\
&= \int_{\mathbb{D}} |f(\eta)|^2 |\varphi_a'(\eta)|^2 dA_\alpha(\eta) \\
&\leq \int_{\mathbb{D}} |f(\eta)|^2 \frac{(1-|a|^2)^2}{|1-\bar{a}\eta|^4} dA_\alpha(\eta) \\
&\leq \int_{\mathbb{D}} |f(\eta)|^2 \frac{(1-|a|)^2(1+|a|)^2}{(1-|a|)^4} dA_\alpha(\eta) \\
&= \int_{\mathbb{D}} |f(\eta)|^2 \frac{(1+|a|)^2}{(1-|a|)^2} dA_\alpha(\eta) \\
&= \left(\frac{1+|a|}{1-|a|}\right)^2 \int_{\mathbb{D}} |f(\eta)|^2 dA_\alpha(\eta) \\
&= \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^2 \int_{\mathbb{D}} |f(\eta)|^2 dA_\alpha(\eta).
\end{aligned}$$

By putting $\alpha = 0$, in the last relations, we get the following result

$$\int_{\mathbb{D}} |f(\varphi(\eta))|^2 d\mu(\eta) \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^2 \int_{\mathbb{D}} |f(\eta)|^2 dA(\eta). \tag{3.7}$$

Therefore, μ is a Carleson measure for $A^2(\mathbb{D})$. This implies that there exists $C > 0$ such that

$$\int_{\mathbb{D}} |f(\varphi(\eta))|^{p_0} d\mu(\eta) \leq C \int_{\mathbb{D}} |f(\eta)|^{p_0} dA(\eta).$$

Note that according to (3.1) and (3.2) we have the following relation

$$\int_{\mathbb{D}} |f(\varphi(\eta))|^{p_0} \omega(\eta) dA(\eta) \leq CC_0 \int_{\mathbb{D}} |f(\eta)|^{p_0} dA(\eta).$$

Now, by using the inequality $\omega(\mathbb{D}) \leq [\omega]_{A_1} \omega(z)$ we can write

$$\int_{\mathbb{D}} |f(\varphi(\eta))|^{p_0} \omega(\eta) dA(\eta) \leq C_1 \int_{\mathbb{D}} |f(\eta)|^{p_0} \omega(\eta) dA(\eta).$$

This inequality implies that the two functions $F(\eta) = |f(\varphi(\eta))|$ and $G(\eta) = |f(\eta)|$ satisfy the assumptions of the Rubio de Francia extrapolation theorem. On the other hand, $p \in \mathcal{P}^{\log}(\mathbb{D})$ therefore, $\frac{p}{p_0} \in \mathcal{P}^{\log}(\mathbb{D})$. Therefore, the result follows. $\square$

Remark 3.4. We close this section by commenting on how to prove a similar statement for the variable exponent Hardy space $H^{p(\cdot)}(\mathbb{D})$. First of all we need to use an estimate for the norm of the evaluation functional $\gamma_z(f) = f(z)$. It is known that

$$\|\gamma_z\| \leq C\|k_z\| \leq \frac{C}{(1-|z|)^{1/p(\theta)}}, \quad z = re^{i\theta},$$

where C is a constant, and

$$k_z(w) = \frac{1}{1-\overline{w}z}$$

is the reproducing kernel. This is used to produce a function whose norm is 1. As in the previous theorem, we can find a necessary condition for the boundedness of C_φ (boundedness of an expression in terms of $p(\cdot)$ and φ). The rest of the proof is essentially the same as that of the preceding theorem. Recall that in this case, $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function and $p(\cdot) \in \mathcal{P}^{log}([0, 2\pi])$ is a 2π-periodic function.

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