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Characterization of 2-cocycles and 2-coboundaries on Direct Sum of Banach Algebras

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ABSTRACT

Let A and B be Banach algebras. In this paper, we investigate the structure of 2-cocycles and 2-coboundaries on $A \oplus B$, when A and B are unital. Actually, we provide a specific criterion for each 2-cocycle map and establish a connection between 2-cocycles and 2-coboundaries on $A \oplus B$ and 2-cocycles and 2-coboundaries on A and B. Finally, our results lead to a connection between $\mathcal{H}^2(A, A^*)$,$\mathcal{H}^2(B, B^*)$ and $\mathcal{H}^2(A \oplus B, A^* \oplus B^*)$.

1. Introduction

A derivation from a Banach algebra A to a Banach A-bimodule X is a bounded linear mapping $D : A \to X$ such that $D(a_1a_2) = a_1 \cdot D(a_2) + D(a_1) \cdot a_2$ $(a_1, a_2 \in A)$. For each $x \in X$ the mapping $\mathbf{ad}_x : a \mapsto a \cdot x - x \cdot a$ $(a \in A)$, is a derivation, called the inner derivation implemented by x. The

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space of all derivations is denoted by $\mathcal{Z}^1(A,X)$ and the space of all ineer derivations is denoted by $\mathcal{B}^1(A,X)$. The first Hochschild cohomology group $\mathcal{H}^1(A,X)$ is defined by the quotient group

$$\mathcal{H}^1(A,X) = \frac{\mathcal{Z}^1(A,X)}{\mathcal{B}^1(A,X)}.$$

A 2-cocycle map from a Banach algebra A to a Banach A-bimodule X is a bounded 2-linear mapping $\varphi : A \times A \to X$ such that

$$a_1 \cdot \varphi(a_2,a_3) - \varphi(a_1a_2,a_3) + \varphi(a_1,a_2a_3) - \varphi(a_1,a_2) \cdot a_3 = 0,$$

for all $a_1,a_2,a_3 \in A$. For each bounded linear map $\psi : A \to X$, the mapping $\mathbf{ad}_\psi : A \times A \to X$ defined with

$$\mathbf{ad}_\psi(a_1,a_2) = a_1 \cdot \psi(a_2) - \psi(a_1a_2) + \psi(a_1) \cdot a_2,$$

is a 2-cocycle map, called the 2-coboundary map. The space of all bounded 2-linear maps from A to X is denoted by $C^2(A,X)$, the space of all 2-cocycle maps is denoted by $\mathcal{Z}^2(A,X)$ and the space of all 2-coboundary maps is denoted by $\mathcal{B}^2(A,X)$. The second Hochschild cohomology group $\mathcal{H}^2(A,X)$ is defined by the quotient group

$$\mathcal{H}^2(A,X) = \frac{\mathcal{Z}^2(A,X)}{\mathcal{B}^2(A,X)}.$$

Let A and B be Banach algebras and M be a Banach A,B-module. Suppose that

$$\mathcal{T} = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} : a \in A, b \in B, m \in M \right\},$$

be the corresponding triangular Banach algebra. Forrest and Marcoux investigated and studied derivations on triangular Banach algebra of $\mathcal{T}$ in [2] and analyzed the first Hochschild cohomology group $\mathcal{H}^1(\mathcal{T},\mathcal{T})$. They in [3] characterize derivation $D : \mathcal{T} \to \mathcal{T}^*$ according to derivations $D_A : A \to A^*$, $D_B : B \to B^*$ and $\gamma_D \in M^*$, as follows

$$D\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} D_A(a) - m\gamma_D & \gamma_D a - b\gamma_D \\ & D_B(b) + \gamma_D m \end{bmatrix}.$$

Indeed, they showed that there exixts derivation $D_{AB} : \mathcal{T} \to \mathcal{T}^*$ defined by $D_{AB}\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) = \begin{bmatrix} D_A(a) & \\ & D_B(b) \end{bmatrix}$, such that for every $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$

$$\begin{aligned} D\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) &= \begin{bmatrix} D_A(a) - m\gamma_D & \gamma_D a - b\gamma_D \\ & D_B(b) + \gamma_D m \end{bmatrix} \\ &= D_{AB}\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) + \begin{bmatrix} -m\gamma_D & -b\gamma_D \\ & 0 \end{bmatrix} + \begin{bmatrix} 0 & \gamma_D a \\ & \gamma_D m \end{bmatrix} \\ &= D_{AB}\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right) - \mathrm{ad}_{\begin{bmatrix} 0 & \gamma_D \\ & 0 \end{bmatrix}}\left(\begin{bmatrix} a & m \\ & b \end{bmatrix}\right), \end{aligned}$$

so $D_{AB} - D = \mathrm{ad}_{\begin{bmatrix} 0 & \gamma_D \\ & 0 \end{bmatrix}} \in \mathcal{B}^1(\mathcal{T}, \mathcal{T}^*)$. Using this characterization, they proved

$$\mathcal{H}^1(\mathcal{T}, \mathcal{T}^*) \simeq \mathcal{H}^1(A, A^*) \oplus \mathcal{H}^1(B, B^*),$$

of course, if we restrict ourselves to the case of unitary A, B and M. In particular, in the case of $M = 0$, they proved that

$$\mathcal{H}^1(A \oplus B, A^* \oplus B^*) \simeq \mathcal{H}^1(A, A^*) \oplus \mathcal{H}^1(B, B^*).$$

In this paper, somehow we want to provide the same characterization for 2-cocycle maps in $\mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. Indeed, we show that if $\varphi = (\varphi_1, \varphi_2) \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$, then there exist $\varphi_A \in \mathcal{Z}^2(A, A^*)$, $\varphi_B \in \mathcal{Z}^2(B, B^*)$ and $\psi = (\psi_1, \psi_2) \in C^1(A \oplus B, A^* \oplus B^*)$ such that for every $\omega_1 = (a_1, b_1), \omega_2 = (a_2, b_2) \in A \oplus B$,

$$\begin{aligned} \varphi_1(\omega_1, \omega_2) &= \varphi_A((a_1, a_2)) + a_1\psi_1(\omega_2) - \psi_1(\omega_1\omega_2) + \psi_1(\omega_1)a_2 \\ \varphi_2(\omega_1, \omega_2) &= \varphi_B(b_1, b_2) + b_1\psi_2(\omega_2) + \psi_2(\omega_1\omega_2) + \psi_2(\omega_1)b_2, \end{aligned}$$

that leads to $\varphi - \varphi_{AB} = \mathbf{ad}_\psi \in \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$, where $\varphi_{AB} : (A \oplus B) \times (A \oplus B) \to A^* \oplus B^*$ is defined by

$$\varphi_{AB}\left((a_1, b_1), (a_2, b_2)\right) = \left(\varphi_A(a_1, a_2), \varphi_B(b_1, b_2)\right).$$

In the following, as an application of our results, we show that

$$\mathcal{H}^2(A \oplus B, A^* \oplus B^*) \simeq \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*).$$

2. Main Results

Let A and B be unital Banach algebras with units $\mathbf{1}_A$ and $\mathbf{1}_B$, respectively. ℓ^1-direct sum of Banach algebras $A \oplus B$ with usual multiplication and the norm $\|(a, b)\| = \|a\| + \|b\|$ is a Banach algebra. It is clear that $(A \oplus B)^* \simeq A^* \oplus B^*$. Let $(a, b) \in A \oplus B$ and $(\mu, \theta) \in A^* \oplus B^*$. Then the action of $A^* \oplus B^*$ upon $A \oplus B$ is given by $(\mu, \theta)((a, b)) = \mu(a) + \theta(b)$. Also, it is easy to check that module action $A \oplus B$ on $A^* \oplus B^*$ are as follows:

$$(a, b) \cdot (\mu, \theta) = (a \cdot \mu, b \cdot \theta) \quad \text{and} \quad (\mu, \theta) \cdot (a, b) = (\mu \cdot a, \theta \cdot b). \tag{2.1}$$

Throughout we will remove the dot (the sign " $\cdot$ ") for simplicity.

Remark 2.1. (i) Note that $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$ if and only if φ satisfies the 2-cocycle equation

$$\omega_1\varphi(\omega_2, \omega_3) - \varphi(\omega_1\omega_2, \omega_3) + \varphi(\omega_1, \omega_2\omega_3) - \varphi(\omega_1, \omega_2)\omega_3 = 0, \tag{2.2}$$

for all $\omega_1, \omega_2, \omega_3 \in A \oplus B$.

(ii) Let $\psi \in C^1(A \oplus B, A^* \oplus B^*)$ and $\varphi \in C^2(A \oplus B, A^* \oplus B^*)$. Assume that $\psi_i := \pi_i \circ \psi$ ($\varphi_i := \pi_i \circ \varphi$), for $i = 1, 2$, denote the coordinate functions associated to ψ (φ), when π_i be the projection msp on arrays. That is

$$\psi(\omega) = (\psi_1(\omega), \psi_2(\omega)) \quad \text{and} \quad \varphi(\omega_1, \omega_2) = \left(\varphi_1(\omega_1, \omega_2), \varphi_2(\omega_1, \omega_2)\right),$$

for all $\omega_1, \omega_2 \in A \oplus B$. We will often write $\psi = (\psi_1, \psi_2)$ and $\varphi = (\varphi_1, \varphi_2)$.

Lemma 2.2. *Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. Suppose that $\varphi_A : A \times A \to A^*$ and $\varphi_B : B \times B \to B^*$ defined by*

$$\varphi_A(a_1, a_2) := \varphi_1((a_1, 0), (a_2, 0)) \quad \text{and} \quad \varphi_B(b_1, b_2) := \varphi_2((0, b_1), (0, b_2)). \tag{2.3}$$

Then $\varphi_A \in \mathcal{Z}^2(A, A^*)$ and $\varphi_B \in \mathcal{Z}^2(B, B^*)$. Conversely, if φ_A and φ_B belong to $\mathcal{Z}^2(A, A^*)$ and $\mathcal{Z}^2(B, B^*)$ respectively, and also $\varphi_{AB} : (A \oplus B) \times (A \oplus B) \to A^* \oplus B^*$ is defined by

$$\varphi_{AB}\left((a_1, b_1), (a_2, b_2)\right) = \left(\varphi_A(a_1, a_2), \varphi_B(b_1, b_2)\right),$$

then $\varphi_{AB} \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. Furthermore, $\varphi_{AB} \in \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$ if and only if $\varphi_A \in \mathcal{B}^2(A, A^*)$ and $\varphi_B \in \mathcal{B}^2(B, B^*)$.

Proof. Left to the reader. □

Proposition 2.3. *Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. Then there exists $\psi \in C^1(A \oplus B, A^* \oplus B^*)$ such that for each $\omega_1 = (a_1, b_1)$ and $\omega_2 = (a_2, b_2)$ belong to $A \oplus B$, we have*

$$\varphi(\omega_1, \omega_2) = \begin{pmatrix} \varphi_A(a_1, a_2) + a_1\psi_1(\omega_2) & b_1\psi_2(\omega_2) - \psi_2(\omega_1\omega_2) \\ -\psi_1(\omega_1\omega_2) + \psi_1(\omega_1)a_2 , & +\psi_2(\omega_1)b_2 + \varphi_B(b_1, b_2) \end{pmatrix}, \tag{2.4}$$

Proof. First define

$$\mu : B \to A^* \quad \text{by} \quad \mu(b) := \varphi_1\left((0, b), (\mathbf{1}_A, 0)\right), \tag{2.5a}$$

$$\theta : A \to B^* \quad \text{by} \quad \theta(a) := \varphi_2\left((0, \mathbf{1}_B), (a, 0)\right). \tag{2.5b}$$

We continue the proof in four steps. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. In each step, we use the actions (2.1) and 2-cocycle equation (2.2) for φ with different elements of $\omega_1, \omega_2, \omega_3$ in $A \oplus B$.

step 1. For $\omega_1 = (0, \mathbf{1}_B), \omega_2 = (a_1, 0)$ and $\omega_3 = (a_2, 0)$, we have

$$(0, \mathbf{1}_B)\varphi\left((a_1, 0), (a_2, 0)\right) = -\varphi\left((0, \mathbf{1}_B), (a_1a_2, 0)\right) + \varphi\left((0, \mathbf{1}_B), (a_1, 0)\right)(a_2, 0),$$

which according to (2.3) and (2.5b), we obtain

$$\varphi\left((a_1, 0), (a_2, 0)\right) = \left(\varphi_A(a_1, a_2), -\theta(a_1a_2)\right). \tag{2.6}$$

step 2. For $\omega_1 = (a_1, 0), \omega_2 = (0, b_2)$ and $\omega_3 = (\mathbf{1}_A, 0)$, we have

$$\varphi((a_1, 0), (0, b_2))(\mathbf{1}_A, 0) = (a_1, 0)\varphi((0, b_2), (\mathbf{1}_A, 0)).$$

Also for $\omega_1 = (0, \mathbf{1}_B), \omega_2 = (a_1, 0)$ and $\omega_3 = (0, b_2)$, we have

$$(0, \mathbf{1}_B)\varphi((a_1, 0), (0, b_2)) = \varphi((0, \mathbf{1}_B), (a_1, 0))(0, b_2).$$

Using (2.5a), (2.5b) and the two previous equalities, we obtain

$$\varphi((a_1, 0), (0, b_2)) = (a_1\mu(b_2), \theta(a_1)b_2). \tag{2.7}$$

step 3. For $\omega_1 = (0, b_1), \omega_2 = (0, \mathbf{1}_B)$ and $\omega_3 = (a_2, 0)$, we have

$$\varphi((0, b_1), (a_2, 0)) = (0, b_1)\varphi((0, \mathbf{1}_B), (a_2, 0)) - \varphi((0, b_1), (0, \mathbf{1}_B))(a_2, 0),$$

which according to (2.5b) and the previous equality, we obtain

$$\varphi((0, b_1), (a_2, 0)) = (\mu(b_1)a_2, b_1\theta(a_2)). \tag{2.8}$$

step 4. For $\omega_1 = (0, b_1), \omega_2 = (0, b_2)$ and $\omega_3 = (\mathbf{1}_A, 0)$, we have

$$\varphi((0, b_1), (0, b_2))(\mathbf{1}_A, 0) = (0, b_1)\varphi((0, b_2), (\mathbf{1}_A, 0)) - \varphi((0, b_1b_2), (\mathbf{1}_A, 0)).$$

Using (2.3), (2.5a) and the previous equaliy, we obtain

$$\varphi((0, b_1), (0, b_2)) = (-\mu(b_1b_2), \varphi_B(b_1, b_2)). \tag{2.9}$$

Now, from the sum of the sides of the relationships (2.6) to (2.9), we get

$$\varphi(\omega_1, \omega_2) = \begin{pmatrix} \varphi_A(a_1, a_2) + a_1\mu(b_2) & & -\theta(a_1a_2) + \theta(a_1)b_2 \\ +\mu(b_1)a_2 - \mu(b_1b_2) & , & +b_1\theta(a_2) + \varphi_B(b_1, b_2) \end{pmatrix}.$$

Next, let us assume that $\psi = (\psi_1, \psi_2) : A \oplus B \longrightarrow A^* \oplus B^*$, to be defined by

$$\begin{aligned} \psi_1 : A \oplus B \to A^* \quad &\text{by} \quad \psi_1((a, b)) := \mu(b), \\ \psi_2 : A \oplus B \to B^* \quad &\text{by} \quad \psi_2((a, b)) := \theta(a). \end{aligned}$$

Proof that ψ is bounded and linear is entrusted to the reader. Therefore, $\psi \in C^1(A \oplus B, A^* \oplus B^*)$ and also

$$\varphi(\omega_1, \omega_2) = \begin{pmatrix} \varphi_A(a_1, a_2) + a_1\psi_1(\omega_2) & & b_1\psi_2(\omega_2) - \psi_2(\omega_1\omega_2) \\ -\psi_1(\omega_1\omega_2) + \psi_1(\omega_1)a_2 & , & +\psi_2(\omega_1)b_2 + \varphi_B(b_1, b_2) \end{pmatrix}.$$

So (2.4) is valid and the proof is complete. □

So far, we have established a connection between the 2-cocycles on A, B and $A \oplus B$. We want to extend this connection to the 2-coboundaries and for this, we need the following Lemma.

Lemma 2.4. *Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. Then $\varphi \in \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$ if and only if $\varphi_A \in \mathcal{B}^2(A, A^*)$ and $\varphi_B \in \mathcal{B}^2(B, B^*)$.*

Proof. Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. By Proposition 2.3, there exists $\psi \in C^1(A \oplus B, A^* \oplus B^*)$, such that (2.4) is valid. For each $\omega_1 = (a_1, b_1), \omega_2 = (a_2, b_2)$ in $A \oplus B$, we have

$$\begin{aligned} (\varphi - \varphi_{AB})(\omega_1, \omega_2) &= \begin{pmatrix} a_1\psi_1(\omega_2) - \psi_1(\omega_1\omega_2) & & b_1\psi_2(\omega_2) - \psi_2(\omega_1\omega_2) \\ +\psi_1(\omega_1)a_2 & , & +\psi_2(\omega_1)b_2 \end{pmatrix} \\ &= (a_1\psi_1(\omega_2), b_1\psi_2(\omega_2)) - (\psi_1(\omega_1\omega_2), \psi_2(\omega_1\omega_2)) \\ &\quad + (\psi_1(\omega_1)a_2, \psi_2(\omega_1)b_2) \\ &= (a_1, b_1)(\psi_1(\omega_2), \psi_2(\omega_2)) - (\psi_1(\omega_1\omega_2), \psi_2(\omega_1\omega_2)) \\ &\quad + (\psi_1(\omega_1), \psi_2(\omega_1))(a_2, b_2) \\ &= \omega_1\psi(\omega_2) - \psi(\omega_1\omega_2) + \psi(\omega_1)\omega_2 \\ &= \mathbf{ad}_\psi(\omega_1, \omega_2). \end{aligned}$$

Hence $\varphi - \varphi_{AB} \in \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$. That this means, φ is 2-coboundary, if and only if φ_{AB} is 2-coboundary, and by Lemma 2.2, if and only if φ_A and φ_B are 2-coboundaries. □

Forrest and Marcoux in [3] have studied the first Hochschild cohomology group of triangular Banach algebra $\mathcal{H}^1(A \oplus B, A^* \oplus B^*)$. They showed that

$$\mathcal{H}^1(A \oplus B, A^* \oplus B^*) \simeq \mathcal{H}^1(A, A^*) \oplus \mathcal{H}^1(B, B^*).$$

In this section, we wish to identify the second Hochschild cohomology group $\mathcal{H}^2(A \oplus B, A^* \oplus B^*)$. We get similar results with the results of Forrest and Marcoux in [3], but in the second order.

Corollary 2.5. *Let A and B be unital Banach algebras. Then*

$$\mathcal{H}^2(A \oplus B, A^* \oplus B^*) \simeq \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*).$$

Proof. Let $\varphi \in \mathcal{Z}^2(A \oplus B, A^* \oplus B^*)$. By Lemma 2.2 and Proposition 2.3, there exists $\varphi_A \in \mathcal{Z}^2(A, A^*)$, $\varphi_B \in \mathcal{Z}^2(B, B^*)$ and $\psi \in C^1(A \oplus B, A^* \oplus B^*)$ such that (2.4) is valid.

Consider the map

$$\begin{aligned} \Gamma : \mathcal{Z}^2(A \oplus B, A^* \oplus B^*) &\to \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*) \\ \varphi \quad &\mapsto (\varphi_A + \mathcal{B}^2(A, A^*),\ \varphi_B + \mathcal{B}^2(B, B^*)). \end{aligned}$$

Using Lemma 2.2 and Proposition 2.3, it can be easily proved that Γ is well-define and surjective. Now we have

$$\frac{\mathcal{Z}^2(A \oplus B, A^* \oplus B^*)}{\ker \Gamma} \simeq \operatorname{Im} \Gamma = \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*).$$

On the other hand, by Lemma 2.2,

$$\varphi \in \ker \Gamma \Leftrightarrow \varphi_A \in \mathcal{B}^2(A, A^*) \text{ and } \varphi_B \in \mathcal{B}^2(B, B^*) \Leftrightarrow \varphi \in \mathcal{B}^2(A \oplus B, A^* \oplus B^*).$$

That is, $\ker \Gamma = \mathcal{B}^2(A \oplus B, A^* \oplus B^*)$. Thus

$$\mathcal{H}^2(A \oplus B, A^* \oplus B^*) = \frac{\mathcal{Z}^2(A \oplus B, A^* \oplus B^*)}{\mathcal{B}^2(A \oplus B, A^* \oplus B^*)} \simeq \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*).$$

This completes the proof. □

We end this section with the final theorem and two rather simple examples. Recall that the action of $A \oplus B$ on $A^{(3)} \oplus B^{(3)}$ is a restriction of the action of $A^{(2)} \oplus B^{(2)}$ on $A^{(3)} \oplus B^{(3)}$. The same way the action of $A^{(2n-4)} \oplus B^{(2n-4)}$ on $A^{(2n-1)} \oplus B^{(2n-1)}$ is a restriction of the action of $A^{(2n-2)} \oplus B^{(2n-2)}$ on $A^{(2n-1)} \oplus B^{(2n-1)}$ for any $n \geq 2$. From this we can again the behavior of a 2-cocycle map $\varphi : A \oplus B \to A^{(2n-1)} \oplus B^{(2n-1)}$. Suppose n is a positive integer. By repeating exactly the same calculations as Proposition 2.3, we find that there exist 2-cocycle maps $\varphi_A : A \to A^{(2n-1)}$ and $\varphi_B : B \to B^{(2n-1)}$ as well as $\psi \in C^1(A \oplus B, A^{(2n-1)} \oplus B^{(2n-1)})$ such that (2.4) is valid. We therefore have:

Theorem 2.6. *Let A and B be unital Banach algebras. Suppose $A \oplus B$ is ℓ^1-direct sum of A and B. Then*

$$\mathcal{H}^2(A \oplus B, A^{(2n-1)} \oplus B^{(2n-1)}) \simeq \mathcal{H}^2(A, A^{(2n-1)}) \oplus \mathcal{H}^2(B, B^{(2n-1)}).$$

3. Examples

Now we recall that semigroup S is an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. The element $s \in S$ is called central and idempotent if $st = ts$ for each $t \in S$ and $s = s^* = s^2$, respectively. The inverse semigroup S is called semilattice if each element of S is central and idempotent. Also the inverse semigroup S is called Clifford semigroup if each idempotent element of S is central. Subsequently, it is assumed that S is an unital semigroup. In this case, semigroup algebra $\ell^1(S)$ is unital Banach algebras with the actions

$$\delta_s \cdot \delta_t = \delta_t \cdot \delta_s = \delta_s * \delta_t = \delta_{st} \qquad (s, t \in S).$$

Example 3.1. Consider $A = B = \ell^1(S)$, where S is semigroup. Using Corollary 2.5, [1, Theorem 3.2], [1, Theorem 3.3] and [4, Theorem 4.6], we have

(i) $\mathcal{H}^2(\ell^1(S) \oplus \ell^1(S), \ell^\infty(S) \oplus \ell^\infty(S)) = 0$, for $S = \mathbb{Z}_+$ or unital semilattice S.

(ii) $\mathcal{H}^2(\ell^1(S) \oplus \ell^1(S), \ell^\infty(S) \oplus \ell^\infty(S))$ is Banach space for Clifford semigroups.

Using Theorem 2.6 and Theorems 2.3 and 3.3 of [5], we have the following examples.

Example 3.2. $\mathcal{H}^2(\ell^1(G) \oplus \ell^1(G), \ell^1(G)^{(2n-1)} \oplus \ell^1(G)^{(2n-1)})$ is Banach space, for every discrete group G and any $n \in \mathbb{N}$.

Example 3.3. $\mathcal{H}^2(L^1(G) \oplus L^1(G), L^1(G)^{(2n-1)} \oplus L^1(G)^{(2n-1)})$ is Banach space, for every locally campact group G and any $n \in \mathbb{N}$.

We end this paper with a question:

Question. Let A and B be Banach algebras and M be a Banach A, B-module. Suppose that

$$\mathcal{T} = \mathrm{Tri}(A, B, M) = \left\{ \begin{bmatrix} a & m \\ & b \end{bmatrix} : a \in A, b \in B, m \in M \right\},$$

be the corresponding triangular Banach algebra. In this paper, in the case of $M = 0$, we proved that $\mathcal{H}^2(\mathcal{T}, \mathcal{T}^*) \simeq \mathcal{H}^2(A, A^*) \oplus \mathcal{H}^2(B, B^*)$, where $\mathcal{T} = A \oplus B$. Now the question is, whether there is a general rule as well. In fact, is the Lemma 2.4 also true for the general case $M \neq 0$? Indeed, is $\varphi - \varphi_{AB} \in \mathcal{B}^2(\mathcal{T}, \mathcal{T}^*)$, for every $\varphi \in \mathcal{Z}^2(\mathcal{T}, \mathcal{T}^*)$?

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