Property (T) for C*-dynamical systems

H. Abbasi\textsuperscript{a,}\textsuperscript{*}, Gh. Haghighatdoost\textsuperscript{a}, I. Sadeqi\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Islamic Republic of Iran.
\textsuperscript{b}Faculty of Sciences, Sahand University of Technology, Tabriz, Islamic Republic of Iran.

Abstract

In this paper, we introduce a notion of property (T) for a C*-dynamical system (\(A, \mathcal{G}, \alpha\)) consisting of a unital C*-algebra \(A\), a locally compact group \(\mathcal{G}\), and an action \(\alpha\) on \(A\). As a result, we show that if \(A\) has strong property (T) and \(\mathcal{G}\) has Kazhdan’s property (T), then the triple (\(A, \mathcal{G}, \alpha\)) has property (T).

\textsuperscript{c}© (2014) Wavelets and Linear Algebra

1. Introduction

A unital C*-algebra \(A\) has property (T) if there exist a finite subset \(F\) of \(A\) and \(\varepsilon > 0\), such that for every Hilbert bimodule on \(A\) with a unit \((F, \varepsilon)\)-central vector, there is a non-zero central
vector (see [1]). This property is similar to the property (T) for locally compact groups, which is defined by D. Kazhdan in [7]. A locally compact group $G$ has property (T) if, whenever a unitary representation $(\pi, \mathcal{H})$ of $G$ almost has invariant vectors, $\mathcal{H}$ has a non-zero invariant vector. It is proved in [1] that a countable discrete group $G$ has property (T) if and only if its full (or equivalently reduced) group $C^*$-algebra has property (T). In [3], property (T) for a von Neumann algebra was introduced, it is shown that a discrete ICC-group $G$ has property (T) if and only if the von Neumann algebra generated by the left regular representation of $G$ has property (T).

In this paper, if $V$ and $W$ are Hilbert spaces, $V \otimes W$ denotes their Hilbert space tensor product. If $V$ and $W$ are algebras, $V \otimes W$ denotes their algebraic tensor product. If $V$ and $W$ are $C^*$-algebras, then $V \otimes_{\text{min}} W$ will denote their $C^*$-tensor product with respect to the minimal (spatial) $C^*$-norm and $V \otimes_{\text{max}} W$ will denote their $C^*$-tensor product with respect to the maximal $C^*$-norm. Also, if $V$ is a Hilbert space we denote by $\mathcal{L}(V)$ the unital $C^*$-algebra of bounded linear operators on $V$.

The paper is organised as follows. In Section 2, we recall some definitions and results in the framework of $C^*$-dynamical systems which are used in this paper.

In Section 3, we define a notion of property (T) for an arbitrary $C^*$-dynamical system $(\mathcal{A}, G, \alpha)$. We show that if $\mathcal{A}$ has strong property (T) and $G$ has property (T), then $(\mathcal{A}, G, \alpha)$ has property (T). We will also show that if $G$ is a discrete group and $(\mathcal{A}, G, \alpha)$ has property (T), then its $C^*$-crossed product has property (T) as a unital $C^*$-algebra. Furthermore, we show that if $\mathcal{A}$ is a commutative unital $C^*$-algebra, $G$ is a countable discrete group such that there exists a faithful representation of $\mathcal{A}$ to the Hilbert space $\ell^2(G)$, then property (T) of $C_r^*(G) \otimes_{\text{min}} \mathcal{A}$ implies property (T) of $G$, where $C_r^*(G)$ is the reduced group $C^*$-algebra of $G$.

Our basic references for $C^*$-algebras are [5, 8, 9]. A good reference for $C^*$-dynamical systems is [10]. For a survey on Kazhdan’s property (T) one can refer to [2].

2. Preliminaries and Basic Concepts

A $C^*$-dynamical system (or a dynamical system) is a triple $(\mathcal{A}, G, \alpha)$, where $\mathcal{A}$ is a unital $C^*$-algebra, $G$ is a locally compact group, and $\alpha$ is a continuous homomorphism from $G$ into the group of all $*$-automorphisms of $\mathcal{A}$. Note that the continuity condition on $\alpha$ amounts to the statement that $g \mapsto \alpha_g(a)$ is continuous for all $a \in \mathcal{A}$.

Let $(\mathcal{A}, G, \alpha)$ be a dynamical system such that $G$ is a discrete group. Let $\mathcal{K}(G, \mathcal{A})$ be the algebra of all $\mathcal{A}$-valued functions with finite support endowed with the following twisted convolution as product, involution and norm:

$$xy(t) = \sum_\gamma x(\gamma) \alpha_\gamma(y(\gamma^{-1}t)), \quad x^*(t) = \alpha_t(x(t^{-1})), \quad ||x||_1 = \sum_\gamma ||x(\gamma)||,$$

where $x, y \in \mathcal{K}(G, \mathcal{A})$ and $t \in G$. The algebra $\mathcal{K}(G, \mathcal{A})$ becomes a normed $*$-algebra and we denote its completion by $\ell^1(G, \mathcal{A})$. The algebra $\mathcal{A}$ is regarded as a subalgebra of $\mathcal{K}(G, \mathcal{A})$ with the same unit element in which each arbitrary element $a \in \mathcal{A}$ can be thought as a function on $G$ subject to the conditions $a(e) = a$ and $a(\gamma) = 0$ for $\gamma \neq e$, where $e$ is the unit of $G$.

The unital Banach $*$-algebra $\ell^1(G, \mathcal{A})$ has a faithful representation and we call the $C^*$-envelope of $\ell^1(G, \mathcal{A})$ the $C^*$-crossed product of $\mathcal{A}$ by $G$ with respect to the action $\alpha$ and write as $\mathcal{A} \rtimes_\alpha G$. Let $\delta_\gamma$ be the unitary element of $\ell^1(G, \mathcal{A})$ such that $\delta_\gamma(\gamma) = 1$ and $\delta_\gamma(t) = 0$ if $t \neq \gamma$. The element
\(\delta_\gamma\) belongs to \(\mathcal{A} \times_{\alpha} \mathcal{G}\) and satisfies \(\delta_\gamma a \delta_\gamma = \alpha_\gamma(a)\). An element \(x\) in \(\mathcal{K}(\mathcal{G}, \mathcal{A})\) can be written as \(x = \sum_\gamma x(\gamma)\delta_\gamma\).

A pair \((\mu, \pi)\) consisting of a representation \(\mu\) of \(\mathcal{A}\) and a unitary representation \(\pi\) of \(\mathcal{G}\) on the same Hilbert space \(\mathcal{H}\) is called a covariant representation of \((\mathcal{A}, \mathcal{G}, \alpha)\) if for all \(a \in \mathcal{A}\) and \(\gamma \in \mathcal{G}\) we have

\[
\pi(\gamma)\mu(a) = \mu(\alpha_\gamma(a))\pi(\gamma).
\]

Consider two covariant representations \((\mu_1, \pi_1)\) and \((\mu_2, \pi_2)\) on the Hilbert spaces \(\mathcal{H}_1, \mathcal{H}_2\), respectively. We say that \((\mu_1, \pi_1)\) and \((\mu_2, \pi_2)\) are equivalent if there exists a unitary operator \(W : \mathcal{H}_1 \to \mathcal{H}_2\) such that

\[
W\mu_1(a) = \mu_2(a)W, \quad W\pi_1(\gamma) = \pi_2(\gamma)W,
\]

for all \(a \in \mathcal{A}\) and \(\gamma \in \mathcal{G}\).

Consider a faithful representation of \(\mathcal{A}\) on a Hilbert space \(\mathcal{H}\). Define a representation of \(\mathcal{A}\) as well as a unitary representation of \(\mathcal{G}\) on the Hilbert space \(\ell^2(\mathcal{G}, \mathcal{H})\) by

\[
\pi_a(\alpha)\tilde{\xi}(\gamma) = \alpha_{\gamma^{-1}}(a) \cdot \tilde{\xi}(\gamma), \quad \lambda_\gamma(\tilde{\xi})(t) = \tilde{\xi}(\gamma^{-1}t),
\]

where \(a \in \mathcal{A}, \tilde{\xi} \in \ell^2(\mathcal{G}, \mathcal{H})\) and \(\gamma, t \in \mathcal{G}\). We say that \((\pi_a, \lambda_\gamma)\) is a regular representation of \((\mathcal{A}, \mathcal{G}, \alpha)\).

The reduced \(C^*\)-crossed product \(\mathcal{A} \times_\alpha \mathcal{G}\) is the \(C^*\)-algebra on \(\ell^2(\mathcal{G}, \mathcal{H})\) generated by the family of \(\{\pi_a(\alpha), \lambda_\gamma(\alpha) \mid a \in \mathcal{A}, \gamma \in \mathcal{G}\}\). Note that this definition is independent of the choice of the space \(\mathcal{H}\).

If \(\mathcal{A} = \mathbb{C}\) and \(\alpha\) is trivial, then \(\ell^1(\mathcal{G}, \mathcal{A})\) coincide with \(\ell^1(\mathcal{G})\) and \(\lambda_\gamma\) is the regular representation on the Hilbert space \(\ell^2(\mathcal{G})\). In this case, \(\mathcal{A} \times_\alpha \mathcal{G}\) is the group \(C^*\)-algebra \(C^*(\mathcal{G})\) and \(\mathcal{A} \times_{ar} \mathcal{G}\) is the reduced group \(C^*\)-algebra \(C^r(\mathcal{G})\).

### 3. Property (T) for a dynamical system

A Hilbert bimodule on a unital \(C^*\)-algebra \(\mathcal{A}\) (or a Hilbert \(\mathcal{A}\)-bimodule) is a Hilbert space \(\mathcal{H}\) carrying two commuting actions, one from \(\mathcal{A}\) and one from the opposite algebra \(\mathcal{A}^0\) (see [1]). In other words, there exists a representation from \(\mathcal{A} \otimes_{\max} \mathcal{A}^0\) to \(L(\mathcal{H})\). If \(\mathcal{H}\) is a Hilbert \(\mathcal{A}\)-bimodule, we will write \(a \cdot \xi \cdot b\) for all \(a, b \in \mathcal{A}\) and \(\xi \in \mathcal{H}\), to denote the module actions.

A tracial state on a unital \(C^*\)-algebra \(\mathcal{A}\) is a positive linear functional \(Tr : \mathcal{A} \to \mathbb{C}\) such that \(Tr(ab) = Tr(ba)\) for all \(a, b \in \mathcal{A}\) and \(Tr(1) = 1\).

**Definition 3.1.** (see [1]) Let \(\mathcal{B} \subset \mathcal{A}\) be a \(C^*\)-subalgebra containing the identity of a unital \(C^*\)-algebra \(\mathcal{A}\). The pair \((\mathcal{A}, \mathcal{B})\) has property \((T)\) if there exist a finite subset \(\mathcal{F}\) of \(\mathcal{A}\) and \(\varepsilon > 0\) such that the following property holds: if a Hilbert bimodule \(\mathcal{H}\) on \(\mathcal{A}\) contains a unit vector \(\xi \in \mathcal{H}\) which is \((\mathcal{F}, \varepsilon)\)-central, that is:

\[
\max_{a \in \mathcal{F}} \|a \cdot \xi - \xi \cdot a\| < \varepsilon,
\]

then \(\mathcal{H}\) has a non-zero \(\mathcal{B}\)-central vector, that is, a non-zero vector \(\eta \in \mathcal{H}\) such that

\[
b \cdot \eta = \eta \cdot b,
\]

for all \(b \in \mathcal{B}\). Moreover, \(\mathcal{A}\) has property \((T)\) if the pair \((\mathcal{A}, \mathcal{A})\) has such property.
It is clear that if \( A \) has property (T), then the pair \((A, B)\) has, too. As an example, if \( H \) is any Hilbert space and \( B \subset L(H) \) a unital \( C^* \)-subalgebra, then \((L(H), B)\) has property (T) (see [4]).

Note that Definition 3.1 comes from the original definition of property (T) for groups. Let \( G \) be a locally compact group and \( N \) a closed subgroup. The pair \((G, N)\) has property (T) if there exist a compact subset \( Q \) of \( G \) and \( r > 0 \) such that the following property holds: if a unitary representation \((\pi, H)\) of \( G \) contains a unit vector \( \xi \in H \) which is \((Q,r)\)-invariant, that is:

\[
\sup_{\gamma \in Q} ||\pi(\gamma)(\xi) - \xi|| < r,
\]

then \( H \) has a non-zero \( N \)-invariant vector, that is, there is a non-zero vector \( \eta \in H \) such that

\[
\pi(\gamma)(\eta) = \eta,
\]

for all \( \gamma \in N \). Moreover, \( G \) has property (T) if the pair \((G, G)\) has property (T). An example of a pair with property (T) is the pair \((S L_2(\mathbb{Z}) \rtimes \mathbb{Z}^2, \mathbb{Z}^2)\), where \( S L_2(\mathbb{Z}) \rtimes \mathbb{Z}^2 \) is the semi-direct product for the natural action of \( S L_2(\mathbb{Z}) \) on \( \mathbb{Z}^2 \).

In the following, we give definition of a covariant birepresentation on a dynamical system and apply it to study the property (T) on the dynamical systems.

Let \((\mathcal{A}, G, \alpha)\) be a dynamical system. A triple \((H, \pi_1, \pi_2)\) consisting of a Hilbert bimodule \( H \) on \( A \) and two commuting unitary representations \( \pi_1, \pi_2 \) of \( G \) on the same Hilbert space \( H \) is called a covariant birepresentation of \((\mathcal{A}, G, \alpha)\) if we have

\[
\pi_1(\gamma_1)\pi_2(\gamma_2)(a \cdot \xi \cdot b) = \alpha_{\gamma_1}(a) \cdot \pi_1(\gamma_1)\pi_2(\gamma_2)(\xi) \cdot \alpha_{\gamma_2}(b),
\]

for all \( a, b \in \mathcal{A}, \gamma_1, \gamma_2 \in G \) and \( \xi \in H \).

Obviously, covariant birepresentations of the dynamical system \((\mathcal{A}, \{e\}, id)\) are in one-to-one correspondence with Hilbert bimodules on \( \mathcal{A} \), where \( \{e\} \) is the trivial group with one element. Covariant birepresentations of the dynamical system \((\mathbb{C}, G, id)\) correspond to commuting unitary representations of \( G \). Note that if \((H, \pi_1, \pi_2)\) is a covariant birepresentation of \((\mathcal{A}, G, \alpha)\), then so is the triple \((H, \pi_2, \pi_1)\).

Let \((\mathcal{A}, G, \alpha)\) be a dynamical system and \( B \subset \mathcal{A} \) an \( \alpha \)-invariant \( C^* \)-subalgebra containing the identity element of \( \mathcal{A} \). Let \((H, \pi_1, \pi_2)\) be a covariant birepresentation of \((\mathcal{A}, G, \alpha)\). We say that \((H, \pi_1, \pi_2)\) has a non-zero \((B, G)\)-central vector if there exists a non-zero vector \( \eta \) in \( H \) such that

\[
b \cdot \eta = \eta \cdot b, \quad \pi_1(\gamma)\pi_2(\gamma)(\eta) = \eta,
\]

for all \( \gamma \in G \) and \( b \in B \).

If \((\mathcal{A}, G, \alpha)\) is a dynamical system and \( G \) is a discrete group, then covariant birepresentations with non-zero central vectors are in one-to-one correspondence with \( \alpha \)-invariant tracial states of the associated \( C^* \)-algebra.

**Lemma 3.2.** (i) Let \((H, \pi_1, \pi_2)\) be a covariant birepresentation of a dynamical system \((\mathcal{A}, G, \alpha)\) with a non-zero \((\mathcal{A}, G)\)-central vector \( \eta \). Then \( \mathcal{A} \) admits an \( \alpha \)-invariant tracial state.
(ii) Let \((\mathcal{A}, \mathcal{G}, \alpha)\) be a dynamical system such that \(\mathcal{G}\) is a discrete group. Let \(Tr : \mathcal{A} \rightarrow \mathbb{C}\) be an \(\alpha\)-invariant tracial state on \(\mathcal{A}\). Then there exists a covariant birepresentation of \((\mathcal{A}, \mathcal{G}, \alpha)\) with a non-zero \((\mathcal{A}, \mathcal{G})\)-central vector.

Proof. (i) Let \(\zeta = \frac{\eta}{\|\eta\|}\). Define \(Tr : \mathcal{A} \rightarrow \mathbb{C}\) by \(Tr(a) = \langle a \cdot \zeta, \zeta \rangle\). Then \(Tr\) is a tracial state on \(\mathcal{A}\), and for all \(a \in \mathcal{A}, \gamma \in \mathcal{G}\) we have
\[
Tr(\alpha_r(a)) = \langle \pi_1(\gamma)\pi_2(\gamma)(a \cdot \zeta), \zeta \rangle = \langle a \cdot \zeta, \pi_2(\gamma^{-1})(\pi_1(\gamma^{-1})(\zeta)) \rangle = Tr(a).
\]

(ii) First, consider the extension of \(\alpha\)-invariant tracial state on \(\mathcal{A} \times_{\alpha} \mathcal{G}\), again denoted by \(Tr\). Setting \(N = \{x \in \mathcal{A} \times_{\alpha} \mathcal{G} \mid Tr(x \cdot x) = 0\}\), it is easy to check that \(N\) is a two-sided ideal of \(\mathcal{A} \times_{\alpha} \mathcal{G}\) and that the map \(\langle x + N, y + N \rangle = Tr(x \cdot y)\) is a well-defined inner product on the quotient space \(\mathcal{A} \times_{\alpha} \mathcal{G}/N\). We denote by \(L^2(Tr)\) the Hilbert space completion of \(\mathcal{A} \times_{\alpha} \mathcal{G}/N\). For each \(a \in \mathcal{A}\), the mappings \(x + N \mapsto ax + N\) and \(x + N \mapsto xa + N\) can be extend to bounded operators on \(L^2(Tr)\), and \((L^2(Tr), \pi_1, \pi_2)\) is a Hilbert bimodule on \(\mathcal{A}\). Also, if \(\gamma \in \mathcal{G}\), define two operators \(\pi_1(\gamma), \pi_2(\gamma) \in \mathcal{L}(L^2(Tr))\) by
\[
\pi_1(\gamma)(x + N) = \delta_{\gamma}x + N, \quad \pi_2(\gamma)(x + N) = x\delta_{\gamma^{-1}} + N.
\]

We obtain two commuting unitary representations \(\pi_1, \pi_2\) of \(\mathcal{G}\) on \(L^2(Tr)\), and \((L^2(Tr), \pi_1, \pi_2)\) is a covariant birepresentation of \((\mathcal{A}, \mathcal{G}, \alpha)\). Moreover, \(\eta = \delta_e + N\) is a non-zero \((\mathcal{A}, \mathcal{G})\)-central vector.

\(\square\)

Let \((\mathcal{H}, \pi_1, \pi_2)\) be a covariant birepresentation of \((\mathcal{A}, \mathcal{G}, \alpha)\). Given a finite subset \(\mathcal{F}\) of \(\mathcal{A}\), a compact subset \(\mathcal{Q}\) of \(\mathcal{G}\) and \(\varepsilon, r > 0\), we say that a unit vector \(\xi \in \mathcal{H}\) is \((\mathcal{F}, \varepsilon, \mathcal{Q}, r)\)-central if:
\[
\max_{a \in \mathcal{F}} \|a \cdot \xi - \xi \cdot a\| < \varepsilon, \quad \sup_{\gamma \in \mathcal{Q}} \|\pi_1(\gamma)\pi_2(\gamma)(\xi) - \xi\| < r.
\]

The covariant birepresentation \((\mathcal{H}, \pi_1, \pi_2)\) almost has invariant vectors if it has \((\mathcal{F}, \varepsilon, \mathcal{Q}, r)\)-central vectors for every finite subset \(\mathcal{F}\) of \(\mathcal{A}\), compact subset \(\mathcal{Q}\) of \(\mathcal{G}\) and every \(\varepsilon, r > 0\).

**Definition 3.3.** Let \((\mathcal{A}, \mathcal{G}, \alpha)\) be a dynamical system, and \(\mathcal{B} \subset \mathcal{A}\) an \(\alpha\)-invariant \(C^*\)-subalgebra containing the identity element of \(\mathcal{A}\). We denote the dynamical system \((\mathcal{A}, \mathcal{G}, \alpha)\) with the \(\alpha\)-invariant \(C^*\)-subalgebra \(\mathcal{B}\), by \(((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)\). We say that \(((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)\) has property (T) if there exist a finite subset \(\mathcal{F}\) of \(\mathcal{A}\), a compact subset \(\mathcal{Q}\) of \(\mathcal{G}\) and \(\varepsilon, r > 0\) such that any covariant birepresentation of \((\mathcal{A}, \mathcal{G}, \alpha)\) with a unit \((\mathcal{F}, \varepsilon, \mathcal{Q}, r)\)-central possesses non-zero \((\mathcal{B}, \mathcal{G})\)-central vectors. Moreover, the dynamical system \((\mathcal{A}, \mathcal{G}, \alpha)\) has property (T) if the system \(((\mathcal{A}, \mathcal{A}), \mathcal{G}, \alpha)\) has such property.

It is clear that if \((\mathcal{A}, \mathcal{G}, \alpha)\) has property (T), then so has \(((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)\). Property (T) of the dynamical system \(((\mathcal{A}, \{e\}, id)\) correspond to property (T) of \(\mathcal{A}\), and property (T) of the dynamical system \(((\mathbb{C}, \mathcal{G}, id)\) correspond to property (T) of \(\mathcal{G}\).
Example 3.5. Let \( q \in M \) represent \( \pi \)-representations of \((A, G, \alpha)\). More precisely, given a Hilbert \( \chi \)-module \( M \), let \( \theta \) be a \( \chi \)-invariant character, that is, a non-zero multiplicative linear map \( \chi : C^*\rightarrow \mathbb{C} \) such that \( \chi(\alpha(a)) = \chi(a) \) for all \( \gamma \in G \) and \( a \in A \). If \((A, G, \alpha)\) has property (T), then \( G \) has property (T).

Let us give an example of a dynamical system which does not have property (T).

Example 3.5. Let \( X \) be a smooth vector field on a compact manifold \( M \). Suppose for each point \( q \in M \) there is a unique integral curve \( \theta^q : \mathbb{R} \rightarrow M \) of \( X \) starting at \( q \), and \( p \) be an element in \( M \) such that \( \theta^p \) is the constant curve \( \theta^q(t) \equiv p \). For each \( t \in \mathbb{R} \), we can define a map \( \theta_t \) from \( M \) to itself by sending each point \( q \in M \) to the point obtained by the curve starting at \( q \) for time \( t \):

\[
\theta_t(q) = \theta^q(t).
\]

This defines a family of maps \( \theta_t : M \rightarrow M \) for \( t \in \mathbb{R} \). Let \( C(M) \) denote the unital \( C^*\)-algebra of continuous complex valued functions on \( M \). We obtain a homomorphism \( \alpha : \mathbb{R} \rightarrow Aut(C(M)) \), defined by

\[
\alpha_t(f)(q) = f(\theta_t^{-1}(q)),
\]

and \((C(M), \mathbb{R}, \alpha)\) is a dynamical system. Define an \( \alpha \)-invariant character \( \chi : C(M) \rightarrow \mathbb{C} \) by \( \chi(f) = f(p) \). We know that \( \mathbb{R} \) does not have property (T) (see [2]), it follows from Remark 3.4 that \((C(M), \mathbb{R}, \alpha)\) does not have property (T).

The notion of property (T) for a dynamical system \((A, G, \alpha)\) and for \( C^*\)-algebras associated to it are related via the correspondence between covariant birepresentations and Hilbert bimodules. More precisely, given a Hilbert \( A \times_\alpha G \)-bimodule \( H \), one can define two commuting unitary representations \( \pi_1, \pi_2 \) of \( G \) on the same Hilbert space \( H \) by

\[
\pi_1(\gamma)(\xi) = \delta_\gamma \cdot \xi, \quad \pi_2(\gamma)(\xi) = \xi \cdot \delta_{\gamma^{-1}}.
\]

Viewing \( A \) as a subalgebra of \( A \times_\alpha G \), it is simple to see that \((H, \pi_1, \pi_2)\) is a covariant birepresentation of \((A, G, \alpha)\).

Conversely, suppose \((H, \pi_1, \pi_2)\) is a covariant birepresentation of \((A, G, \alpha)\). Take \( x \in K(G, A) \) and define two operators \( \pi(x) \) and \( \rho(x) \) on \( H \) by

\[
\pi(x)\xi = \sum_\gamma x(\gamma) \cdot \pi_1(\gamma)(\xi), \quad \rho(x)\xi = \sum_\gamma \pi_2(\gamma^{-1})(\xi \cdot x(\gamma)).
\]

Since \( \pi \) is obviously norm decreasing, it extends to a representation of \( \ell^1(G, A) \), hence to that of \( A \times_\alpha G \). Similarly, \( \rho \) extends to a representation of the opposite algebra of \( A \times_\alpha G \). Two representations \( \pi \) and \( \rho \) are commuting, so that \( H \) is a Hilbert bimodule on \( A \times_\alpha G \).

Hence, a non-zero \((A, G)\)-central vector for a covariant birepresentation of \((A, G, \alpha)\) is a non-zero \( A \times_\alpha G \)-central vector.

By the argument of Remark 15 in [1], we know that every unital \( C^*\)-algebra without tracial states has property (T). We will show that a similar fact is true for dynamical systems which the associated \( C^*\)-algebra does not admit \( \alpha \)-invariant tracial states.
Theorem 3.6. Let \((\mathcal{A}, \mathcal{G}, \alpha)\) be a dynamical system such that the unital C*-algebra \(\mathcal{A}\) does not admit \(\alpha\)-invariant tracial states. Then \((\mathcal{A}, \mathcal{G}, \alpha)\) has property (T).

Proof. Assume that \((\mathcal{A}, \mathcal{G}, \alpha)\) does not have property (T). Then, there is a covariant birepresentation \((\mathcal{H}, \pi_1, \pi_2)\) almost has invariant vectors. This implies that there is a net of unit vectors \((\xi_i)_{i \in I}\) in \(\mathcal{H}\) such that:

\[
\lim_i \|a \cdot \xi_i - \xi_i \cdot a\| = 0, \quad \lim_i \|\pi_1(\gamma)\pi_2(\gamma)(\xi_i) - \xi_i\| = 0,
\]

for all \(a \in \mathcal{A}, \gamma \in \mathcal{G}\). For each \(T \in \mathcal{L}(\mathcal{H})\), let \(D_T\) be the closed disc in \(\mathbb{C}\) of radius \(\|T\|\), and consider the product space

\[X = \prod_{T \in \mathcal{L}(\mathcal{H})} D_T,\]

endowed with the product topology. By Tychonoff’s Theorem, \(X\) is compact. Since \(((T\xi_i, \xi_i))_{T \in \mathcal{L}(\mathcal{H})}\) is an element of \(X\) for all \(i \in I\), there exists a subnet \((\xi_j)_{j \in J}\) such that, for all \(T \in \mathcal{L}(\mathcal{H})\), the limit

\[\varphi(T) = \lim_j (T\xi_j, \xi_j)\]

exists. It is clear that \(T \mapsto \varphi(T)\) is a positive linear functional on \(\mathcal{L}(\mathcal{H})\) with \(\varphi(id_{\mathcal{H}}) = 1\). Moreover, for every \(\gamma \in \mathcal{G}\) and \(T \in \mathcal{L}(\mathcal{H})\), we have

\[\varphi(\pi_1(\gamma)\pi_2(\gamma)T) = \varphi(T) = \varphi(T\pi_1(\gamma)\pi_2(\gamma)).\]

Then \(Tr : \mathcal{A} \to \mathbb{C}\) defined by \(Tr(a) = \varphi(\mu(a))\) is an \(\alpha\)-invariant tracial state on \(\mathcal{A}\), where \(\mu\) is the representation on \(\mathcal{H}\) given by, say, the left action of \(\mathcal{A}\).

Example 3.7. Let \(\mathcal{H}\) be an infinite-dimensional Hilbert space and \(\mathcal{U}(\mathcal{H})\) be its unitary group. Suppose \(\mathcal{B} \subset \mathcal{L}(\mathcal{H})\) is a C*-subalgebra containing the identity element of \(\mathcal{L}(\mathcal{H})\), and that \(u \in \mathcal{U}(\mathcal{H})\) is such that \(u^*Bu \subset \mathcal{B}\). Then \(\varphi(a) = uau^*\) is an automorphism of \(\mathcal{L}(\mathcal{H})\). Therefore, we obtain a homomorphism \(\alpha : \mathbb{Z} \to \text{Aut}(\mathcal{L}(\mathcal{H}))\), defined by \(\alpha_n = \phi^n\), and \((\mathcal{L}(\mathcal{H}), \mathbb{Z}, \alpha)\) is a dynamical system. Using Theorem 3.6, so \(((\mathcal{L}(\mathcal{H}), \mathcal{B}, \mathbb{Z}, \alpha)\) has property (T).

Let \(\mathcal{G}_1 \to \mathcal{G}_2\) be a surjective continuous homomorphism between locally compact groups. It is well-known that if \(\mathcal{G}_1\) has property (T), then \(\mathcal{G}_2\) has property (T). Similarly, let \(\mathcal{A} \to \mathcal{B}\) be a surjective \(*\)-homomorphism between unital C*-algebras. If \(\mathcal{A}\) has property (T), then so has \(\mathcal{B}\). The corresponding statement for dynamical systems is as follows and its proof is straightforward.

Lemma 3.8. Let \((\mathcal{A}, \mathcal{G}, \alpha)\) and \((\mathcal{B}, \mathcal{G}, \beta)\) be two dynamical systems with actions \(\alpha\) and \(\beta\) of a fixed group \(\mathcal{G}\) on \(\mathcal{A}\) and \(\mathcal{B}\), respectively. Let \(f : \mathcal{A} \to \mathcal{B}\) be a surjective \(*\)-homomorphism between \(\mathcal{A}\) and \(\mathcal{B}\) such that

\[\beta_\gamma(f(a)) = f(\alpha_\gamma(a)),\]

for all \(\gamma \in \mathcal{G}, a \in \mathcal{A}\). If \((\mathcal{A}, \mathcal{G}, \alpha)\) has property (T), then \((\mathcal{B}, \mathcal{G}, \beta)\) has also property (T).

Let \(\mathcal{H}\) be a Hilbert bimodule on a C*-algebra \(\mathcal{A}\) and \(\mathcal{B} \subset \mathcal{A}\) a C*-subalgebra containing the identity of \(\mathcal{A}\). Let

\[\mathcal{H}^\mathcal{B} = \{\eta \in \mathcal{H} \mid b \cdot \eta = \eta \cdot b, \forall b \in \mathcal{B}\},\]
and $P^B_H : \mathcal{H} \to \mathcal{H}^B$ be the orthogonal projection from $\mathcal{H}$ over the closed subspace $\mathcal{H}^B$.

Let us recall a notion of strong property (T) in [4]. The pair $(\mathcal{A}, \mathcal{B})$ has strong property (T) if for any $r > 0$, there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$ and $\varepsilon > 0$ such that the following property holds: if a Hilbert bimodule $\mathcal{H}$ on $\mathcal{A}$ contains a unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{F}, r, \varepsilon)$-central, then $||\xi - P^B_{\mathcal{H}}(\xi)|| < r$. Also, $\mathcal{A}$ has strong property (T) if $(\mathcal{A}, \mathcal{A})$ has such property.

By taking $r < \frac{1}{2}$, we see that strong property (T) implies property (T). If $\mathcal{A}$ has no tracial state, then $\mathcal{A}$ has strong property (T), and so does $(\mathcal{A}, \mathcal{B})$ (see [4]).

Also, suppose $(\pi, \mathcal{H})$ is a unitary representation of a locally compact group $\mathcal{G}$ and $\mathcal{N}$ is a closed subgroup of $\mathcal{G}$. Let

$$\mathcal{H}^N = \{ \eta \in \mathcal{H} \mid \pi(\gamma)(\eta) = \eta, \forall \gamma \in \mathcal{N} \},$$

and $P^N_H : \mathcal{H} \to \mathcal{H}^N$ be the orthogonal projection from $\mathcal{H}$ over the closed subspace $\mathcal{H}^N$.

**Theorem 3.9.** Suppose $(\mathcal{A}, \mathcal{B})$ has strong property (T) and $\mathcal{G}$ has property (T). Then $((\mathcal{A}, \mathcal{B}), \mathcal{G}, \alpha)$ has property (T).

**Proof.** Since $\mathcal{G}$ has property (T), there exist a compact subset $\mathcal{Q}$ of $\mathcal{G}$ and $\varepsilon > 0$ such that for any unitary representation $(\pi, \mathcal{H})$ and unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{Q}, \varepsilon)$-invariant, one has a non-zero vector $\eta \in \mathcal{H}$ such that

$$\pi(\gamma)(\eta) = \eta,$$

for all $\gamma \in \mathcal{G}$. Let $h = \min\{\frac{1}{2}, \frac{\varepsilon}{3}\}$. Since $(\mathcal{A}, \mathcal{B})$ has strong property (T), there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$ and $r > 0$ such that for any Hilbert bimodule $\mathcal{H}$ and unit vector $\xi \in \mathcal{H}$ which is $(\mathcal{F}, r)$-central, one has

$$||\xi - P^B_{\mathcal{H}}(\xi)|| < h.$$

Let $k = \min\{r, \frac{\varepsilon}{4}\}$, and $(\mathcal{H}, \pi_1, \pi_2)$ be a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$ with a unit vector $\xi \in \mathcal{H}$ such that:

$$\max_{\gamma \in \mathcal{F}} ||a \cdot \xi - \xi \cdot a|| < k, \quad \sup_{\gamma \in \mathcal{Q}} ||\pi_1(\gamma)\pi_2(\gamma)(\xi) - \xi|| < k.$$

Then $||\xi - P^B_{\mathcal{H}}(\xi)|| < h$ and $||P^B_{\mathcal{H}}(\xi)|| > \frac{1}{2}$. For all $b \in \mathcal{B}, \gamma \in \mathcal{G}$ and $\zeta \in \mathcal{H}^B$ we have:

$$b \cdot \pi_1(\gamma)\pi_2(\gamma)(\zeta) = \pi_1(\gamma)(\alpha_{\gamma^{-1}}(b) \cdot \pi_2(\gamma)(\zeta)) = \pi_1(\gamma)\pi_2(\gamma)(\alpha_{\gamma^{-1}}(b) \cdot \zeta) = \pi_1(\gamma)\pi_2(\gamma)(\zeta) \cdot \alpha_{\gamma^{-1}}(b)) = \pi_1(\gamma)(\pi_2(\gamma)(\zeta) \cdot b) = \pi_1(\gamma)\pi_2(\gamma)(\zeta) \cdot b.$$

Hence, $\pi(\gamma) = \pi_1(\gamma)\pi_2(\gamma)$ is a unitary representation of $\mathcal{G}$ on $\mathcal{H}^B$. If we take $\zeta = \frac{P^B_{\mathcal{H}}(\xi)}{||P^B_{\mathcal{H}}(\xi)||}$, then we have

$$\sup_{\gamma \in \mathcal{Q}} ||\pi(\gamma)(\zeta) - \zeta|| < \frac{k}{||P^B_{\mathcal{H}}(\xi)||} + \frac{2h}{2||P^B_{\mathcal{H}}(\xi)||} < \frac{\varepsilon}{2\cdot 2} = \varepsilon.$$
Therefore, there exists a non-zero vector $\eta \in \mathcal{H}^B$ such that

$$\pi_1(\gamma)\pi_2(\gamma)(\eta) = \pi(\gamma)(\eta) = \eta,$$

for all $\gamma \in \mathcal{G}$, which implies that $\eta$ is a non-zero $(\mathcal{B}, \mathcal{G})$-central vector, as required. 

We need the following proposition from [6] to prove the next lemma.

**Proposition 3.10.** Let $\mathcal{G}$ be a locally compact and $\sigma$-compact group and let $N$ be a closed subgroup of $\mathcal{G}$. The following properties are equivalent:

(i) $(\mathcal{G}, \mathcal{N})$ has property (T),

(ii) for every $r > 0$, there exists a pair $(Q, \varepsilon)$ of compact subset $Q$ of $\mathcal{G}$ and $\varepsilon > 0$ such that for any unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ which has a $(Q, \varepsilon)$-invariant unit vector $\xi$, then we have $\|\xi - P_{\mathcal{H}}^Q(\xi)\| \leq r$. 

**Lemma 3.11.** Let $\mathcal{G}$ be a countable discrete group with property (T). Then $(C^*(\mathcal{G}), \mathcal{G}, \alpha)$ has property (T) for any action $\alpha$ of $\mathcal{G}$ on $C^*(\mathcal{G})$.

**Proof.** By Theorem 3.9, it suffices to prove that $C^*(\mathcal{G})$ has strong property (T). Let $r > 0$. Since $\mathcal{G}$ has property (T), by Proposition 3.10 there exist a finite subset $Q$ of $\mathcal{G}$ and $\varepsilon > 0$ such that for any unitary representation $(\pi, \mathcal{H})$ and unit vector $\xi \in \mathcal{H}$ which is $(Q, \varepsilon)$-invariant, one has $\|\xi - P_{\mathcal{H}}^Q(\xi)\| \leq \varepsilon$. Let $F = \{\delta_\gamma \mid \gamma \in Q\}$ be the finite subset of $C^*(\mathcal{G})$, and $\mathcal{H}$ a Hilbert bimodule on $C^*(\mathcal{G})$ contains a unit vector $\xi \in \mathcal{H}$ which is $(F, \varepsilon)$-central. Define a unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ by

$$\pi(\gamma)(\xi) = \delta_\gamma \cdot \xi \cdot \delta_{\gamma^{-1}}.$$ 

Hence, $\xi$ is $(Q, \varepsilon)$-invariant, and we have

$$\|\xi - P_{\mathcal{H}}^{C^*(\mathcal{G})}(\xi)\| = \|\xi - P_{\mathcal{H}}^F(\xi)\| < r.$$ 

In the following, we show that property (T) of a dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ such that $\mathcal{G}$ is a discrete group implies property (T) of its $C^*$-crossed product.

**Theorem 3.12.** Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system, and that $\mathcal{G}$ is a discrete group. If $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T), then $\mathcal{A} \rtimes_\alpha \mathcal{G}$ has property (T) (and so does $\mathcal{A} \rtimes_{ar} \mathcal{G}$).

**Proof.** Since $(\mathcal{A}, \mathcal{G}, \alpha)$ has property (T), there exist a finite subset $F$ of $\mathcal{A}$, a finite subset $Q$ of $\mathcal{G}$ and $\varepsilon, r > 0$ such that for every covariant birepresentation $(\mathcal{H}, \pi_1, \pi_2)$ of $(\mathcal{A}, \mathcal{G}, \alpha)$ contains a unit vector $\xi \in \mathcal{H}$ which is $(F, \varepsilon, Q, r)$-central, then $\mathcal{H}$ has a non-zero $(\mathcal{A}, \mathcal{G})$-central vector.

Let $D = F \cup \{\delta_\gamma \mid \gamma \in Q\}$ and $\ell = \min(r, \varepsilon)$. Let $\mathcal{H}$ be a Hilbert bimodule on $\mathcal{A} \rtimes_\alpha \mathcal{G}$ contains a unit vector $\xi \in \mathcal{H}$ which is $(D, \ell)$-central. Define two commuting unitary representations $(\pi_1, \mathcal{H})$ and $(\pi_2, \mathcal{H})$ of $\mathcal{G}$ by

$$\pi_1(\gamma)(\xi) = \delta_\gamma \cdot \xi, \quad \pi_2(\gamma)(\xi) = \xi \cdot \delta_{\gamma^{-1}}.$$ 

In the following, we show that property (T) of a dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ such that $\mathcal{G}$ is a discrete group implies property (T) of its $C^*$-crossed product.
Viewing $\mathcal{A}$ as a subalgebra of $\mathcal{A} \times_{\alpha} \mathcal{G}$, it is clear that $(\mathcal{H}, \pi_1, \pi_2)$ is a covariant birepresentation of $(\mathcal{A}, \mathcal{G}, \alpha)$, and $\xi$ is a $(\mathcal{F}, \varepsilon, \mathcal{Q}, r)$-central. Therefore, there exists a non-zero vector $\eta \in \mathcal{H}$ such that

$$a \cdot \eta = \eta \cdot a, \quad \delta_{\gamma} \cdot \eta = \eta \cdot \delta_{\gamma},$$

for all $a \in \mathcal{A}$ and $\gamma \in \mathcal{G}$. Then for any $x = \sum_{\gamma} x(\gamma)\delta_{\gamma} \in \mathcal{K}(\mathcal{G}, \mathcal{A})$, we have

$$x \cdot \eta = \sum_{\gamma} x(\gamma)\delta_{\gamma} \cdot \eta = \sum_{\gamma} x(\gamma) \cdot \eta \cdot \delta_{\gamma} = \sum_{\gamma} \eta \cdot x(\gamma)\delta_{\gamma} = \eta \cdot x.$$

Since $\mathcal{K}(\mathcal{G}, \mathcal{A})$ is dense in $\ell^1(\mathcal{G}, \mathcal{A})$ and $\ell^1(\mathcal{G}, \mathcal{A})$ is dense in $\mathcal{A} \times_{\alpha} \mathcal{G}$, we obtain $x \cdot \eta = \eta \cdot x$ for all $x \in \mathcal{A} \times_{\alpha} \mathcal{G}$. Since $\mathcal{A} \times_{ar} \mathcal{G}$ is a quotient of $\mathcal{A} \times_{\alpha} \mathcal{G}$, it follows that $\mathcal{A} \times_{ar} \mathcal{G}$ also has property (T).

**Remark 3.13.** If $(\mathcal{A}, \mathcal{G}, \alpha)$ is a dynamical system, $\mathcal{G}$ is a discrete group and $\alpha$ trivial, then:

$$\mathcal{A} \times_{ar} \mathcal{G} \cong C^*_r(\mathcal{G}) \otimes_{\text{min}} \mathcal{A}, \quad \mathcal{A} \times_{\alpha} \mathcal{G} \cong C^*(\mathcal{G}) \otimes_{\text{max}} \mathcal{A}.$$

By Theorems 3.9 and 3.12 for a discrete group $\mathcal{G}$ with property (T) and a unital $C^*$-algebra $\mathcal{A}$ with strong property (T), $C^*_r(\mathcal{G}) \otimes_{\text{min}} \mathcal{A}$ and $C^*(\mathcal{G}) \otimes_{\text{max}} \mathcal{A}$ have property (T).

If a locally compact group with property (T) is amenable, then it is compact, a similar fact is true for $C^*$-algebras with property (T) which are nuclear. A $C^*$-algebra $\mathcal{A}$ is nuclear if, for any $C^*$-algebra $\mathcal{B}$, there is a unique pre-$C^*$-norm on $\mathcal{A} \otimes \mathcal{B}$. Let $Tr$ be a tracial state on the unital $C^*$-algebra $\mathcal{A}$. By the GNS-construction, $Tr$ defines a Hilbert $\mathcal{A}$-bimodule, denoted by $L^2(Tr)$. In [1], it is shown that if $\mathcal{A}$ is a unital $C^*$-algebra with property (T) which is nuclear, then for any tracial state $Tr$ on $\mathcal{A}$, the left action of $\mathcal{A}$ on the Hilbert space $L^2(Tr)$ is completely atomic, that is, $L^2(Tr)$ decomposes as a direct sum of finite dimensional $\mathcal{A}$-submodules. This implies that if $\mathcal{A}$ is a unital $C^*$-algebra with property (T), and that there exists a tracial state $Tr$ on $\mathcal{A}$ such that $L^2(Tr)$ is not completely atomic, then $\mathcal{A}$ is not nuclear.

**Corollary 3.14.** Let $(\mathcal{A}, \mathcal{G}, \alpha)$ be a dynamical system such that $\mathcal{G}$ is a discrete group and $\mathcal{A}$ is nuclear and $\mathcal{G}$ amenable. Suppose that there exists a tracial state $Tr$ of $\mathcal{A} \times_{\alpha} \mathcal{G}$ such that $L^2(Tr)$ is not completely atomic. Then $(\mathcal{A}, \mathcal{G}, \alpha)$ does not have property (T).

**Proof.** Since $\mathcal{G}$ is amenable and $\mathcal{A}$ is nuclear, so $\mathcal{A} \times_{\alpha} \mathcal{G}$ is nuclear (see [10]). As cited above $\mathcal{A} \times_{\alpha} \mathcal{G}$ does not have property (T). So by Theorem 3.12, $(\mathcal{A}, \mathcal{G}, \alpha)$ does not have property (T). \qed

Note that even if $\mathcal{A} \times_{\alpha} \mathcal{G}$ has strong property (T) and $\alpha$ is trivial, it does not follow that $\mathcal{G}$ has property (T).

**Proposition 3.15.** Let $\mathcal{G}$ be a locally compact and $\sigma$-compact group and $\mathcal{N}$ a closed subgroup of $\mathcal{G}$. The following properties are equivalent:

(i) $(\mathcal{G}, \mathcal{N})$ has property (T),

(ii) if a unitary representation $(\pi, \mathcal{H})$ of $\mathcal{G}$ almost has invariant vectors, that is, if it has $(\mathcal{Q}, \varepsilon)$-invariant vectors for every compact subset $\mathcal{Q}$ of $\mathcal{G}$ and every $\varepsilon > 0$, then $\mathcal{H}$ contains a non-zero finite dimensional subspace which is invariant under $\mathcal{N}$.
We will now use the same technique as in the proof of Theorem 6 in [1] to obtain the following theorem, using the above proposition from [1].

**Theorem 3.16.** Let $\mathcal{A}$ be a commutative unital $C^*$-algebra, and $\mathcal{G}$ a countable discrete group such that there exists a faithful representation of $\mathcal{A}$ to the Hilbert space $\ell^2(\mathcal{G})$. If $C_r^*(\mathcal{G}) \otimes_{\text{min}} \mathcal{A}$ has property (T), then $\mathcal{G}$ has property (T).

**Proof.** Viewing $C_r^*(\mathcal{G}) \otimes_{\text{min}} \mathcal{A}$ as $\mathcal{A} \times_{ar} \mathcal{G}$ in dynamical system $(\mathcal{A}, \mathcal{G}, \alpha)$ with $\alpha$ trivial, suppose $\mathcal{A} \times_{ar} \mathcal{G}$ has property (T). Choose a finite subset $\mathcal{F}$ of $\mathcal{A} \times_{ar} \mathcal{G}$ and $\varepsilon > 0$ as in Definition 3.1. We may assume that $\|y\| \leq 1$ for all $y \in \mathcal{F}$. Take an element $\xi_0 \in \ell^2(\mathcal{G})$ such that $\|\xi_0\| = 1$. One can check that there exists a finite subset $\mathcal{Q}$ of $\mathcal{G}$ such that:

$$\sum_{\gamma \in \mathcal{G} \setminus \mathcal{Q}} \|y(\delta_e \otimes \xi_0)(\gamma)\|^2 < \frac{\varepsilon^2}{9},$$

for all $y \in \mathcal{F}$. Assume that $(\pi, \mathcal{H})$ is a unitary representation of $\mathcal{G}$ almost has invariant vectors. Choose a unit vector $\xi \in \mathcal{H}$ such that is $(\mathcal{Q}, \xi)$-invariant vector. Define a representation $\mu$ of $\mathcal{A}$ as well as two unitary representations $\pi_1, \pi_2$ of $\mathcal{G}$ on the Hilbert space tensor product $\ell^2(\mathcal{G}, \ell^2(\mathcal{G})) \otimes \mathcal{H}$ by

$$\mu(a) = \pi_a(a) \otimes \text{id},$$

and,

$$\pi_1(\gamma) = \lambda_\gamma(\gamma) \otimes \text{id}, \quad \pi_2(\gamma) = \mu_\gamma(\gamma) \otimes \pi(\gamma),$$

for all $a \in \mathcal{A}, \gamma \in \mathcal{G}$, where $\mu_\gamma$ is a representation of $\mathcal{G}$ on the Hilbert space $\ell^2(\mathcal{G}, \ell^2(\mathcal{G}))$ defined by $\mu_\gamma(\gamma)\xi(s) = \xi(s \gamma)$ for all $\gamma, s \in \mathcal{G}$ and $\xi \in \ell^2(\mathcal{G}, \ell^2(\mathcal{G}))$.

Since $(\mu, \pi_1)$ and $(\mu, \pi_2)$ are covariant representations, are equivalent to multiples of the regular representation $(\pi_a, \lambda_\gamma)$, they extend to commuting representations of $\mathcal{A} \times_{ar} \mathcal{G}$, so that $\ell^2(\mathcal{G}, \ell^2(\mathcal{G})) \otimes \mathcal{H}$ is a Hilbert bimodule on $\mathcal{A} \times_{ar} \mathcal{G}$.

Let $\bar{\eta} = \bar{\xi} \otimes \xi$, where $\bar{\xi} \in \ell^2(\mathcal{G}, \ell^2(\mathcal{G}))$ is defined by $\bar{\xi}(e) = \xi_0$ and $\bar{\xi}(\gamma) = 0$ otherwise. For any $y \in \mathcal{F}$, we have

$$\|y \cdot \bar{\eta} - \bar{\eta} \cdot y\|^2 = \sum_{\gamma \in \mathcal{G}} \|y(\delta_e \otimes \xi_0)(\gamma)\|^2 \|\pi(\gamma)(\xi) - \xi\|^2$$

$$\leq \frac{4\varepsilon^2}{9} + \sum_{\gamma \in \mathcal{Q}} \|y(\delta_e \otimes \xi_0)(\gamma)\|^2 \|\pi(\gamma)(\xi) - \xi\|^2$$

$$\leq \frac{4\varepsilon^2}{9} + \frac{\varepsilon^2}{9} < \varepsilon^2.$$

Therefore, there exists a non-zero vector $\eta$ in $\ell^2(\mathcal{G}, \ell^2(\mathcal{G})) \otimes \mathcal{H}$ which is $\mathcal{A} \times_{ar} \mathcal{G}$-central. Viewing $\eta$ as a non-zero vector in the Hilbert space $\ell^2(\mathcal{G}, \ell^2(\mathcal{G}, \mathcal{H}))$, in particular, we have

$$\eta(\gamma \tau \gamma^{-1})(s) = \pi(\gamma)(\eta(t)(s)),$$
for all $\gamma, t, s \in G$. Then $\gamma \mapsto ||\eta(\gamma)||$ is a non-zero function in $\ell^2(G)$ which is invariant under conjugation by elements of $G$. Let $t_0 \in G$ be such that $\eta(t_0) \neq 0$. It follows that $\{\gamma t_0 \gamma^{-1} \mid \gamma \in G\}$ is a finite subset of $G$. Let $s_0 \in G$ be such that $\eta(t_0)(s_0) \neq 0$. Then $\{\eta(\gamma t_0 \gamma^{-1})(s_0) \mid \gamma \in G\}$ is finite, hence $\{\pi(\gamma)(\eta(t_0))(s_0) \mid \gamma \in G\}$ is a finite subset of $\mathcal{H}$ and its linear span defines a non-zero finite dimensional invariant subspace under $G$. It follows from Proposition 3.15 that $G$ has property (T).

Remark 3.17. (i) Let $G$ be a countable discrete group. Since all finite dimensional $C^*$-algebras have strong property (T) (see [4]), using Theorems 3.9, 3.12 and 3.16, $G$ has property (T) if and only if $C^*_r(G)$ has property (T). This is a well-known result of Bekka (see [1]).

(ii) Let $G$ be a countable discrete abelian group. Since $G$ is amenable there exists a faithful representation of $C^*(G)$ in the Hilbert space $\ell^2(G)$. In fact the regular representation can be extended to an $^*$-isomorphism between the group $C^*$-algebra $C^*(G)$ and the reduce group $C^*$-algebra $C^*_r(G)$, and we have $C^*(G) \cong C^*_r(G)$. Using Lemma 3.11 and Theorems 3.12, 3.16 it follows that $G$ has property (T) if and only if $C^*_r(G) \otimes_{\text{min}} C^*_r(G)$ has property (T), by choosing $\mathcal{A} = C^*(G)$ and $\alpha$ trivial in dynamical system $(\mathcal{A}, G, \alpha)$.

References