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The structure of the set of all C^*-convex maps in $*$-rings

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Abstract

In this paper, for every unital $*$-ring $\mathcal{S}$, we investigate the Jensen's inequality preserving maps on C^*-convex subsets of $\mathcal{S}$, which we call them C^*-convex maps on $\mathcal{S}$. We consider an involution for maps on $*$-rings, and we show that for every C^*-convex map f on the C^*-convex set B in $\mathcal{S}$, f^* is also a C^*-convex map on B. We prove that in the unital commutative $*$-rings, the set of all C^*-convex maps (C^*-affine maps) on a C^*-convex set B, is also a C^*-convex set. In addition, we prove some results for increasing C^*-convex maps. Moreover, it is proved that the set of all C^*-affine maps on B, is a C^*-face of the set of all C^*-convex maps on B in the unital commutative $*$-rings. Finally, some examples of C^*-convex maps and C^*-affine maps in $*$-rings are given.

1. Introduction and Preliminaries

One of the form of non-commutative convexity is C^*-convexity. The official study of this type of convexity was started by Loebl and Paulsen in [6]. Farenick and Morenz continued this work

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in several papers such as [5] and [9] and they obtained the C^*-extreme points, the relative extreme points, and a kind of the Krein Milman theorem for M_n. Also Morenz has generalised the notion of face from the classical convexity to C^*-face in C^*-convexity in C^*-algebras [9]. Magajna has worked on this subject especially on C^*-convex subsets of C^*-algebras and operator bimodules in several papers such as [7], and [8]. The author and G. H. Esslamzadeh generalised the concepts of C^*-convexity and C^*-extreme points to $*$-rings [4]. Recently, the subjects of C^*-convex maps and C^*-affine maps has studied in [2]. Indeed, the C^*-extreme points of the graph and epigraph of C^*-affine maps has investigated and, also it has shawn there, that for a C^*-convex map f defined on a unital $*$-ring $\mathcal{S}$ with some conditions the graph of f is a C^*-face of the epigraph of f, and some other results about the C^*-faces of C^*-convex sets in $*$-rings.

In the whole of this paper, $\mathcal{S}$ denotes a unital $*$-ring, that is, an involutive ring with an identity element. As mentioned in [1], we say that $a \in \mathcal{S}$ is positive if a is of the form $a = x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n$ for some $x_1, x_2, \cdots, x_n$ in $\mathcal{S}$, and we write $a \geq 0$. As usual, $a \in \mathcal{S}$ is called hermitian or self-adjoint if $a = a^*$. We use the notation $\mathcal{S}_{sa}$ for denoting the set of all self-adjoint elements of $\mathcal{S}$, and it can be ordered by writing $a \leq b$ in case $b - a \geq 0$.

A subset B of a unital $*$-ring $\mathcal{S}$ is called C^*-convex, if it is closed under the summations of the form $\sum_{i=1}^n a_i^* x_i a_i$, where, $a_i \in \mathcal{S}$, and $\sum_{i=1}^n a_i^* a_i = 1_{\mathcal{S}}$, $x_i \in B$, and $n \in \mathbb{N}$. In this case, the summation $\sum_{i=1}^n a_i^* x_i a_i$ is called a C^*-convex combination of elements $x_i \in B$, and it is called a proper C^*-convex combination of elements $x_i \in B$ if moreover, a_i is invertible in $\mathcal{S}$ for each i. Also, a nonempty subset E of a C^*-convex set $B \subseteq \mathcal{S}$ is called a C^*-face of B, if the condition $x \in E$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ as a proper C^*-convex combination of elements $x_i \in B$, implies that $x_i \in E$ for all i.

If $M_n(\mathcal{S})$ be the set of all $n \times n$ matrices with elements in a $*$-ring $\mathcal{S}$ then $M_n(\mathcal{S})$ can be considered as a $*$-ring with the usual matrix operations and $*$-transposition as an involution, i.e. $[a_{ij}]^* = [(a_{ji})^*]$. The notions of $\mathcal{S}$-convexity, $\mathcal{S}$-face, and $\mathcal{S}$-extreme point in $M_n(\mathcal{S})$ has defined in [3], as following.

For every subset K of the unital $*$-ring $\mathcal{S}$, $M_n(K)$, the set of all $n \times n$ matrices with entries in K, is called an $\mathcal{S}$-convex subset of the unital $*$-ring $M_n(\mathcal{S})$ if $\sum\limits_{i=1}^{m} a_i^* X_i a_i \in M_n(K)$ whenever, $X_i \in M_n(K)$, $a_i \in \mathcal{S}$ for each i, $\sum\limits_{i=1}^{m} a_i^* a_i = 1_{\mathcal{S}}$ and $m \in \mathbb{N}$. In this case, an element $X \in M_n(K)$ is called an $\mathcal{S}$-extreme point of $M_n(K)$ if the condition

$$X = \sum_{i=1}^{m} a_i^* X_i a_i,\ \sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{S}}, X_i \in M_n(K), a_i \text{ is invertible in } \mathcal{S}, n \in \mathbb{N},$$

implies that all X_is are unitarily equivalent to X in $M_n(\mathcal{S})$, in the sense that, there exist unitaries $u_i \in \mathcal{S}$ such that $X_i = u_i^* X u_i$ for all i. The set of all $\mathcal{S}$-extreme points of $M_n(K)$ is denoted by $\mathcal{S}\text{-}ext(M_n(K))$.

A nonempty subset $M_n(E)$ of an $\mathcal{S}$-convex set $M_n(K)$ in the unital $*$-ring $M_n(\mathcal{S})$ is called an $\mathcal{S}$-face of $M_n(K)$, if the conditions $X \in M_n(E)$ and $X = \sum_{i=1}^n a_i^* X_i a_i$, where $X_i \in M_n(K)$, $a_i \in \mathcal{S}$, $\sum_{i=1}^n a_i^* a_i = 1_{\mathcal{S}}$, $n \in \mathbb{N}$, and a_i is invertible in $\mathcal{S}$, implies that $X_i \in M_n(E)$ for all i.

Also, in [3] the relation between the C^*-convex subsets of $\mathcal{S}$ and $\mathcal{S}$-convex subsets of $M_n(\mathcal{S})$, as well as, the relation between the C^*-faces of these C^*-convex sets and $\mathcal{S}$-faces of $\mathcal{S}$-convex subsets of $M_n(\mathcal{S})$ has shawn, as following;

Theorem 1.1. *If $\mathcal{S}$ is a unital $*$-ring. Then*
(1) *B is a C^*-convex subset of $\mathcal{S}$ iff $M_n(B)$ is an $\mathcal{S}$-convex subset of $M_n(\mathcal{S})$.*
(2) *E is a C^*-face of C^*-convex set B in $\mathcal{S}$ if and only if $M_n(E)$ is an $\mathcal{S}$-face of $M_n(B)$ in $M_n(\mathcal{S})$.*
(3) *If B is a C^*-convex subset of $\mathcal{S}$ such that for each $x \in B$, $C^* - co(\{x\}) = \{x\}$, then $M_n(C^* - ext(B)) = \mathcal{S} - ext(M_n(B))$.*

Moreover, in [3] we have proved that, if the entries restricted to the positive elements in $\mathcal{S}$, then the set of all diagonal matrices is an $\mathcal{S}$-face of the set of all lower (upper) triangular matrices, and all of these sets are $\mathcal{S}$-faces of $M_n(\mathcal{S}^+)$, where, $\mathcal{S}^+$ denotes the set of all positive elements in $\mathcal{S}$.

In the next section, we give the main results of this paper, and the last section is devoted to some examples.

2. Main Results

Let $\mathcal{S}$ be a unital $*$-ring and B be a $*$-subring of $\mathcal{S}$. Then the set $\mathcal{F}(\mathcal{B})$ of all maps $f : B \to B$, is a ring by pointwise operations. We can define an involution on $\mathcal{F}(\mathcal{B})$ by $f^*(x) = (f(x^*))^*$, for all $x \in B$. To see this, we investigate the conditions of an involution. For every $x \in B$, and $f, g \in \mathcal{F}(\mathcal{B})$ we have
1) $(f^*)^*(x) = (f^*(x^*))^* = (f(x^{**}))^{**} = f(x)$;
2) $(f + g)^*(x) = ((f + g)(x^*))^* = (f(x^*) + g(x^*))^* = (f(x^*))^* + (g(x^*))^* = (f^* + g^*)(x)$;
3) $(f.g)^*(x) = ((f.g)(x^*))^* = (f(x^*)g(x^*))^* = (g(x^*))^*.(f(x^*))^* = (g^*.f^*)(x)$.
Also, if B is a $*$-subalgebra of the algebra $\mathcal{A}$, then for each scalar γ we have
4) $(\gamma f)^*(x) = ((\gamma f)(x^*))^* = (\gamma.f(x^*))^* = \bar{\gamma}(f(x^*)^* = \bar{\gamma} f^*(x)$.

Also, note that $f = f^*$ if and only if $f(x) = f^*(x) = (f(x^*))^*$ and so $f(x^*) = (f(x))^*$ for all $x \in \mathcal{S}$. Moreover, for each $n \in \mathbb{N}$, $(f^n)^* = (f^*)^n$, and hence f is a nilpotent map on B iff f^* is a nilpotent map on B too.

The author and Esslamzadeh introduced the notion of C^*-convex map as a map which satisfies the Jensen's inequality in $*$-rings ([4]) as following;

Definition 2.1. Let B be a C^*-convex subset of the unital $*$-ring $\mathcal{S}$ and f be a map on B. Then f is called a C^*-convex map on B, if

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i,$$

whenever, $n \in \mathbb{N}$, $x_i \in B$, $a_i \in \mathcal{S}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{S}}$.

In the next definition, we define the notion of C^*-affine map by replacing the equality instead of the less or equality in the above definition.

Definition 2.2. Let B be a C^*-convex subset of the unital $*$-ring $\mathcal{S}$ and f be a map on B. Then f is called a C^*-affine map on B, if

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) = \sum_{i=1}^{n} a_i^* f(x_i) a_i,$$

whenever, $n \in \mathbb{N}$, $x_i \in B$, $a_i \in \mathcal{S}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{S}}$.

Remark 2.3. We can consider the classical convexity as a C^*-convexity by consideration of $\lambda x = \sqrt{\lambda} x \sqrt{\lambda}$, and hence every classical convex function can be considered as a C^*-convex function. Similarly, every affine function can be considered as a C^*-affine function.

In the next theorem we prove that the involution of every C^*-convex map is also a C^*-convex map.

Theorem 2.4. *Let f be a C^*-convex map on a $*$-closed C^*-convex set B in $\mathcal{S}$. Then f^* is also a C^*-convex map on B.*

Proof. For each C^*-convex combination $\sum_{i=1}^{n} a_i^* x_i a_i$ of elements $x_i \in B$, we have

$$\begin{aligned} f^*\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) &= \left(f\left(\sum_{i=1}^{n} a_i^* x_i a_i\right)^*\right)^* \\ &= \left(f\left(\sum_{i=1}^{n} a_i^* x_i^* a_i\right)\right)^* \\ &\leq \left(\sum_{i=1}^{n} a_i^* f(x_i^*) a_i\right)^* \\ &= \sum_{i=1}^{n} a_i^* (f(x_i^*))^* a_i \\ &= \sum_{i=1}^{n} a_i^* f^*(x_i) a_i. \end{aligned}$$

Note that, the implication $x \leq y \Longrightarrow x^* \leq y^*$ has used in the proof. □

Corollary 2.5. *The same conclusion holds if we replace the C^*-affine map instead of C^*-convex map in the previous theorem.*

For every C^*-convex map f defined on a C^*-convex subset B of the unital $*$-ring $\mathcal{S}$, and $b \in \mathcal{S}$, we define $b^* f(.) b$ on B by $(b^* f(.) b)(x) = b^* f(x) b$ for all $x \in B$.

Lemma 2.6. *Let f be a C^*-convex map on a C^*-convex set B in $\mathcal{S}$, and $b \in Z(\mathcal{S})$, the center of $\mathcal{S}$, then $b^* f(.) b$ is also a C^*-convex map on B.*

Proof. It is straightforward by using the facts that, for each $x, y \in \mathcal{S}$, $x \leq y$, implies that $b^* x b \leq b^* y b$, and b commutes with all elements of $\mathcal{S}$. □

Theorem 2.7. *Let $\mathcal{S}$ be a unital commutative $*$-ring, and B be a C^*-convex subset of $\mathcal{S}$. Then the set of all C^*-convex maps on B is a C^*-convex set, in the sense that, $\sum_{i=1}^{n} a_i^* f_i a_i$ is a C^*-convex map on B, whenever, f_i is a C^*-convex map on B for each i, $a_i \in \mathcal{S}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{S}}$.*

Proof. Suppose that, $\sum_{j=1}^{m} b_j^* f_j(.)b_j$ is a C^*-convex combination of C^*-convex maps f_j on B, and $\sum_{i=1}^{n} a_i^* x_i a_i$ is a C^*-convex combination of elements $x_i \in B$. Then,

$$\begin{aligned}
(\sum_{j=1}^{m} b_j^* f_j(.)b_j)(\sum_{i=1}^{n} a_i^* x_i a_i) &= \sum_{j=1}^{m} b_j^* f_j(\sum_{i=1}^{n} a_i^* x_i a_i) b_j \\
&\leq \sum_{j=1}^{m} b_j^* (\sum_{i=1}^{n} a_i^* f_j(x_i) a_i) b_j \\
&= \sum_{j=1}^{m} (\sum_{i=1}^{n} a_i^* b_j^* f_j(x_i) b_j a_i) \\
&= \sum_{i=1}^{n} (\sum_{j=1}^{m} a_i^* b_j^* f_j(x_i) b_j a_i) \\
&= \sum_{i=1}^{n} a_i^* (\sum_{j=1}^{m} b_j^* f_j(x_i) b_j) a_i \\
&= \sum_{i=1}^{n} a_i^* (\sum_{j=1}^{m} b_j^* f_j b_j)(x_i) a_i.
\end{aligned}$$

Thus, the set of all C^*-convex maps on B is a C^*-convex set. □

Corollary 2.8. *Let B be a C^*-convex subset of the unital commutative $*$-ring $\mathcal{S}$, then the set of all C^*-affine maps on B is a C^*-convex set, in the sense of the previous theorem.*

Note that, the condition commutativity is essential in the above theorem and its corollary (see Example 3.6.).

Theorem 2.9. *Suppose that $\mathcal{S}$ is a unital $*$-ring and B is a C^*-convex additive subset of $\mathcal{S}$. Then the set $\mathcal{M}$ of all C^*-convex maps on B is an additive set. Also, if $\mathcal{S}$ is a unital $*$-algebra, then $\mathcal{M}$ is a convex set. Moreover, if B is a positive cone and C^*-convex subset of $\mathcal{S}$, then $\mathcal{M}$ is a positive cone and specially a convex set. Furthermore, if B is a $*$-subring of $\mathcal{S}$, then the set of all C^*-affine maps on B is an abelian group and $*$-closed subset of $\mathcal{F}(\mathcal{B})$.*

Proof. It is straightforward. □

Corollary 2.10. *The same conclusion holds if we replace the C^*-affine map instead of C^*-convex map in the previous theorem.*

Remark 2.11. Let $\mathcal{S}$ be a unital $*$-ring and B be a C^*-convex and additive $*$-closed subset of $\mathcal{S}$. Then the set of all increasing C^*-convex maps on B is closed under the following operations:
$(f+g)(x) = f(x) + g(x)$, $fog(x) = f(g(x))$, and $f^*(x) = (f(x^*))^*$.

To see this, note that if f and g are increasing maps on B, then $f + g$, fog, and f^* are also increasing maps on B. To see the last, suppose that $x, y \in B$, and $x \leq y$. Then $y - x \geq 0$, and so $(y - x)^* = (y - x) \geq 0$. Hence, $y^* \geq x^*$. Thus, $f(y^*) \geq f(x^*)$, and similarly $(f(y^*))^* \geq (f(x^*))^*$. Therefore, $f^*(y) \geq f^*(x)$. Now, by use of the Theorem 2.4, and the fact that the summation and the composition of C^*-convex maps is also a C^*-convex map, the result is concluded.

By the same operations of the above remark, and the usual scalar multiplication, we have the following corollaries;

Corollary 2.12. *1) The set of all increasing C^*-convex maps on the C^*-convex cone B of a unital $*$-algebra is a positive cone.*
2) The set of all C^*-affine maps on the C^*-convex cone B of a unital $*$-algebra is a cone.

In the next theorem, we show that, the set of all C^*-affine maps is a C^*-face of the set of all C^*-convex maps in the commutative $*$-ring $\mathcal{S}$.

Theorem 2.13. *Let $\mathcal{S}$ be a unital commutative $*$-ring, and B be a C^*-convex subset of $\mathcal{S}$. Then the set of all C^*-affine maps on B is a C^*-face of the set of all C^*-convex maps on B.*

Proof. suppose that, f is a C^*-affine map on B, and $f = \sum_{j=1}^{m} b_j^* f_j(.) b_j$ is a proper C^*-convex combination of the C^*-convex maps f_j on B. We must show that, each f_j is a C^*-affine map on B. If there exists index $j_0 \in \{1, 2, \cdots, n\}$ such that f_{j_0} is not a C^*-affine map on B, then there exists a C^*-convex combination $\sum_{i=1}^{n} a_i^* x_i a_i$ of elements $x_i \in B$, such that

$$f_{j_0}(\sum_{i=1}^{n} a_i^* x_i a_i) < \sum_{i=1}^{n} a_i^* f_{j_0}(x_i) a_i.$$

Hence,

$$\begin{aligned} f(\sum_{i=1}^{n} a_i^* x_i a_i) &= (\sum_{j=1}^{m} b_j^* f_j b_j)(\sum_{i=1}^{n} a_i^* x_i a_i) = \sum_{j=1}^{m} b_j^* f_j(\sum_{i=1}^{n} a_i^* x_i a_i) b_j \\ &< \sum_{j=1}^{m} b_j^*(\sum_{i=1}^{n} a_i^* f_j(x_i) a_i) b_j = \sum_{j=1}^{m}(\sum_{i=1}^{n} a_i^* b_j^* f_j(x_i) b_j a_i) \\ &= \sum_{i=1}^{n}(\sum_{j=1}^{m} a_i^* b_j^* f_j(x_i) b_j a_i) = \sum_{i=1}^{n} a_i^*(\sum_{j=1}^{m} b_j^* f_j(x_i) b_j) a_i \\ &= \sum_{i=1}^{n} a_i^* f(x_i) a_i. \end{aligned}$$

Therefore, $f(\sum_{i=1}^{n} a_i^* x_i a_i) < \sum_{i=1}^{n} a_i^* f(x_i) a_i$, which is a contradiction. □

3. Examples

Example 3.1. The center $Z(\mathcal{S})$ of the $*$-ring $\mathcal{S}$, is a $*$-subring of $\mathcal{S}$, which is not in general a C^*-face of $\mathcal{S}$. For example if $\mathcal{S}$ is the $*$-ring of all 2×2 complex matrices with the usual operations, then $Z(\mathcal{S})$ is the set of all matrices of the form $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ where, $x \in \mathbb{C}$, which is not a C^*-face of $\mathcal{S}$. To see this, consider the following proper C^*-convex combination,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} \end{bmatrix}.$$

Example 3.2. If f_i is a C^*-convex map on B for each i $(1 \leq i \leq n)$. Then the map $f = \max_{i=1}^{n} f_i$ is also a C^*-convex map on B, since for every C^*-convex combination $\sum_{j=1}^{m} a_j^* x_j a_j$ of elements $x_j \in B$, we have

$$f(\sum_{j=1}^{m} a_j^* x_j a_j) = \max_{i=1}^{n} f_i(\sum_{j=1}^{m} a_j^* x_j a_j) \leq \max_{i=1}^{n}(\sum_{j=1}^{m} a_j^* f_i(x_j) a_j) \leq \sum_{j=1}^{m} a_j^* f(x_j) a_j.$$

Example 3.3. It is well known that, the affine function in the usual sense, is of the form $f(x) = a_0 x + b_0$, where $a_0, b_0 \in \mathbb{R}$. We can extend this result to the $*$-rings as following;
If $a, b \in Z(\mathcal{S})$, the center of $\mathcal{S}$, then the map $f(x) = ax + b$ is a C^*-affine map on $\mathcal{S}$. Moreover, if a be invertible in $\mathcal{S}$, then f is also an invertible C^*-affine map on $\mathcal{S}$.

Example 3.4. (1) Every bi-sided ideal of $\mathcal{S}$ is a C^*-convex subset of $\mathcal{S}$.
(2) Let Y be a Banach space, and $\mathcal{B}(Y)$ be the $*$-algebra of all bounded operators on Y. We known that, the set $\mathcal{B}_0(Y)$, of all compact operators on Y, is a closed two sided ideal of $\mathcal{B}(Y)$. Therefore, $\mathcal{B}_0(Y)$ is a C^*-convex subset of $\mathcal{B}(Y)$.

Example 3.5. We know that the trace function is an affine function in the classical sense. Here, we give a similar conclusion in the context of $*$-rings. Let $M_n(\mathcal{S})$ be the $*$-ring of all n by n matrices with elements in $\mathcal{S}$ as mentioned in preliminaries. Also, we consider the trace map as usual

$$\begin{aligned} tr : & M_n(\mathcal{S}) \longrightarrow \mathcal{S} \\ & [a_{ij}] \longmapsto \sum_{i=1}^{n} a_{ii}, \end{aligned}$$

and for each $a, b \in \mathcal{S}$, $a.[a_{ij}].b = [a.a_{ij}.b]$. Then, the trace map, tr, is a C^*-affine map on $M_n(\mathcal{S})$, in the following sense,

$$tr(\sum_{i=1}^{n} a_i^* A_i a_i) = \sum_{i=1}^{n} a_i^* tr(A_i) a_i,$$

where, $A_i \in M_n(\mathcal{S})$, $a_i \in \mathcal{S}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{S}}$.

In the next example we show that the condition commutativity is essential in the Theorem 2.7 and Corollary 2.8.

Example 3.6. Let $\mathcal{S}$ be the $*$-ring of all 2×2 real matrices. Then $f(x) = x^*$ is a C^*-affine map on $\mathcal{S}$. Also, suppose that $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, then $A^* A = I_2$. We show that $g(.) = A^* f(.) A$ is not neither a C^*-affine map nor a C^*-convex map on $\mathcal{S}$, and hence the set of all C^*-affine maps (C^*-convex maps) is not a C^*-convex set in the sense of the theorem 2.7. To see this, we consider the following C^*-convex combination of elements $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, and $Y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$ in $\mathcal{S}$, with the condition $x_1 \neq y_1$.

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then,

$$\begin{aligned} g(Z) &= A^* f(Z) A \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 & y_3 \\ y_2 & y_4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{2}\left(\begin{bmatrix} x_1 & -x_1 \\ -x_1 & x_1 \end{bmatrix} + \begin{bmatrix} y_1 & y_1 \\ y_1 & y_1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} x_1 + y_1 & -x_1 + y_1 \\ -x_1 + y_1 & x_1 + y_1 \end{bmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} g\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} g\left(\begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} y_1 & y_3 \\ y_2 & y_4 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\right)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \frac{1}{2}\left(\begin{bmatrix} x_1 + x_2 + x_3 + x_4 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & y_1 + y_2 + y_3 + y_4 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} x_1 + x_2 + x_3 + x_4 & 0 \\ 0 & y_1 + y_2 + y_3 + y_4 \end{bmatrix}. \end{aligned}$$

By the assumption $x_1 \neq y_1$, g is not a C^*-affine map on $\mathcal{S}$. Also, g is not a C^*-convex map on $\mathcal{S}$.

Open problem

Give an example of a noncommutative unital $*$-ring $\mathcal{S}$ and a C^*-convex subset B of $\mathcal{S}$ such that the set of all C^*-affine maps on B is not a C^*-face of the set of all C^*-convex maps on B.

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