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The Banach algebras with generalized matrix representation

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ABSTRACT

A Banach algebra $\mathfrak{A}$ has a generalized Matrix representation if there exist the algebras A, B, (A, B)-module M and (B, A)-module N such that $\mathfrak{A}$ is isomorphic to the generalized matrix Banach algebra $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$. In this paper, the algebras with generalized matrix representation will be characterized. Then we show that there is a unital permanently weakly amenable Banach algebra A without generalized matrix representation such that $H^1(A, A) = \{0\}$. This implies that there is a unital Banach algebra A without any triangular matrix representation such that $H^1(A, A) = \{0\}$ and gives a negative answer to the open question of [2].

1. Introduction and Preliminaries

Let A be a Banach algebra and let $\mathcal{X}$ be a Banach A-module. The dual $\mathcal{X}^*$ of $\mathcal{X}$ together with the actions $(fa)(x) = f(ax)$ and $(af)(x) = f(xa)$ is a Banach A-module where $a \in A$, $x \in \mathcal{X}$ and

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$f \in \mathcal{X}^*$ and the second dual $\mathcal{X}^{**}$ of $\mathcal{X}$ under the actions $(aF)(f) = F(fa)$ and $(Fa)(f) = F(af)$ is a Banach A-module, where $a \in A$, $f \in \mathcal{X}^*$ and $F \in \mathcal{X}^{**}$. Similarly the n-th dual $\mathcal{X}^{(n)}$ of $\mathcal{X}$ is a Banach A-module. In particular when $\mathcal{X} = A$, $A^{(n)}$ is a Banach A-module.

A derivation D is a bounded linear operator $D : A \to \mathcal{X}$ such that $D(ab) = D(a)b + aD(b)$, for all $a, b \in A$. A derivation D is called inner if there exists $x \in \mathcal{X}$ such that $D(a) = \delta_x(a) = ax - xa$, for all $a \in A$. The cohomology group of A with coefficients in $\mathcal{X}$ is denoted by $H^1(A, \mathcal{X}) = \frac{Z^1(A,\mathcal{X})}{N^1(A,\mathcal{X})}$, where $Z^1(A, \mathcal{X})$ is the linear space of bounded derivations from A into $\mathcal{X}$, and $N^1(A, \mathcal{X})$ is the linear subspace of bounded inner derivations from A into $\mathcal{X}$.

The notion of weak amenability for a commutative Banach algebra A was defined by Bade et al [1]. In 1998, the concept of n-weak amenability and permanent weak amenability were introduced by Dales et al [7]. They called A, n-weakly amenable if $H^1(A, A^{(n)}) = 0$, and A is said to be permanently weakly amenable if it is n-weakly amenable for all $n \geq 1$.

Let A and X be Banach algebras and $\Theta : X \times X \to A$ be a bounded bilinear mapping. If also X is an algebraic Banach $A-$module with respect to Θ, which is a Banach $A-$module with compatible operations, that is for each $a, a' \in A$ and $x, x', x'' \in X$

$$a\Theta(x, x') = \Theta(ax, x'), \Theta(x, x')a = \Theta(x, x'a), \Theta(xa, x') = \Theta(x, ax'), \Theta(xx', x'') = \Theta(x, x'x'') \quad \text{in A},$$

and

$$(xx')a = x(x'a), a(xx') = (ax)x', (xa)x' = x(ax'), \Theta(x, x')x'' = x\Theta(x', x'') \quad \text{in X},$$

then the direct product $A \times X$ as a linear space with the product and the norm defined by

$$(a, x)(a', x') = (aa' + \Theta(x, x'), ax' + xa' + xx') \text{ and } \|(a, x)\| = \|a\| + \|x\| \qquad (a, a' \in A, x, x' \in X),$$

is a Banach algebra which is called the generalized bi-amalgamated Banach algebra with respect to Θ and it is denoted by $A \boxtimes_\Theta X$. For more details about this Banach algera see [9]. The generalized module extension Banach algebras, Lau product of Banach algebras, module extension Banach algebras, unitization of Banach algebras and direct product of Banach algebras are the main examples of generalized bi-amalgamated Banach algebras; see for more details about this Banach algebras [12], [14], [10], [5] and [6]. When the product of X is zero, we denote $A \boxtimes_\Theta X$ by $A \oplus_\Theta X$.

One another example of the generalized bi-amalgamated Banach algebras is the generalized matrix Banach algebra. Consider two Banach algebras A and B, two (A, B)-module M (i.e. a left A-module and a right B-module with compatible actions) and (B, A)-module N (i.e. a left B-module and a right A-module with compatible actions) and two bilinear mappings $\Phi : M \times N \to A$ and $\Psi : N \times M \to B$ which are bimodule morphisms on each of their coordinates and satisfying the following equalities.

$$m(nm') = (mn)m' \quad and \quad n(mn') = (nm)n' \qquad (n, n' \in N, m, m' \in M);$$

Where $mn := \Phi(m, n)$ and $nm = \Psi(n, m)$. If we impose the additional assumption $\|m.n\| \leq \|m\|\|n\|$, then $G = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ becomes a Banach algebra with norm given by

$$\left\| \begin{matrix} a & m \\ n & b \end{matrix} \right\|_G = \|a\| + \|m\| + \|b\| + \|n\|,$$

which is called a generalized matrix Banach algebra. Then we can identify G under matrix-like addition and matrix-like multiplication with the Banach algebra $(A \times B) \oplus_{\Theta} (M \times N)$, where $A \times B$ is the direct product of Banach algebras A and B, and $M \times N$ as a direct product of modules is a Banach $(A \times B)-$module with the module actions

$$(a,b).(m,n) = (am, bn), \quad (m,n).(a,b) = (mb, na) \quad (a \in A, b \in B, m \in M, n \in N),$$

and $\Theta : (M\times N)\times(M\times N) \to A\times B$ is defined by $\Theta((m,n),(m',n')) = (mn', nm')$. For more details see [9].
Generalized matrix algebras were introduced by Sands in [13]. Obviously, when $M = 0$ or $N = 0$, G exactly degenerates to the so-called triangular algebra. When $\Phi = 0$ and $\Psi = 0$ such kind of generalized matrix algebras are called *trivial generalized matrix algebras*. For example the module extension Banach algebra $(A \dot{+} B) \oplus (M \dot{+} N)$, which is defined in [14], where $(A \dot{+} B)$ is the direct l_1-sum of Banach algebras A and B, and $(M \dot{+} N)$ as an l_1-direct sum of modules is an $(A \dot{+} B)-$module with the module actions

$$(a,b).(m,n) = (am, bn), \quad (m,n).(a,b) = (mb, na) \quad (a \in A, b \in B, m \in M, n \in N),$$

is a trivially generalized matrix algebra.
Another common examples of generalized matrix algebras are full matrix algebras and inflated algebras over a unital algebra. See [11] for more examples. These generalized matrix algebras mainly come from matrix theory and operator theory.

An algebra $\mathfrak{A}$ is called to have a generalized Matrix representation if there exist the algebras A, B, (A,B)-module M and (B,A)-module N such that $\mathfrak{A}$ is isomorphic to $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$ as an algebra.

We can consider every Banach algebra as a (trivial) generalized matrix Banach algebra with $M = 0, N = 0$ and $B = 0$. In this paper, we investigate the Banach algebras which have the (non-trivial) generalized matrix representation.

2. The Banach algebras with generalized matrix representation

Theorem 2.1. *A unital Banach algebra $\mathfrak{A}$ has a generalized Matrix representation if and only if it has a non trivial idempotent.*

Proof. Let the algebra $\mathfrak{A}$ have a non trivial idempotent e. Then since $(1-e)^2 = (1-e)$, the linear mapping

$$\psi : \mathfrak{A} \to \begin{bmatrix} e\mathfrak{A}e & e\mathfrak{A}(1-e) \\ (1-e)\mathfrak{A}e & (1-e)\mathfrak{A}(1-e) \end{bmatrix}$$

which is defined by

$$\Psi(a) = \begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix}, \quad (a \in \mathfrak{A}),$$

is a homomorphism. Now, since for each $a \in \mathfrak{A}$ we have

$$\begin{aligned} a &= ae + a(1-e) \\ &= eae + (1-e)ae + ea(1-e) + (1-e)a(1-e), \end{aligned}$$

Ψ is one to one. Also since $(1-e)^2 = (1-e)$ and $e(1-e) = 0 = (1-e)e$, we have for each $\alpha, \beta, \gamma, \lambda \in \mathfrak{A}$ and for $a = e\alpha e + e\beta(1-e) + (1-e)\gamma e + (1-e)\lambda(1-e)$,

$$\begin{bmatrix} e\alpha e & e\beta(1-e) \\ (1-e)\gamma e & (1-e)\lambda(1-e) \end{bmatrix} = \begin{bmatrix} eae & ea(1-e) \\ (1-e)ae & (1-e)a(1-e) \end{bmatrix}.$$

Therefore Ψ is onto and so $\mathfrak{A}$ is isomorphic to $\begin{bmatrix} e\mathfrak{A}e & e\mathfrak{A}(1-e) \\ (1-e)\mathfrak{A}e & (1-e)\mathfrak{A}(1-e) \end{bmatrix}$.
Conversely if $\mathfrak{A}$ is isomorphic to the unital algebra $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$, for some algebras A, B, (A, B)-module M and (B, A)-module N, then we can consider $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as a non trivial idempotent of $\mathfrak{A}$. □

Note that in Theorem 2.1, we have for each $a \in \mathfrak{A}$

$$\begin{aligned} \|a\| &= \|eae + (1-e)ae + ea(1-e) + (1-e)a(1-e)\| \\ &\leq \|eae\| + \|(1-e)ae\| + \|ea(1-e)\| + \|(1-e)a(1-e)\| \\ &= \|\Psi(a)\| \\ &\leq 4M^2\|a\|. \end{aligned}$$

Where $M = Max\{\|e\|, \|(1-e)\|\}$. That is these two norms can be considered equivalent.
Also if $eae = a$ for each $a \in \mathfrak{A}$, and $e\mathfrak{A}(1-e), (1-e)\mathfrak{A}e$ and $(1-e)\mathfrak{A}(1-e)$ are zero, then $\mathfrak{A}$ has the trivial generalized Matrix representation. In this case, the equalities $ea(1-e) = 0 = (1-e)ae$ and $eae = a$, for each $a \in \mathfrak{A}$, imply that $ea = ae = eae = a$, and so $e = 1_A$.

Corollary 2.2. *[3, Theorem 5.1.4] A unital Banach algebra $\mathfrak{A}$ has a triangular representation if and only if it has a non trivial idempotent e such that $(1-e)\mathfrak{A}e = 0$.*

Theorem 2.3. *Let A be a unital Banach algebra. Then the generalized bi-amalgamated Banach algebra $A \boxtimes_\Theta X$ has a generalized Matrix representation if and only if there is a non trivial element $(e, x) \in A \boxtimes_\Theta X$ such that $e^2 + \Theta(x, x) = e$ and $ex + xe + x^2 = x$.*

Proof. Since A is unital, $A \boxtimes_\Theta X$ is also unital with the unit $(1_A, 0)$. Now Theorem 2.1 implies that $A \boxtimes_\Theta X$ has a generalized Matrix representation if and only if it has a non trivial idempotent (e, x) if and only if

$$(e, x) = (e, x)(e, x) = (e^2 + \Theta(x, x), ex + xe + x^2),$$

if and only if $e^2 + \Theta(x, x) = e$ and $ex + xe + x^2 = x$. □

Corollary 2.4. *Let A be a unital Banach algebra and let X be a unital Banach A-module. Then the module extension Banach algebra $A \oplus X$ has a generalized Matrix representation if and only if A has a generalized Matrix representation.*

Proof. If the module extension Banach algebra $A \oplus X$ has a generalized Matrix representation, then since $\Theta = 0$, by Theorem 2.3 there are two elements $e \in A$ and $x \in X$ such that $e^2 = e$ and $ex + xe = x$. But obviously if e is trivial, then the second equality does not hold. Therefore A has a non trivial idempotent e and by Theorem 2.1 A has a generalized Matrix representation.
Conversely if A has a generalized Matrix representation, then by Theorem 2.1 it has a non trivial idempotent e. Now $(e, 0)$ is a non trivial idempotent of $A \oplus X$ and Theorem 2.1 implies that $A \oplus X$ has a generalized Matrix representation. □

Denote the idempotents of the algebra A by $I(A)$. Then we get the following corollary.

Corollary 2.5. *$I(A \boxtimes_\Theta X) = \emptyset$ if and only if $a^2 + \Theta(x, x) \neq a$, for each $a \in A$ and $x \in X$.*

Proof. Similar to Theorem 2.3 $I(A \boxtimes_\Theta X) = \emptyset$ if and only if for each $a \in A$ and $x \in X$, $a^2 + \Theta(x, x) \neq a$ or $ax + xa + x^2 \neq x$. But the second equation is not true for some $x \in X$ such as $x = 0$. Therefore $I(A \boxtimes_\Theta X) = \emptyset$ if and only if for each $a \in A$ and $x \in X$, $a^2 + \Theta(x, x) \neq a$. □

The latter corollary implies that if $I(A \boxtimes_\Theta X) = \emptyset$, then in particular for each $a \in A$ and $x = 0$ we have $a^2 \neq a$. Therefore if $I(A \boxtimes_\Theta X) = \emptyset$, then $I(A) = \emptyset$. In particular if $\Theta = 0$ we have $I(A \boxtimes_\Theta X) = \emptyset$ if and only if $I(A) = \emptyset$.

Proposition 2.6. *If e is a non trivial idempotent of A, then $A \boxtimes_\Theta X$ is isomorphic to*

$$\begin{bmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{bmatrix} \oplus \begin{bmatrix} eXe & eX(1-e) \\ (1-e)Xe & (1-e)X(1-e) \end{bmatrix}.$$

Proof. If e is a non trivial idempotent of A, then $E = (e, 0)$ is a non trivial idempotent of $S = A \boxtimes_\Theta X$ and so it is isomorphic to

$$\begin{bmatrix} ESE & ES(1-E) \\ (1-E)SE & (1-E)S(1-E) \end{bmatrix} = \begin{bmatrix} (eAe, eXe) & (eA(1-e), eX(1-e)) \\ ((1-e)Ae, (1-e)Xe) & ((1-e)A(1-e), (1-e)X(1-e)) \end{bmatrix}$$
$$= \begin{bmatrix} eAe & eA(1-e) \\ (1-e)Ae & (1-e)A(1-e) \end{bmatrix} \oplus \begin{bmatrix} eXe & eX(1-e) \\ (1-e)Xe & (1-e)X(1-e) \end{bmatrix}$$

□

3. A permanently weekly amenable Banach algebra without generalized matrix representation

At the beginning of this section, we prove the following Lemma. For roughly similar results of this lemma see [8, Propositions 2.3 and 2.4].

Lemma 3.1. *For each unimodular locally compact qroup G, $L^1(G) * C_0(G) \subseteq C_0(G)$ and $C_0(G) * L^1(G) \subseteq C_0(G)$.*

Proof. for each $x \in G$, $f \in L^1(G)$ and $g \in C_0(G)$ we have

$$\begin{aligned} |f * g(x)| &= |\int_G f(y)g(y^{-1}x)dy| \\ &\leq \int_G |f(y)||g(y^{-1}x)|dy \\ &\leq ||g||_u \int_G |f(y)|dy \\ &= ||g||_u ||f||_1. \end{aligned}$$

Where $||g||_u = sup\{|g(x)|; x \in G\}$ and $||f||_1 = \int_G |f(y)|dy$. Therefore we conclude that

$$||f * g||_u \leq ||f||_1 ||g||_u \quad (f \in L^1(G), g \in C_0(G)).$$

Similarly since G is unimodular, we can show that

$$||g * f||_u \leq ||f||_1 ||g||_u \quad (f \in L^1(G), g \in C_0(G)).$$

Now since $C_0(G) = \overline{C_c(G)}^{||\cdot||_u}$ and $L^1(G) = \overline{C_c(G)}^{||\cdot||_1}$, for each $f \in L^1(G)$ and $g \in C_0(G)$ there are the bounded subnets $\{f_\alpha\}$ and $\{g_\beta\}$ in $C_c(G)$ such that $f_\alpha \xrightarrow{||\cdot||_1} f$ and $g_\beta \xrightarrow{||\cdot||_u} g$. Now we have

$$\begin{aligned} ||f * g - f_\alpha * g_\beta||_u &= ||(f - f_\alpha) * g - f_\alpha * (g_\beta - g)||_u \\ &\leq ||(f - f_\alpha) * g||_u + ||f_\alpha * (g_\beta - g)||_u \\ &\leq ||f - f_\alpha||_1 ||g||_u + ||f_\alpha||_1 ||g_\beta - g||_u \\ &\to 0 \end{aligned}$$

That is $f_\alpha * g_\beta \xrightarrow{||\cdot||_u} f * g$ and so $f * g \in \overline{C_c(G)}^{||\cdot||_u} = C_0(G)$. Similarly we may show that $g * f \in C_0(G)$. □

Now since each discerete group is unimodular, we get the following theorem.

Theorem 3.2. *For each group algebra $\ell^1(G)$ of a discerete group G (with convolution) and each integer $n \geq 0$, we have $H^1(\ell^1(G), \ell^1(G)^{(n)}) = \{0\}$.*

Proof. For each $n \geq 1$ it follows by [7] and [15]. Since $\ell^1(G) = c_0(G)^*$, by [7] we have $\ell^1(G)^{**} = \ell^1(G) \oplus c_0(G)^\perp$. Now let $D : \ell^1(G) \to \ell^1(G)$ be a derivation and $J : \ell^1(G) \to \ell^1(G)^{**}$ be the inclusion map. Then $J \circ D : \ell^1(G) \to \ell^(G)^{**}$ is a derivation and since $\ell^1(G)$ is 2-weakly amenable, there is $F \in \ell^1(G)^{**}$ such that $J \circ D = \delta_F$. On the other hand we can consider $F = t + s$, for some $t \in \ell^1(G)$ and $s \in c_0(G)^\perp$. So Lemma 3.1 implies that for each $f \in \ell^1(G)$ and $g \in c_0(G)$,

$$\begin{aligned} \langle D(f), g \rangle &= \langle J \circ D(f), g \rangle \\ &= \langle \delta_F(f), g \rangle \\ &= \langle \delta_{t+s}(f), g \rangle \\ &= \langle \delta_t(f), g \rangle. \end{aligned}$$

Therefore $H^1(\ell^1(G), \ell^1(G)) = \{0\}$. □

The following theorem gives a negative answer to the open question raised in [2].

Theorem 3.3. *There is a unital Banach algebra A without any generalized matrix representation such that* $H^1(A, A^{(n)}) = \{0\}$*, for each integer* $n \geq 0$.

Proof. Let G be a free group on 2 generators. Then $\ell^1(G)$ does not have any nontrivial idempotent by [4, Theorem 1]. Therefore Theorem 2.1 implies that $\ell^1(G)$ does not have a generalized matrix representation. On the other hand, Theorem 3.2 says that $H^1(\ell^1(G), \ell^1(G)^{(n)}) = \{0\}$, for each integer $n \geq 0$. □

In particular, If we put $N = 0$ in the matrix representation, then we conclude that there is a unital Banach algebra A without any triangular matrix representation such that $H^1(A, A) = \{0\}$, which answers to the question of [2].

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