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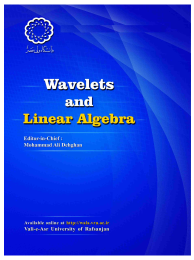

Convex functions on compact C^*-convex sets

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Abstract

It is well known that if a real valued convex function on a compact convex domain contained in the real numbers attains its maximum, then it does so at least at one extreme point of its domain. In this paper, we consider a matrix convex function on a compact and C^*-convex set generated by self–adjoint matrices. An important issue is so that this function on a compact and C^*-convex domain attains its maximum at a C^*-extreme point.

1. Introduction and Preliminaries

Matrix functions have played an important role in scientific computing and engineering. Well known examples of matrix function include $\sqrt{A}$ (the square root function of a positive matrix), and

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e^A (the exponential function of a square matrix). For any real valued function f, a corresponding matrix valued function $f(A)$ can be defined on the space of self–adjoint matrices by the spectral theorem and applying f to the eigenvalues in the spectral decomposition of A.

Let M_n denote the set of $n \times n$ matrices with complex entries, and let $H_n(\mathbb{I})$ denote the set of $n \times n$ self–adjoint matrices with eigenvalues in the interval $\mathbb{I}$. The identity matrix I_n will be denoted simply by 1, and correspondingly, a scalar λ will represent $\lambda 1$. For self–adjoint matrices A, B the order relation $A \le B$ means that $B - A$ is positive.

We consider the spectral representation of A given by $A = \sum_{i=1}^{m} \lambda_i P_i$, where $\lambda_1, ..., \lambda_m$ are the eigenvalues of A (not counting multiplicity) and $P_1, ..., P_m$ are orthogonal projections with the identity matrix as sum. This representation is unique and $f(A) = \sum_{i=1}^{m} f(\lambda_i) P_i$. The functional calculus can be extended to self–adjoint operators acting on an infinite-dimensional Hilbert space, but we will consider the theory only for matrices.

Let f be a real valued function defined on the interval $\mathbb{I}$ of the real line. The function f is said to be convex if

$$f(ca + (1-c)b) \le cf(a) + (1-c)f(b) \ \ (a, b \in \mathbb{I}, 0 \le c \le 1).$$

Let $A \in H_n(\mathbb{I})$ and its eigenvalues λ_j contained in $\mathbb{I}$. We can choose a unitary matrix $U \in M_n$ such that $A = U^* D U$ and $D = diag(\lambda_1, ..., \lambda_n)$ is diagonal and then define $f(A) = U^* f(D) U$, where $f(D) = diag(f(\lambda_1), ..., f(\lambda_n))$. The function f is said to be matrix convex if the map $A \longmapsto f(A)$ is convex on $H_n(\mathbb{I})$ in the sense that

$$f(cA + (1-c)B) \le cf(A) + (1-c)f(B) \ \ (A, B \in H_n(\mathbb{I}), 0 \le c \le 1).$$

Also, f is matrix concave if $-f$ is matrix convex (see [7], [1]).

2. C^*-convex sets

Recently, a notion of C^*-convexity has been studied by Farenick, Morenz [4, 6] and Magajna [8, 9]. C^*-convexity is the natural extension of the classical scalar-valued convex combination to include C^*-algebra valued coefficient. It therefore makes sense in a C^*-algebra and, more generally, for bimodules over C^*-algebras. In particular, there is a rich class of such C^*-convex sets in the $n \times n$ complex matrices. The matrix state spaces of a C^*-algebra are another class of examples.

A set $S \subset M_n$ is called C^*-convex, if S is closed under the formation of finite sums of the type $\sum_i T_i^* A_i T_i$, where $T_i \in M_n$, $A_i \in S$ and $\sum_i T_i^* T_i = 1$. This formation of finite sums is called C^*-convex combination in S and the T_i are called C^*-convex coefficients. If the coefficients T_i are invertible in M_n, then they are called proper C^*-convex coefficients and the C^*-convex combination is called a proper C^*-convex combination. A point A in the C^*-convex set S is a C^*-extreme point, if $A = \sum_i T_i^* A_i T_i$ is a proper C^*-convex combination of elements $A_i \in S$, then every A_i comes from the unitary orbit of A, i. e., for every i there exists a unitary element $U_i \in M_n$ such that $A = U_i^* A_i U_i$ (see [6]).

A point A in a compact and C^*-convex set $S \subset M_n$ is a structural element of size n, if whenever $A = \sum_i T_i^* A_i T_i$ is a C^*-convex combination of elements of S, then there exist unitary elements $U_i \in M_n$ and scalars $\lambda_i \in [0, 1]$ such that, $A = U_i^* A_i U_i$, $T_i = \lambda_i U_i$ and $\sum_i \lambda_i^2 = 1$. Following

[10] we write $A \in \mathsf{str}(S, n)$. It is an immediate consequence that the structural elements of S of size n coincide with the irreducible C^*-extreme points. It is possible for a C^*-convex set has no structural elements of size n. So, we need to define structural elements of size less than n. Let S_k be the compression of S to M_k. The point $A \in M_k$ is called a structural element of size k, if A is a structural element of S_k of size k and A is not equal with the compressions of structural elements of size j, $(k < j \leq n)$ to M_k. We show the structure set of S by $\mathsf{str}(S) = \bigcup_{k=1}^{n} \mathsf{str}(S, k)$. The definition of $\mathsf{str}(S)$ implies the elements of $\mathsf{str}(S)$ may not be all the same size. Thus, we extend the structural elements to $n \times n$ matrices. If $A \in M_k$ is a structural element of size k $(k < n)$, then we denote the extension of A to M_n by $A(n)$ and we define, $A(n) = A \oplus \lambda 1_{n-k}$, where λ is extreme in the numerical range of A. The numerical range of A is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\},$$

where $\langle \cdot, \cdot \rangle$ is the inner product on the Hilbert space $\mathbb{C}^n$. By [10, Corollary 5.3], any structural element can be extended to a C^*-extreme point. Therefore, $A(n)$ is a C^*-extreme point of S (for more details we refer to [10]).

The C^*-convex hull of a subset $\mathsf{L} \subset M_n$ is the smallest C^*-convex set containing L. An essential fact is that C^*-convex hull of L is compact whenever L is a compact subset of M_n [4]. A Carathèodory theorem for convex sets in finite dimensions says that every point in a convex set S contained in an n-dimensional (real) linear space is a convex combination of at most $n + 1$ extreme points of S. Morenz showed in [10] the following Carathèodory type theorem for C^*-convex sets in M_n.

Proposition 2.1. *Let $S \subset M_n$ be compact and C^*-convex and let $A \in S$. Then, A is a C^*-convex combination of at most $3n^2$ elements of $\mathsf{str}(S)$.*

3. The Main Results

In this section we are going to consider a matrix convex function on a C^*-convex set generated by a compact set of self–adjoint matrices and we obtain an upper bound for this function at a scalar C^*-extreme point. In the usual manner the same result holds for a matrix concave function.

Theorem 3.1. *Suppose that $S \subset M_n$ is the C^*-convex hull of a compact set L of self–adjoint matrices and let the closed interval $[\alpha, \beta]$ be the convex hull of the spectra of L. Then, $A \in S$ is C^*-extreme in S if and only if A is either a scalar matrix with scalars α, β or unitary equivalent to the diagonal matrix $\begin{pmatrix} \alpha 1 & 0 \\ 0 & \beta 1 \end{pmatrix}$.*

Proof. This follows from [6, Corollary 4.2]. □

We consider the standard norm $\|\cdot\|$ on the Hilbert space $\mathbb{C}^n$. For $A \in M_n$, we denote by $\|A\|$ the operator (bound) norm of $\|A\|$ defined as

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$

We are recalling the following two theorems from [2]. Note that $T \in M_n$ is a contraction whenever $\|T\| \leq 1$.

Theorem 3.2. *Let f be a convex function, let T be a contraction and set $X = f(T^*AT)$ and $Y = T^*f(A)T$ for $A \in H_n(\mathbb{I})$. Then, there exist unitaries U, V such that*

$$X \leq \frac{UYU^* + VYV^*}{2}.$$

Theorem 3.3. *Let f be a convex function and set $X = f(\sum_{i=1}^m T_i^* A_i T_i)$ and $Y = \sum_{i=1}^m T_i^* f(A_i) T_i$ for $\{A_i\}_{i=1}^m \subset H_n(\mathbb{I})$, where $\sum_{i=1}^m T_i^* T_i = 1$. Then, there exist unitaries U, V such that*

$$X \leq \frac{UYU^* + VYV^*}{2}.$$

We now prove the main results.

Theorem 3.4. *Suppose that f is a convex function on $[\alpha, \beta]$.*

(a) If $S \subset M_n$ is a compact and C^*-convex set of self–adjoint matrices with spectra in the closed interval $[\alpha, \beta]$, then f has an upper bound on S.

(b) If $S \subset M_n$ is the C^*-convex hull of a compact set L of self–adjoint matrices and the closed interval $[\alpha, \beta]$ is the convex hull of the spectra of L, then f attains its upper bound at a scalar C^*-extreme point of S.

Proof. (a) Let A be an arbitrary element of S. Then, the Caratheòdory type theorem shows that $A = \sum_{i \in I} T_i^* A_i(n) T_i$ provided that $\sum_{i \in I} T_i^* T_i = 1$, $T_i \in M_n$, each $A_i(n)$ is C^*-extreme in S and $I \subseteq \{1, 2, ..., 3n^2\}$. It follows from Theorem 3.1 that either $A_i(n) = \alpha 1, \beta 1$ or there exist unitaries U_i such that $A_i(n) = U_i^* \begin{pmatrix} \alpha 1 & 0 \\ 0 & \beta 1 \end{pmatrix} U_i$. Define

$$I_1 := \{i \in I : A_i(n) = \alpha 1\}, \ I_2 := \{i \in I : A_i(n) = \beta 1\},$$

and

$$I_3 := \{i \in I : A_i(n) = U_i^* \begin{pmatrix} \alpha 1 & 0 \\ 0 & \beta 1 \end{pmatrix} U_i\}.$$

It is clear that $f(\alpha 1) = f(\alpha)1$. Without loss of generality we may assume that $f(\beta) \leq f(\alpha)$. It then follows that $f(A_i(n)) \leq f(\alpha)1$ for $i \in I_1 \cup I_2$. Assume that $i \in I_3$. Apply Theorem 3.2 and the convexity of f to obtain unitaries V_i and W_i such that

$$\begin{aligned} f(A_i(n)) &= f(U_i^* \begin{pmatrix} \alpha 1 & 0 \\ 0 & \beta 1 \end{pmatrix} U_i) \\ &\leq \frac{1}{2}\left\{ V_i U_i^* \begin{pmatrix} f(\alpha)1 & 0 \\ 0 & f(\beta)1 \end{pmatrix} U_i V_i^* + W_i U_i^* \begin{pmatrix} f(\alpha)1 & 0 \\ 0 & f(\beta)1 \end{pmatrix} U_i W_i^* \right\} \\ &\leq \frac{1}{2}\left\{ V_i U_i^* \begin{pmatrix} f(\alpha)1 & 0 \\ 0 & f(\alpha)1 \end{pmatrix} U_i V_i^* + W_i U_i^* \begin{pmatrix} f(\alpha)1 & 0 \\ 0 & f(\alpha)1 \end{pmatrix} U_i W_i^* \right\} \\ &= f(\alpha)1. \end{aligned}$$

Hence, $f(A_i(n)) \leq f(\alpha)1$ for $i \in I_3$. By using Theorem 3.3 and the self–adjointness of $A_i(n)$ we get there exist unitaries U and V such that

$$\begin{aligned} f(A) &= f(\sum_{i\in I} T_i^* A_i(n) T_i) \\ &\leq \frac{1}{2}\left\{ U(\sum_{i\in I} T_i^* f(A_i(n))T_i)U^* + V(\sum_{i\in I} T_i^* f(A_i(n))T_i)V^* \right\} \\ &\leq f(\alpha)1. \end{aligned}$$

This means that $f(\alpha)1$ is the upper bound of f on S.

(b) Note that α is extreme in the convex hull of the spectra of L and so Theorem 3.1 asserts that the scalar matrix $\alpha 1$ is C^*-extreme in S. On the other hand, $f(\alpha)1 = f(\alpha 1)$, i.e., f attains this upper bound at $\alpha 1$. □

Corollary 3.5. *Suppose that $S \subset M_n$ is the C^*-convex hull of a compact set L of self–adjoint matrices and let the close interval $[\alpha, \beta]$ be the convex hull of the spectra of L. If f is convex on $[\alpha, \beta]$, then*

$$\|f(A)\| \leq \min\{|f(\alpha)|, |f(\beta)|\}$$

for all $A \in S$. Moreover, there exists $A_0 \in S$ such that A_0 is C^*-extreme in S and

$$\|f(A_0)\| = \min\{|f(\alpha)|, |f(\beta)|\}.$$

Proof. According to Theorem 3.4(a) either $f(\alpha)1$ or $f(\beta)1$ is the upper bound of f on S. Hence, $\|f(A)\| \leq \|f(\alpha)1\| = |f(\alpha)|$ or $\|f(A)\| \leq |f(\beta)|$ for every $A \in S$ and so $\|f(A)\| \leq \min\{|f(\alpha)|, |f(\beta)|\}$ for every $A \in S$. We remarked in Theorem 3.4(b) that f attains its upper bound at either $\alpha 1 \in S$ or $\beta 1 \in S$. Without loss of generality we may assume that $f(\beta) \leq f(\alpha)$. Define $A_0 = \beta 1$. Then, $\|f(A_0)\| = \|f(\beta 1)\| = |f(\beta)| = \min\{|f(\alpha)|, |f(\beta)|\}$. By the same reasoning as in the proof of Theorem 3.4(b), the matrix A_0 is C^*-extreme in S. □

Bourin [3] remarked that the inequalities in Theorem 3.2 and 3.3 reverse for concave functions. The next corollaries list some consequences of our results for concave functions.

Corollary 3.6. *Suppose that f is a concave function on $[\alpha, \beta]$.*

(a) If $S \subset M_n$ is a compact and C^*-convex set of self–adjoint matrices with spectra in the closed interval $[\alpha, \beta]$, then f has a lower bound on S.

(b) If $S \subset M_n$ is the C^*-convex hull of a compact set L of self–adjoint matrices and the closed interval $[\alpha, \beta]$ is the convex hull of the spectra of L, then f attains its lower bound at a scalar C^*-extreme point of S.

Corollary 3.7. *Suppose that $S \subset M_n$ is the C^*-convex hull of a compact set L of self–adjoint matrices and let the close interval $[\alpha, \beta]$ be the convex hull of the spectra of L. If f is concave on $[\alpha, \beta]$, then*

$$\max\{|f(\alpha)|, |f(\beta)|\} \leq \|f(A)\|$$

for all $A \in S$. Moreover, there exists $A_0 \in S$ such that A_0 is C^*-extreme in S and

$$\|f(A_0)\| = \max\{|f(\alpha)|, |f(\beta)|\}.$$

Proof. According to Corollary 3.6(a) either $f(\alpha)1$ or $f(\beta)1$ is the lower bound of f on S. Hence, $|f(\alpha)| = \|f(\alpha)1\| \leq \|f(A)\|$ or $|f(\beta)| \leq \|f(A)\|$ for every $A \in S$ and so $\max\{|f(\alpha)|, |f(\beta)|\} \leq \|f(A)\|$ for every $A \in S$. We remarked in Corollary 3.6(b) that f attains its lower bound at either $\alpha 1 \in S$ or $\beta 1 \in S$. Without loss of generality we may assume that $f(\beta) \leq f(\alpha)$. Define $A_0 = \alpha 1$. Then, $\|f(A_0)\| = \|f(\alpha 1)\| = |f(\alpha)| = \max\{|f(\alpha)|, |f(\beta)|\}$ and the matrix A_0 is C^*-extreme in S. □

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