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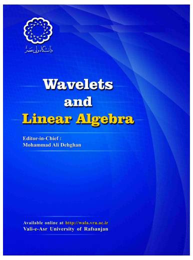

Some New Hermite-Hadamard Type Inequalities for Convex Functions

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Abstract

Convex sets and convex functions play a fundamental role in the development of various fields of pure and applied mathematics. Recently, many new generalizations of inequalities with respect to Hermite-Hadamard have been proposed in the literature. In this paper, some new inequalities of the Hermite-Hadamard type for differentiable convex functions are given. These new inequalities are based on the second derivative functions.

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1. Introduction and Preliminaries

Theory of inequalities is one of the most important applications of convex and abstract analysis, while the common usage within inequalities in convex analysis is Hermite-Hadamard's inequality. The Hermite-Hadamard's inequalities give us an estimate of the (integral) mean value of a continuous convex function. Moreover, equality holds in either side only for affine functions (i.e., for functions of the form $mx + n$). In recent years, much attention have given to develop various inequalities for several classes of convex functions and their generalizations using novel ideas (see [1, 3, 2, 7, 8] and the references therein). Hermite-Hadamard's inequality is given as follows:

Let $f : I \to \mathbb{R}$ be a convex function, and let $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where I is an interval (finite or infinite) in $\mathbb{R}$ Throughout the paper we denote by I^o the interior of I.

The following lemma has been proved in [6], and therefore we omit its proof.

Lemma 1.1. *([6]) Let $f : I^o \to \mathbb{R}$ be a differentiable function, and let $a, b \in I^o$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{(b-x)f(b)+(x-a)f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(t)dt$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 (t-1)f'(tx+(1-t)a)dt + \frac{(b-x)^2}{b-a}\int_0^1 (1-t)f'(tx+(1-t)b)dt.$$

We refer the reader to [1, 3, 2, 5, 4, 7, 8] and the references therein for studying the obtained results and their generalizations related to Hermite-Hadamard's inequality.

2. Main results

In this section, we present our main results. We start with the following lemma.

Lemma 2.1. *Let $f : I^o \to \mathbb{R}$ be a differentiable function, and let $a, b \in I^o$ with $a < b$. If $f'' \in L[a, b]$, then the following equality holds:*

$$\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt$$
$$= \frac{(x-a)^3}{2(b-a)}\int_0^1 t(1-t)f''(tx+(1-t)a)dt + \frac{(x-b)^3}{2(b-a)}\int_0^1 t(t-1)f''(tx+(1-t)b)dt.$$

Proof. By some calculations we obtain the desired result.

$$\begin{aligned}
&\frac{(x-a)^3}{2(b-a)}\int_0^1 t(1-t)f''(tx+(1-t)a)dt + \frac{(x-b)^3}{2(b-a)}\int_0^1 t(t-1)f''(tx+(1-t)b)dt\\
=&\frac{(x-a)^3}{2(b-a)}[\frac{t-t^2}{x-a}f'(tx+(1-t)a)|_0^1 - \frac{1-2t}{(x-a)^2}f(tx+(1-t)a)|_0^1\\
&-\frac{2}{(x-a)^2}\int_0^1 f(tx+(1-t)a)dt]\\
&+\frac{(x-b)^3}{2(b-a)}[\frac{t^2-t}{x-b}f'(tx+(1-t)b)|_0^1 - \frac{(2t-1)}{(x-b)^2}f(tx+(1-t)b)|_0^1\\
&+\frac{2}{(x-b)^2}\int_0^1 f(tx+(1-t)b)dt]\\
&=\frac{(x-a)^3}{2(b-a)}[\frac{f(x)+f(a)}{(x-a)^2} - \frac{2}{(x-a)^3}\int_a^x f(t)dt]\\
&+\frac{(x-b)^3}{2(b-a)}[-\frac{f(x)+f(b)}{(x-b)^2} + \frac{2}{(x-b)^3}\int_b^x f(t)dt]\\
&=\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt.
\end{aligned}$$

□

In the sequel, we present the following result related to Hermite-Hadamard type inequality.

Theorem 2.2. *Let $f : I^o \to \mathbb{R}$ be a differentiable function, and let $a, b \in I^o$ with $a < b$. If $|f''|$ is a convex function on $[a, b]$, then the following inequality holds:*

$$\begin{aligned}
&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt|\\
&\le |\frac{(x-a)^3}{2(b-a)}|(\frac{|f''(x)|+|f''(a)|}{12}) + |\frac{(x-b)^3}{2(b-a)}|(\frac{|f''(x)|+|f''(b)|}{12}).
\end{aligned}$$

Proof. By using Lemma 2.1, we get

$$\begin{aligned}
&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt|\\
&\leq |\frac{(x-a)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)a)|dt\\
&+|\frac{(x-b)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)b)|dt\\
&\leq |\frac{(x-a)^3}{2(b-a)}|\int_0^1 t(1-t)(t|f''(x)|+(1-t)|f''(a)|)dt\\
&+|\frac{(x-b)^3}{2(b-a)}|\int_0^1 t(1-t)(t|f''(x)|+(1-t)|f''(b)|)dt\\
&\leq |\frac{(x-a)^3}{2(b-a)}|(B(3,2)|f''(x)|+B(2,3)|f''(a)|)\\
&+|\frac{(x-b)^3}{2(b-a)}|(B(3,2)|f''(x)|+B(2,3)|f''(b)|)\\
&\leq |\frac{(x-a)^3}{2(b-a)}|(\frac{|f''(x)|+|f''(a)|}{12})+|\frac{(x-b)^3}{2(b-a)}|(\frac{|f''(x)|+|f''(b)|}{12}).
\end{aligned}$$

It should be noted that in the above inequalities we used the following fact:

$$\int_0^1 t^2(1-t)dt = B(3,2) = B(2,3) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{1}{12},$$

where, we have

$$\begin{aligned}
&B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt,\ \ \Gamma(x) := \int_0^{+\infty} e^{-t}t^{x-1}dt,\ \ x>0, y>0, and\\
&B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\end{aligned}$$

□

Corollary 2.3. *In Theorem 2.2, if we take $x := b$ or $x := a$, then we obtain the following inequality.*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{(b-a)^2}{24}(|f''(a)|+|f''(b)|).$$

Corollary 2.4. *In Theorem 2.2, if we put $x := \frac{a+b}{2}$, then one has*

$$|\frac{f(a)+f(b)+2f(\frac{a+b}{2})}{4}-\frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{(b-a)^2}{192}(|f''(a)|+2|f''(\frac{a+b}{2})|+|f''(b)|).$$

Corollary 2.5. *In Corollary 2.4, by using convexity of $|f''|$, we deduce that*

$$|\frac{f(a)+f(b)+2f(\frac{a+b}{2})}{4}-\frac{1}{b-a}\int_a^b f(t)dt| \le \frac{(b-a)^2}{96}(|f''(a)|+|f''(b)|).$$

We now give a similar result to Theorem 2.2 whenever $|f''|^q$ is convex for $q>1$.

Theorem 2.6. *Let $f: I^o \to \mathbb{R}$ be a differentiable function. Let $a,b \in I^o$ with $a<b$. If $|f''|^q$ is a convex function on $[a,b]$ with $q>1$. Then the following inequality holds:*

$$\begin{aligned}&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt|\\&\le |\frac{(x-a)^3}{12(b-a)}|(\frac{|f''(x)|^q+|f''(a)|^q}{2})^{\frac{1}{q}}+|\frac{(x-b)^3}{12(b-a)}|(\frac{|f''(x)|^q+|f''(b)|^q}{2})^{\frac{1}{q}}.\end{aligned}$$

Proof. Again, by using Lemma 2.1, we have

$$\begin{aligned}&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt|\\
&\le |\frac{(x-a)^3}{2(b-a)}|\int_0^1([t(1-t)]^{1-\frac{1}{q}}[t(1-t))]^{\frac{1}{q}}|f''(tx+(1-t)a)|dt\\
&+|\frac{(x-b)^3}{2(b-a)}|\int_0^1[t(1-t)]^{1-\frac{1}{q}}[t(1-t))]^{\frac{1}{q}}|f''(tx+(1-t)b)|dt\\
&\le |\frac{(x-a)^3}{2(b-a)}|[\int_0^1 t(1-t)dt]^{1-\frac{1}{q}}[\int_0^1 t(1-t)|f''(tx+(1-t)a)|^q dt]^{\frac{1}{q}}\\
&+|\frac{(x-b)^3}{2(b-a)}|[\int_0^1 t(1-t)dt]^{1-\frac{1}{q}}[\int_0^1 t(1-t)|f''(tx+(1-t)b)|^q dt]^{\frac{1}{q}}\\
&\le |\frac{(x-a)^3}{2(b-a)}|[B(2,2)]^{1-\frac{1}{q}}[B(3,2)|f''(x)|^q+B(2,3)|f''(a)|^q]^{\frac{1}{q}}\\
&+|\frac{(x-b)^3}{2(b-a)}|[B(2,2)]^{1-\frac{1}{q}}[B(3,2)|f''(x)|^q+B(2,3)|f''(b)|^q]^{\frac{1}{q}}\\
&\le |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{6})^{1-\frac{1}{q}}[\frac{|f''(x)|^q+|f''(a)|^q}{12}]^{\frac{1}{q}}\\
&+|\frac{(x-b)^3}{2(b-a)}|(\frac{1}{6})^{1-\frac{1}{q}}[\frac{|f''(x)|^q+|f''(b)|^q}{12}]^{\frac{1}{q}}\\
&=|\frac{(x-a)^3}{12(b-a)}|[\frac{|f''(x)|^q+|f''(a)|^q}{2}]^{\frac{1}{q}}+|\frac{(x-b)^3}{12(b-a)}[\frac{|f''(x)|^q+|f''(b)|^q}{2}]^{\frac{1}{q}}.\end{aligned}$$

□

Corollary 2.7. *In Theorem 2.6, by choosing $x := b$ or $x := a$, we have the following inequality.*

$$|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt| \le \frac{(b-a)^2}{12}(\frac{|f''(a)|^q+|f''(b)|^q}{2})^{\frac{1}{q}}.$$

Corollary 2.8. *In Theorem 2.6, if we set* $x := \frac{a+b}{2}$*, then we conclude that*

$$
\begin{aligned}
&|\frac{f(a)+f(b)+2f(\frac{a+b}{2})}{4} - \frac{1}{b-a}\int_a^b f(t)dt| \\
&\leq \frac{(b-a)^2}{96}\Big([\frac{|f''(\frac{a+b}{2})|^q + |f''(a)|^q}{2}]^{\frac{1}{q}} + [\frac{|f''(\frac{a+b}{2})|^q + |f''(b)|^q}{2}]^{\frac{1}{q}}\Big).
\end{aligned}
$$

We now present a more general result.

Theorem 2.9. *Let* $f : I \subseteq [0, +\infty) \to \mathbb{R}$ *be a differentiable function such that* $f'' \in L[a, b]$*, where* $a, b \in I^o$ *with* $a < b$*. If* $|f''|^q$ *is a convex function on* $[a, b]$ *with* $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then the following inequality is satisfied.*

$$
\begin{aligned}
&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt| \\
&\leq |\frac{(x-a)^3}{8(b-a)}|[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}(\frac{|f''(x)|^q + |f''(a)|^q}{2})^{\frac{1}{q}} \\
&+ |\frac{(x-b)^3}{8(b-a)}|[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}(\frac{|f''(x)|^q + |f''(b)|^q}{2})^{\frac{1}{q}}.
\end{aligned}
$$

Proof. In view of convexity of f'' and Hölder's inequality, we obtain

$$
\begin{aligned}
&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt| \\
&\leq |\frac{(x-a)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)a)|dt \\
&+ |\frac{(x-b)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)b)|dt \\
&\leq |\frac{(x-a)^3}{2(b-a)}|(\int_0^1 (t^p(1-t)^p))^{\frac{1}{p}}(\int_0^1 |f''(tx+(1-t)a)|^q dt)^{\frac{1}{q}} \\
&+ |\frac{(x-b)^3}{2(b-a)}|(\int_0^1 (t^p(1-t)^p))^{\frac{1}{p}}(\int_0^1 |f''(tx+(1-t)b)|^q dt)^{\frac{1}{q}} \\
&\leq |\frac{(x-a)^3}{2(b-a)}|(B(p+1,p+1))^{\frac{1}{p}}(\frac{|f''(x)|^q+|f''(a)|^q}{2})^{\frac{1}{q}} \\
&+ |\frac{(x-b)^3}{2(b-a)}|(B(p+1,p+1))^{\frac{1}{p}}(\frac{|f''(x)|^q+|f''(b)|^q}{2})^{\frac{1}{q}} \\
&\leq |\frac{(x-a)^3}{8(b-a)}|(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}[\frac{|f''(x)|^q+|f''(a)|^q}{2}]^{\frac{1}{q}} \\
&+ |\frac{(x-b)^3}{8(b-a)}|(\frac{\sqrt{\pi}}{2})^{\frac{1}{p}}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}[\frac{|f''(x)|^q+|f''(b)|^q}{2}]^{\frac{1}{q}} \\
&\leq |\frac{(x-a)^3}{8(b-a)}|[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}[\frac{|f''(x)|^q+|f''(a)|^q}{2}]^{\frac{1}{q}} \\
&+ |\frac{(x-b)^3}{8(b-a)}|[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}[\frac{|f''(x)|^q+|f''(b)|^q}{2}]^{\frac{1}{q}}.
\end{aligned}
$$

Note that in the above we used the following inequality:

$$
\begin{aligned}
\int_0^1 |f''(ta+(1-t)b)|^q dt &\leq |f''(a)|^q \int_0^1 tdt + m|f''(b))|^q \int_0^1 (1-t)dt \\
&= \frac{|f''(a)|^q+|f''(b)|^q}{2}.
\end{aligned}
$$

Moreover, in view of [7], we used the following assertions.

$$
\begin{aligned}
&B(x,x) = 2^{1-2x}B(\frac{1}{2},x) \textit{ and} \\
&B(p+1,p+1) = 2^{1-2(p+1)}B(\frac{1}{2},p+1) = \frac{2^{1-2(p+1)}\Gamma(\frac{1}{2})\Gamma(p+1)}{\Gamma(\frac{3}{2}+p)}.
\end{aligned}
$$

□

Example 2.10. Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be $f(x) = x^n$, $n \in \mathbb{N}$. It is easy to show that $|f''|^q, q > 1$ is a convex function. Now, Let $a, b \in I$ such that $a < b$ then, in view of Theorem 2.9 we have

$$
\begin{aligned}
&|\frac{(n-1)(b^{n+1}-a^{n+1}) + x(n+1)(a^n - b^n) + (n+1)(b-a)x^n)}{(n+1)(b-a)}| \\
&\leq \frac{n(n-1)|x-a|^3}{4(b-a)}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}[\frac{x^{(n-2)q}+a^{(n-2)q}}{2}]^{\frac{1}{q}} \\
&+ \frac{n(n-1)|x-b|^3}{4(b-a)}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}[\frac{x^{(n-2)q}+b^{(n-2)q}}{2}]^{\frac{1}{q}}
\end{aligned}
$$

Corollary 2.11. *In Theorem 2.9, if we put $x := b$ or $x := a$, we get*

$$
|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt| \leq \frac{(b-a)^2}{8}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}(\frac{|f''(a)|^q + |f''(b)|^q}{2})^{\frac{1}{q}}.
$$

Corollary 2.12. *In Theorem 2.9, by choosing $x := \frac{a+b}{2}$, we obtain*

$$
\begin{aligned}
|\frac{f(a)+f(b)+2f(\frac{a+b}{2})}{4} - \frac{1}{b-a}\int_a^b f(t)dt| &\leq \frac{(b-a)^2}{64}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}(\frac{|f''(a)|^q + |f''(\frac{a+b}{2})|^q}{2})^{\frac{1}{q}} \\
&+ \frac{(b-a)^2}{64}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}(\frac{|f''(\frac{a+b}{2})|^q + |f''(b)|^q}{2})^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 2.13. *Let $f : I \subseteq [0,+\infty) \to \mathbb{R}$ be a differentiable function such that $f'' \in L[a,b]$, where $a, b \in I^o$ with $a < b$. If $|f''|^q$ is a convex function on $[a,b]$ with $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then we get the following inequality.*

$$
\begin{aligned}
&|\frac{(b-x)f(b) + (x-a)f(a) + f(x)(b-a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt| \\
&\leq |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}[\frac{|f''(x)|^q + (q+1)|f''(a)|^q}{(q+1)(q+2)}]^{\frac{1}{q}} \\
&+ |\frac{(x-b)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}[\frac{|f''(x)|^q + (q+1)|f''(b)|^q}{(q+1)(q+2)}]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. In view of convexity of f'' and Hölder's inequality, we conclude that

$$
\begin{aligned}
&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt| \\
&\le |\frac{(x-a)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)a)|dt \\
&+|\frac{(x-b)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)b)|dt \\
&\le |\frac{(x-a)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)a)|dt+|\frac{(x-b)^3}{2(b-a)}|\int_0^1 t(1-t)|f''(tx+(1-t)b)|dt \\
&\le |\frac{(x-a)^3}{2(b-a)}|(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 (1-t)^q|f''(tx+(1-t)a)|^q dt)^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\int_0^1 t^p dt)^{\frac{1}{p}}(\int_0^1 (1-t)^q|f''(tx+(1-t)b)|^q dt)^{\frac{1}{q}} \\
&\le |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}(|f''(x)|^q\int_0^1 (1-t)^q t dt+|f''(a)|^q\int_0^1 (1-t)^q(1-t)dt)^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}(|f''(x)|^q\int_0^1 (1-t)^q t dt+|f''(b)|^q\int_0^1 (1-t)^q(1-t)dt)^{\frac{1}{q}} \\
&\le |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}[|f''(x)|^q B(2,q+1)+|f''(a)|^q(\frac{1}{q+2})]^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}[|f''(x)|^q B(2,q+1)+|f''(b)|^q(\frac{1}{q+2})]^{\frac{1}{q}} \\
&\le |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}[\frac{|f''(x)|^q+(q+1)|f''(a)|^q}{(q+1)(q+2)}]^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\frac{1}{p+1})^{\frac{1}{p}}[\frac{|f''(x)|^q+(q+1)|f''(b)|^q}{(q+1)(q+2)}]^{\frac{1}{q}}.
\end{aligned}
$$

Example 2.14. Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be $f(x) = x^n$, $n \in \mathbb{N}$. It is easy to show that $|f''|^q, q > 1$ is a convex function. Now, Let $a, b \in I$ such that $a < b$ then, in view of Theorem 2.13 we have

$$
\begin{aligned}
&|\frac{(n-1)(b^{n+1}-a^{n+1})+x(n+1)(a^n-b^n)+(n+1)(b-a)x^n}{(n+1)(b-a)}| \\
&\le \frac{n(n-1)|x-a|^3}{(b-a)}[\frac{1}{p+1}]^{\frac{1}{p}}[\frac{x^{(n-2)q}+(q+1)a^{(n-2)q}}{(q+1)(q+2)}]^{\frac{1}{q}} \\
&+\frac{n(n-1)|x-b|^3}{(b-a)}[\frac{1}{p+1}]^{\frac{1}{p}}[\frac{x^{(n-2)q}+(q+1)b^{(n-2)q}}{(q+1)(q+2)}]^{\frac{1}{q}}
\end{aligned}
$$

□

Theorem 2.15. *Let $f : I \subseteq [0,+\infty) \to \mathbb{R}$ be a differentiable function such that $f'' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is a convex function on $[a,b]$ with $q > 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$. Then the* *following inequality holds.*

$$
\begin{aligned}
&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt| \\
&\leq |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{2})^{1-\frac{1}{q}}[|f^{''}(x)|^q B(3,q+1)+|f^{''}(a)|^q B(2,q+3)]^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\frac{1}{2})^{1-\frac{1}{q}}[|f^{''}(x)|^q B(3,q+1)+|f^{''}(b)|^q B(2,q+3)]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. By using convexity of f'' and Hölder's inequality, one has

$$
\begin{aligned}
&|\frac{(b-x)f(b)+(x-a)f(a)+f(x)(b-a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt| \\
&\leq |\frac{(x-a)^3}{2(b-a)}|\int_0^1 t(1-t)|f^{''}(tx+(1-t)a)|dt \\
&+|\frac{(x-b)^3}{2(b-a)}|\int_0^1 t(1-t)|f^{''}(tx+(1-t)b)|dt \\
&\leq |\frac{(x-a)^3}{2(b-a)}|(\int_0^1 tdt)^{1-\frac{1}{q}}(\int_0^1 t(1-t)^q|f^{''}(tx+(1-t)a)|^q dt)^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\int_0^1 tdt)^{1-\frac{1}{q}}(\int_0^1 t(1-t)^q|f^{''}(tx+(1-t)b)|^q dt)^{\frac{1}{q}} \\
&\leq |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{2})^{1-\frac{1}{q}}(|f^{''}(x)|^q\int_0^1 t^2(1-t)^q dt+|f^{''}(a)|^q\int_0^1 t(1-t)^{q+1}dt)^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\frac{1}{2})^{1-\frac{1}{q}}(|f^{''}(x)|^q\int_0^1 t^2(1-t)^q dt+|f^{''}(b)|^q\int_0^1 t(1-t)^{q+1}dt)^{\frac{1}{q}} \\
&\leq |\frac{(x-a)^3}{2(b-a)}|(\frac{1}{2})^{1-\frac{1}{q}}[|f^{''}(x)|^q B(3,q+1)+|f^{''}(a)|^q B(2,q+3)]^{\frac{1}{q}} \\
&+|\frac{(x-b)^3}{2(b-a)}|(\frac{1}{2})^{1-\frac{1}{q}}[|f^{''}(x)|^q B(3,q+1)+|f^{''}(b)|^q B(2,q+3)]^{\frac{1}{q}}.
\end{aligned}
$$

□

Example 2.16. Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be $f(x)=x^n,\ \ n\in\mathbb{N}$. It is easy to show that $|f''|^q, q>1$ is a convex function. Now, Let $a,b\in I$ such that $a<b$ then, in view of Theorem 2.15 we have

$$
\begin{aligned}
&|\frac{(n-1)(b^{n+1}-a^{n+1})+x(n+1)(a^n-b^n)+(n+1)(b-a)x^n)}{(n+1)(b-a)}| \\
&\leq \frac{n(n-1)|x-a|^3}{(b-a)}[\frac{1}{2}]^{1-\frac{1}{q}}[\frac{x^{(n-2)q}B(3,q+1)+B(2,q+2)a^{(n-2)q}}{(q+1)(q+2)}]^{\frac{1}{q}} \\
&+\frac{n(n-1)|x-b|^3}{(b-a)}[\frac{1}{2}]^{1-\frac{1}{q}}[\frac{x^{(n-2)q}B(3,q+2)+B(2,q+3)b^{(n-2)q}}{(q+1)(q+2)}]^{\frac{1}{q}}
\end{aligned}
$$

3. Applications to special means

Consider the following special means for two nonnegative real numbers α,β with $\alpha \neq \beta$ as follows:(see [5, 4]).

(1): The arithmetic mean:

$$A = A(\alpha,\beta) = \frac{\alpha+\beta}{2},\ \alpha,\beta \in \mathbb{R},$$

with $\alpha,\beta > 0$.

(2): The logarithmic mean:

$$\overline{L} = \overline{L}(\alpha,\beta) = \frac{\beta-\alpha}{\ln\beta - \ln\alpha},\quad \alpha \neq \beta, \alpha,\beta \in \mathbb{R},$$

with $\alpha,\beta > 0$.

(3): The generalized logarithmic mean:

$$L_n(\alpha,\beta) = [\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}]^{\frac{1}{n}},\ n \in \mathbb{R}\backslash\{-1,0\}, \alpha \neq \beta, \alpha,\beta \in \mathbb{R},$$

with $\alpha,\beta > 0$.
These means are often used in numerical approximation and in other areas.

Proposition 3.1. *Let $a, b \in \mathbb{R}$ such that $0 < a < b, n \in \mathbb{N}$. Then we have the following inequality*

$$|L_n^n(a,b)| \leq C_1[A(x^{(n-2)q}, a^{(n-2)q})]^{\frac{1}{q}} + C_2[A(x^{(n-2)q}, b^{(n-2)q})]^{\frac{1}{q}} + C_3.$$

Where,

$$C_1 = \frac{n(n-1)|x-a|^3}{4(b-a)}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}$$
$$C_2 = \frac{n(n-1)|x-b|^3}{4(b-a)}[\frac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}]^{\frac{1}{p}}$$
$$C_3 = |\frac{(b-x)b^n + (x-a)a^n + x^n(b-a)}{2(b-a)}|$$

Proof. It follows from the Theorem 2.13 for $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^n,\ n \in \mathbb{N}$. □

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