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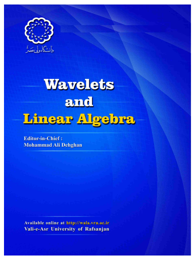

A Class of Nested Iteration Schemes for Generalized Coupled Sylvester Matrix Equation

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Abstract

Global Krylov subspace methods are the most efficient and robust methods to solve generalized coupled Sylvester matrix equation. In this paper, we propose nested splitting conjugate gradient process for solving this equation. This method has inner and outer iterations, which employs the generalized conjugate gradient method as inner iteration to approximate each outer iterate, while each outer iteration is induced by a convergence and symmetric positive definite splitting of the coefficient matrices. Convergence properties of this method are investigated. Finally, the effectiveness of the nested splitting conjugate gradient method is explained by some numerical examples.

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1. Introduction

Consider the following generalized coupled Sylvester matrix equation

$$\begin{cases} AXB + CYD = M, \\ EXF + GYH = N, \end{cases} \tag{1.1}$$

where $A, C, E, G \in \mathbb{R}^{n\times n}$, $B, D, F, H \in \mathbb{R}^{s\times s}$ and $M, N \in \mathbb{R}^{n\times s}$ are defined matrices and $X, Y \in \mathbb{R}^{n\times s}$ are unknown matrices. These matrix equations, and their transpose arise in computing error bounds for computed eigenvalues and eigenspaces of the generalized eigenvalue problem, in computing deflating subspaces of the same problem, and in computing certain decompositions of transfer matrices arising in control theory. Also, they play an important role in image restoration and other problems, see [3, 6, 7, 15, 16].

Note that the coupled linear matrix equation (1.1) can be reformulated by the following $2ns \times 2ns$ linear system:

$$\mathcal{A}x = c, \tag{1.2}$$

where

$$\mathcal{A} = \begin{pmatrix} B^T \otimes A & D^T \otimes C \\ F^T \otimes E & H^T \otimes G \end{pmatrix}, x = \begin{pmatrix} vec(X) \\ vec(Y) \end{pmatrix}, c = \begin{pmatrix} vec(M) \\ vec(N) \end{pmatrix}. \tag{1.3}$$

However, it seems quite costly and ill-conditioned to solve the linear equation system(1.2).

During the previous years, several iterative methods have been proposed for solving the generalized coupled Sylvester matrix equation. In [13, 14], the authors have introduced two global Krylov subspace methods, the global full orthogonalization method (GlFOM) and global generalized minimum residual method (GlGMRES) for solving the following general coupled linear matrix equations:

$$\sum_{j=1}^{p} A_{ij}X_jB_{ij} = C_i, \qquad i = 1, 2, ..., p.$$

Also, Dehghan [4] presented an algorithm based on conjugate gradient to solve the generalized coupled Sylvester equation (1.1) where the coeffcient matrices are bisymmetric.

In this paper, we present an iterative method for solving the generalized coupled Sylvester matrix equation (1.1) by using the symmetric and skew-symmetric splitting of the matrix $\mathcal{A}$ in a matrix variant of the Nested splitting conjugate gradient (NSCG) method, and give sufficient conditions for convergence. At first, this method was introduced for solving the system of linear equations, $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is a large sparse nonsingular matrix and $x, b \in \mathbb{R}^{n\times n}$ by [1], and then was extended to solve the matrix equation $AXB = C$ and the continuous Sylvester matrix equation AX + XB = C, see [8, 12, 11, 10]. The NSCG method for linear system $Ax = b$ is efficient when the coeffcient matrix A is non-symmetric positive definite and the symmetric part H dominate than skew-symmetric part S, i.e. $\|H\| > \|S\|$. If we have $\|H\| < \|S\|$, the NSCG is not a good choice and we must use the other iterative methods.

This paper contains five sections. In section 2, we review the definitions and lemmas that are used throughout this paper. In section 3, we describe a frame work for NSCG method for solving matrix equation (1.1). Section 4 is devoted to decribing the convergence of the NSCG method. Finally, some numerical examples are presented.

2. Preliminaries

Thoroughout this paper, we use the following notations. Let $\mathbb{R}^{p\times n}$ be the set of $p \times n$ real matrices. The symbols A^T, $||A||_2$, $trace(A)$, $\lambda(A)$, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ will denote transpose, 2-norm, trace, eigenvalue, maximum eigenvalue and minimum eigenvalue, respectively, of a matrix $A \in \mathbb{R}^{p\times n}$. For any matrices A and B in $\mathbb{R}^{p\times n}$, the notation $< A, B >_F = trace(A^T B)$ denotes the inner product. The associated norm is the Frobenius norm obtained by $||.||_F$.
Further, $vec(A)$ is the vector of $\mathbb{R}^{pn}$ obtained by stacking the columns of the matrix $A \in \mathbb{R}^{p\times n}$ and $\otimes$ denotes the Kronecker product, i.e. $A \otimes B = (a_{ij}B)$. For a nonsingular matrix $\mathcal{H}$, we denote by $\kappa(\mathcal{H}) = ||\mathcal{H}||_2||\mathcal{H}^{-1}||_2$, where $\mathcal{H}$ is a symmetric positive definite matrix, define the $||x||_{\mathcal{H}}$ for $x \in \mathbb{R}^n$ as $||x||_{\mathcal{H}} = \sqrt{x^T\mathcal{H}x}$. Then the induced $||.||_{\mathcal{H}}$ of a matrix $A \in \mathbb{R}^{n\times n}$ is defined as $||A||_{\mathcal{H}} = ||\mathcal{H}^{\frac{1}{2}}A\mathcal{H}^{\frac{-1}{2}}||_2$. In addition, it holds that $||Ax||_{\mathcal{H}} \le ||A||_{\mathcal{H}}||x||_{\mathcal{H}}$, $||A||_{\mathcal{H}} \le \sqrt{\kappa(\mathcal{H})}||A||_2$.
Next, we need the following lemmas.

Lemma 2.1. *([5]). Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices and denote the minimum and the maximum eigenvalues of A by $\lambda_{min}(A)$ and $\lambda_{max}(A)$, respectively. Then*

$$\lambda_{max}(A + B) \le \lambda_{max}(A) + \lambda_{max}(B),$$
$$\lambda_{min}(A + B) \ge \lambda_{min}(A) + \lambda_{min}(B).$$

Lemma 2.2. *([9]). Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Then for all $x \in \mathbb{R}^n$, we have $||A^{\frac{1}{2}}x||_2 = ||x||_A$ and*

$$\sqrt{\lambda_{min}(A)}||x||_A \le ||Ax||_2 \le \sqrt{\lambda_{max}(A)}||x||_A.$$

3. The NSCG method for coupled matrix equation (1.1)

In this section, we consider the scheme of the NSCG iteration method and its convergence property. $\mathcal{A} = \mathcal{H} - \mathcal{S}$ is called a splitting of the matrix $\mathcal{A}$ if $\mathcal{H}$ is nonsingular. This splitting is convergent if $\rho(\mathcal{H}^{-1}\mathcal{S}) < 1$, a contractive splitting if $||\mathcal{H}^{-1}\mathcal{S}|| < 1$ for some matrix norm and symmetric positive definite splitting(spd) if $\mathcal{H}$ is spd matrix.
In [1], the authors proposed an efficient nested iterative method for solving the system of linear equation (1.2). This method is called the nested splitting conjugate gradient (NSCG) method, which is described as follows:
Let $\mathcal{A} = \mathcal{H} - \mathcal{S}$ be a splitting symmetric positive definite of matrix $\mathcal{A}$. Then write the linear system (1.2) in an equivalent form:

$$\mathcal{H}x = \mathcal{S}x + c.$$

Starting with an initial guess $x^{(0)} \in \mathbb{R}^{2ns}$ of the solution $x^* \in \mathbb{R}^{2ns}$, we can compute the approximate solutions $x^{(1)}, x^{(2)}, ..., x^{(l)}$ to the solution x^*, where the next approximation $x^{(l+1)}$ is obtained through solving the linear equation systems:

$$\mathcal{H}x = \mathcal{S}x^{(l)} + c, \tag{3.1}$$

iteratively, with the CG (Conjugate gradient) method.
Inspired by these ideas in [1, 11, 12, 8, 10], we next apply the NSCG method for the coupled linear matrix equation (1.1). Consider the symmetric and skew-symmetric splitting for matrix $\mathcal{A}$ in (1.3), i.e, $\mathcal{A} = \mathcal{H} - \mathcal{S}$, where

$$\mathcal{H} = \frac{\mathcal{A}^T + \mathcal{A}}{2} = \frac{1}{2}\begin{pmatrix} B \otimes A^T + B^T \otimes A & F \otimes E^T + D^T \otimes C \\ D \otimes C^T + F^T \otimes E & H \otimes G^T + H^T \otimes G \end{pmatrix}, \tag{3.2}$$

$$\mathcal{S} = \frac{\mathcal{A}^T - \mathcal{A}}{2} = \frac{1}{2}\begin{pmatrix} B \otimes A^T - B^T \otimes A & F \otimes E^T - D^T \otimes C \\ D \otimes C^T - F^T \otimes E & H \otimes G^T - H^T \otimes G \end{pmatrix}. \tag{3.3}$$

We will show how matrix $\mathcal{H}$ could be a positive definite matrix. Matrix $\mathcal{H}$ can be expressed as:

$$\mathcal{H} = \mathcal{M}_1 + \mathcal{N}_1,$$

where

$$\mathcal{M}_1 = \frac{1}{2}\begin{pmatrix} B \otimes A^T + B^T \otimes A & 0 \\ 0 & H \otimes G^T + H^T \otimes G \end{pmatrix}, \tag{3.4}$$

$$\mathcal{N}_1 = \frac{1}{2}\begin{pmatrix} 0 & F \otimes E^T + D^T \otimes C \\ D \otimes C^T + F^T \otimes E & 0 \end{pmatrix}. \tag{3.5}$$

Since $\mathcal{M}_1$ and $\mathcal{N}_1$ are symmetric, by using Lemma 2.1, we obtain the following expression:

$$\lambda_{min}(\mathcal{H}) \geq \lambda_{min}(\mathcal{M}_1) + \lambda_{min}(\mathcal{N}_1) = t.$$

If $t > 0$, then matrix $\mathcal{H}$ is a symmetric positive definite matrix. Assume that $t > 0$, therefore we can solve the following positive definite system of linear equations by using NSCG method

$$\mathcal{H}x = \mathcal{S}x^{(l)} + c,$$

which can be rewritten as matrix form

$$\begin{cases} \frac{1}{2}(A^T X B^T + AXB + E^T Y F^T + CYD) = C_1, \\ \frac{1}{2}(C^T X D^T + EXF + G^T Y H^T + GYH) = C_2, \end{cases} \tag{3.6}$$

where

$$C_1 = M + \frac{1}{2}(A^T X^{(l)} B^T - AX^{(l)} B + E^T Y^{(l)} F^T - CY^{(l)} D),$$
$$C_2 = N + \frac{1}{2}(C^T X^{(l)} D^T - EX^{(l)} F + G^T Y^{(l)} H^T - GY^{(l)} H).$$

To obtain the next approximation solution $[X^{(l+1)}, Y^{(l+1)}]$, we can solve the matrix equation system (3.6) iteratively by the CG-like method. The NSCG algorithm for computing approximation solution of linear equation system (3.6) is summarized in Algorithm 1.

4. Convergence analysis

In this section, we consider convergence properties of NSCG method for solving coupled matrix equation (1.1).
Firstly, we present the following theorem about the convergence properties of this method. This theorem was established by Axelsson, Bai and Qiu [1].

Theorem 4.1. *Let $\mathcal{A} \in \mathbb{R}^{2ns\times 2ns}$ be a nonsingular and nonsymmetric matrix and $\mathcal{A} = \mathcal{H} - \mathcal{S}$ be a contractive (with respect to $\|.\|_{\mathcal{H}}$ norm) and symmetrix positive definite splitting. Suppose that the NSCG method is started from an initial guess $x^{(0)} \in \mathbb{R}^{2ns}$ of the solution $x^* \in \mathbb{R}^{2ns}$, and compute an iterative sequence $\{x^{(l)}\}_{l=0}^{\infty}$, where $x^{(l)}$ is lth aproximation solution of the linear equation system* (3.1) *obtained with k_l steps of CG iterations. Then*

(a) $\|x^{(l)} - x^*\|_{\mathcal{H}} \leq \gamma^{(l)} \|x^{(l-1)} - x^*\|_{\mathcal{H}}, l = 0, 1, 2, ...,$

(b) $\|c - \mathcal{A}x^{(l)}\|_{\mathcal{H}} \leq \hat{\gamma}^{(l)} \|c - \mathcal{A}x^{(l-1)}\|_{\mathcal{H}}, l = 0, 1, 2, ...,$

where

$$\gamma^{(l)} = 2(\frac{\sqrt{\kappa(\mathcal{H})} - 1}{\sqrt{\kappa(\mathcal{H})} + 1})^{k_l}(1 + \varrho) + \varrho, \;\; \hat{\gamma}^{(l)} = \gamma^{(l)} \frac{1 + \varrho}{1 - \varrho}, \;\; l = 0, 1, 2, ...,$$

and $\varrho = \|\mathcal{H}^{-1}\mathcal{S}\|_{\mathcal{H}}$.
Moreover, for some $\gamma \in (\varrho, 1)$, and

$$k_l \geq \frac{ln((\gamma - \varrho)/(2(1 + \varrho)))}{ln((\sqrt{\kappa(\mathcal{H})} - 1)/(\sqrt{\kappa(\mathcal{H})} + 1))}, \;\; l = 0, 1, 2, ...,$$

we have $\gamma^{(l)} \leq \gamma, (l = 0, 1, 2, ...)$ and the sequence $\{x^{(l)}\}_{l=0}^{\infty}$ converges to the solution x^* of the system of linear equations (1.2)*. For $\varrho \in (0, \sqrt{2} - 1)$ and some $\hat{\gamma}^{(l)} \in ((1 + \varrho)\varrho/(1 - \varrho), 1)$, and*

$$k_l \geq \frac{ln(((1 - \varrho)\hat{\gamma} - \varrho(1 + \varrho))/(2(1 + \varrho)^2))}{ln((\sqrt{\kappa(\mathcal{H})} - 1)/(\sqrt{\kappa(\mathcal{H})} + 1))}, \;\; l = 0, 1, 2, ...,$$

we have $\hat{\gamma}^{(l)} \leq \hat{\gamma}, l = 0, 1, 2, ...,$ and the residual sequence $\{c - \mathcal{A}x^{(l)}\}_{l=0}^{\infty}$ converges to zero.

Algorithm 1 The NSCG algorithm for coupled matrix equation (1.1)

Require: $A, C, E, G \in \mathbb{R}^{n\times n}, B, D, F, H \in \mathbb{R}^{s\times s}, M, N, X^{(0)}, Y^{(0)} \in \mathbb{R}^{n\times s}, l_{max}, j_{max}, \eta, \epsilon$;

Ensure: $[X, Y]$;

1: Set $[X^{(0,0)}, Y^{(0,0)}]^T = [X^{(0)}, Y^{(0)}]^T$;

2: Compute $R^{(0)} = \begin{pmatrix} M - AX^{(0,0)}B - CY^{(0,0)}D \\ N - EX^{(0,0)}F - GY^{(0,0)}H \end{pmatrix}$;

3: **for** $l = 0, 1, ..., l_{max}$ **do**

4: Compute $\hat{C} = \begin{pmatrix} M + \frac{1}{2}(A^T X^{(l,0)} B^T - AX^{(l,0)}B + E^T Y^{(l,0)} F^T - CY^{(l,0)}D) \\ N + \frac{1}{2}(C^T X^{(l,0)} D^T - EX^{(l,0)}F + G^T Y^{(l,0)} H^T - GY^{(l,0)}H) \end{pmatrix}$;

5: Compute $\begin{pmatrix} \hat{R}_1^{(l,0)} \\ \hat{R}_2^{(l,0)} \end{pmatrix} = \hat{C} - \frac{1}{2}\begin{pmatrix} A^T X^{(l,0)} B^T + AX^{(l,0)}B + E^T Y^{(l,0)} F^T + CY^{(l,0)}D \\ C^T X^{(l,0)} D^T + EX^{(l,0)}F + G^T Y^{(l,0)} H^T + GY^{(l,0)}H \end{pmatrix}$;

6: Set: $\begin{pmatrix} P_1^{(0)} \\ P_2^{(0)} \end{pmatrix} = \begin{pmatrix} \hat{R}_1^{(l,0)} \\ \hat{R}_2^{(l,0)} \end{pmatrix}$;

7: **for** j=0,1,...,jmax **do**

8: Compute $\begin{pmatrix} W_1^{(j)} \\ W_2^{(j)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} A^T P_1^{(j)} B^T + AP_1^{(j)}B + E^T P_2^{(j)} F^T + CP_2^{(j)}D \\ C^T P_1^{(j)} D^T + EP_1^{(j)}F + G^T P_2^{(j)} H^T + GP_2^{(j)}H \end{pmatrix}$;

9: Compute $\alpha_j = \frac{\|\hat{R}_1^{(l,j)}\|_F^2 + \|\hat{R}_2^{(l,j)}\|_F^2}{<W_1^{(j)}, P_1^{(j)}>_F + <W_2^{(j)}, P_2^{(j)}>_F}$;

10: Compute $\begin{pmatrix} X^{(l,j+1)} \\ Y^{(l,j+1)} \end{pmatrix} = \begin{pmatrix} X^{(l,j)} \\ Y^{(l,j)} \end{pmatrix} + \alpha_j \begin{pmatrix} P_1^{(j)} \\ P_2^{(j)} \end{pmatrix}$;

11: Compute $\begin{pmatrix} \hat{R}_1^{(l,j+1)} \\ \hat{R}_2^{(l,j+1)} \end{pmatrix} = \begin{pmatrix} \hat{R}_1^{(l,j)} \\ \hat{R}_2^{(l,j)} \end{pmatrix} - \alpha_j \begin{pmatrix} W_1^{(j)} \\ W_2^{(j)} \end{pmatrix}$;

12: **if** $\frac{\sqrt{\|\hat{R}_1^{(l+1,j)}\|_F^2 + \|\hat{R}_2^{(l+1,j)}\|_F^2}}{\sqrt{\|\hat{R}_1^{(l,0)}\|_F^2 + \|\hat{R}_2^{(l,0)}\|_F^2}} \leq \eta$, **then**

13: Go to 18

14: **end if**

15: Compute $\beta_j = \frac{\|\hat{R}_1^{(l+1,j)}\|_F^2 + \|\hat{R}_2^{(l+1,j)}\|_F^2}{\|\hat{R}_1^{(l,j)}\|_F^2 + \|\hat{R}_2^{(l,j)}\|_F^2}$;

16: Compute $\begin{pmatrix} P_1^{(j+1)} \\ P_2^{(j+1)} \end{pmatrix} = \begin{pmatrix} \hat{R}_1^{(l,j+1)} \\ \hat{R}_2^{(l,j+1)} \end{pmatrix} + \beta_j \begin{pmatrix} P_1^{(j)} \\ P_2^{(j)} \end{pmatrix}$;

17: **end for**

18: Set $\begin{pmatrix} X^{(l+1)} \\ Y^{(l+1)} \end{pmatrix} = \begin{pmatrix} X^{(l,j+1)} \\ Y^{(l,j+1)} \end{pmatrix}$;

19: Compute $R^{(l+1)} = \begin{pmatrix} M - AX^{(l+1)}B - CY^{(l+1)}D \\ N - EX^{(l+1)}F - GY^{(l+1)}H \end{pmatrix}$;

20: **if** $\frac{\|R^{(l+1)}\|_F}{\|R^{(0)}\|_F} \leq \epsilon$ **then**

21: Stop

22: **end if**

23: $\begin{pmatrix} X^{(l+1,0)} \\ Y^{(l+1,0)} \end{pmatrix} = \begin{pmatrix} X^{(l+1)} \\ Y^{(l+1)} \end{pmatrix}$;

24: **end for**

In the sequel, we need the following lemma proving $\mathcal{A} = \mathcal{H} - \mathcal{S}$ is a contractive splitting.

Lemma 4.2. *Let $\mathcal{H} = \mathcal{M}_1 + \mathcal{N}_1$, where*

$$\mathcal{M}_1 = \frac{1}{2}\begin{pmatrix} B \otimes A^T + B^T \otimes A & 0 \\ 0 & H \otimes G^T + H^T \otimes G \end{pmatrix}, \tag{4.1}$$

$$\mathcal{N}_1 = \frac{1}{2}\begin{pmatrix} 0 & F \otimes E^T + D^T \otimes C \\ D \otimes C^T + F^T \otimes E & 0 \end{pmatrix}, \tag{4.2}$$

and $\mathcal{S} = \mathcal{M}_2 + \mathcal{N}_2$, where

$$\mathcal{M}_2 = \frac{1}{2}\begin{pmatrix} B \otimes A^T - B^T \otimes A & 0 \\ 0 & H \otimes G^T - H^T \otimes G \end{pmatrix}, \tag{4.3}$$

$$\mathcal{N}_2 = \frac{1}{2}\begin{pmatrix} 0 & F \otimes E^T - D^T \otimes C \\ D \otimes C^T - F^T \otimes E & 0 \end{pmatrix}. \tag{4.4}$$

If $\frac{\tau\theta^3}{t} < 1$, then $\mathcal{A} = \mathcal{H} - \mathcal{S}$ is a contractive splitting with respect to the $\|.\|_{\mathcal{H}}$, where $\theta = (\frac{s}{t})^{\frac{1}{2}}$,

$$\begin{aligned} t :&= min_{i,j}\{\lambda_i(B)\lambda_j(A), \lambda_i(H)\lambda_j(G))\} \\ &+\frac{1}{2}min_{i,j}[\lambda_i(DF)\lambda_j(EC) + \sigma_i(F)\sigma_j(E) + \sigma_i(D)\sigma_j(C)]^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} s :&= \max_{i,j}\{|\lambda_i(B)||\lambda_j(A)|, |\lambda_i(H)||\lambda_j(G)|\} \\ &+\frac{1}{2}\max_{i,j}[(2\lambda_i(DF)\lambda_j(EC) + \sigma_i(F)\sigma_j(E) + \sigma_i(D)\sigma_j(C)]^{\frac{1}{2}}, \end{aligned}$$

and

$$\tau := \frac{1}{2}[\max_{i,j}(|2\lambda_i(DF)\lambda_j(EC) - \sigma_i(D)\sigma_j(C) - \sigma_i(F)\sigma_j(E))|^{\frac{1}{2}}.$$

Proof. By Lemma 2.1 and $\mathcal{H} = \mathcal{M}_1 + \mathcal{N}_1$, we have

$$\begin{aligned} \lambda_{min}(\mathcal{H}) &\geq \lambda_{min}(\mathcal{M}_1) + \lambda_{min}(\mathcal{N}_1) \\ &\geq \frac{1}{2}[min_i\{\lambda_i(B^T \otimes A + B \otimes A^T), \lambda_i(H^T \otimes G + H \otimes G^T)\}] \\ &+\frac{1}{2}[\lambda_{min}((F^T \otimes E + D \otimes C^T)(D^T \otimes C + F \otimes E^T))]^{\frac{1}{2}} \\ &\geq min_{i,j}\{\lambda_i(B)\lambda_j(A), \lambda_i(H)\lambda_j(G)\} \\ &+\frac{1}{2}min_{i,j}[2\lambda_i(DF)\lambda_j(EC) + \sigma_i(F)\sigma_j(E) + \sigma_i(D)\sigma_j(C)]^{\frac{1}{2}} := t, \end{aligned}$$

and

$$\begin{aligned}|\lambda_{max}(\mathcal{H})| &\le |\lambda_{max}(\mathcal{M}_1) + \lambda_{max}(\mathcal{N}_1)| \\ &\le \frac{1}{2} max_i\{|\lambda_i(B^T \otimes A + B \otimes A^T)|, |\lambda_i(H^T \otimes G + H \otimes G^T)|\} \\ &+ \frac{1}{2}(\lambda_{max}((F^T \otimes E + D \otimes C^T)(D^T \otimes C + F \otimes E^T)))^{\frac{1}{2}} \\ &\le \max_{i,j}\{|\lambda_i(B)||\lambda_j(A)|, |\lambda_i(H)||\lambda_j(G)|\} \\ &+ \frac{1}{2}\max_{i,j}[(2\lambda_i(DF)\lambda_j(EC) + \sigma_i(F)\sigma_j(E) + \sigma_i(D)\sigma_j(C)]^{\frac{1}{2}} := s.\end{aligned}$$

Therefore, we obtain

$$\kappa(\mathcal{H}) = \|\mathcal{H}\|_2\|\mathcal{H}^{-1}\|_2 = \frac{|\lambda_{max}(\mathcal{H})|}{|\lambda_{min}(\mathcal{H})|} \le \frac{s}{t} := \theta^2, (\theta > 0). \tag{4.5}$$

By Spectral theorem, every real skew- symmetric can be written in the form

$$\mathcal{S} = Q\Sigma Q^T, \tag{4.6}$$

where Q is an orthogonal matrix and

$$\Sigma = diag\left\{\begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & \lambda_r \\ -\lambda_r & 0 \end{pmatrix}, 0, \ldots, 0\right\},$$

for real $\lambda_k, k = 1, 2, ..., r$. The nonzero eigenvalues of matrix Σ are $\pm i\lambda_k, k = 1, 2, ..., r$. By substituting (4.6) into $\mathcal{S} = \mathcal{M}_2 + \mathcal{N}_2$, we have

$$\Sigma = Q^T(\mathcal{M}_2 + \mathcal{N}_2)Q.$$

Since Q is an orthogonal matrix and by triangle inequality property of 2- norm,

$$\|\Sigma\|_2 \le \|\mathcal{M}_2\|_2 + \|\mathcal{N}_2\|_2.$$

Then

$$\begin{aligned}\|\mathcal{S}\|_2 &= \max_i |\lambda_i(\mathcal{S})| \le \max_i |\lambda_i(\mathcal{M}_2)| + \max_i |\lambda_i(\mathcal{N}_2)| \\ &= \frac{1}{2}[\max\{\max_i |\lambda_i(B \otimes A^T - B^T \otimes A)|, \max_i |\lambda_i(H \otimes G^T - H^T \otimes G)|\}] \\ &+ \frac{1}{2}[|\lambda_{max}(D \otimes C^T - F^T \otimes E)(F \otimes E^T - D^T \otimes C)|]^{\frac{1}{2}} \\ &= \frac{1}{2}\max_{i,j}(|2\lambda_i(DF)\lambda_j(EC) - \sigma_i(D)\sigma_j(C) - \sigma_i(F)\sigma_j(E)|)^{\frac{1}{2}} = \tau.\end{aligned} \tag{4.7}$$

Therefore, it follows that:

$$\|\mathcal{H}^{-1}\mathcal{S}\|_{\mathcal{H}} \le \sqrt{\kappa(\mathcal{H})}\|\mathcal{H}^{-1}\mathcal{S}\|_2 \le (\kappa(\mathcal{H}))^{\frac{3}{2}}\frac{\|\mathcal{S}\|_2}{\|\mathcal{H}\|_2} \le (\frac{s}{t})^{\frac{3}{2}}\frac{\tau}{t} = \frac{\tau\theta^3}{t} := \eta. \tag{4.8}$$

This clearly completes the proof. □

Theorem 4.3. *Let $\mathcal{H}$ and $\mathcal{S}$ are symmetric and skew-symmetric parts of the nonsingular and nonsymmetric matrix $\mathcal{A}$, respectively , where they are defined by the relations (3.2) and (3.3). Let $\|\mathcal{H}^{-1}\mathcal{S}\|_{\mathcal{H}} < 1$. Suppose that NSCG method with initial guess $[X^{(0)}, Y^{(0)}]^T$ obtains an iterative sequence $\{[X^{(l)}, Y^{(l)}]^T\}_{l=0}^{\infty}$, where $\{[X^{(l)}, Y^{(l)}]^T\}$ is lth approximation solution of the linear equation system (1.1) computed by k_l steps of CG method with*

$$R^{(l)} = \begin{pmatrix} M - AX^{(l)}B - CY^{(l)}D \\ N - EX^{(l)}F - GY^{(l)}H \end{pmatrix},$$

and $r^{(l)} = \begin{pmatrix} vec(M - AX^{(l)}B - CY^{(l)}D) \\ vec(N - EX^{(l)}F - GY^{(l)}H) \end{pmatrix}$. *Then*

(a) $\|\mathcal{H}\begin{pmatrix} vec(X^{(l)} - X^*) \\ vec(Y^{(l)} - Y^*) \end{pmatrix}\|_2 \le \omega^{(l)}\|\mathcal{H}\begin{pmatrix} vec(X^{(l-1)} - X^*) \\ vec(Y^{(l-1)} - Y^*) \end{pmatrix}\|_2, l = 0, 1, 2, ...,$

(b) $\|\mathcal{H}r^{(l)}\|_2 \le \hat{\omega}^{(l)}\|\mathcal{H}r^{(l-1)}\|_2, l = 0, 1, 2, ...,$
where

$$\omega^{(l)} = (2(\frac{\theta - 1}{\theta + 1})^{k_l}(1 + \eta) + \eta)\theta,$$
$$\hat{\omega}^{(l)} = \omega^{(l)}\frac{1 + \eta}{1 - \eta}, \quad l = 0, 1, 2, ...,$$
$$\theta = (\frac{s}{t})^2, \eta = \frac{\tau\theta^3}{t},$$

(Consider t, s, and τ as defined in Lemma 4.2) and $\varrho = \|\mathcal{H}^{-1}\mathcal{S}\|_{\mathcal{H}}$.

(c) If $\eta \in (0, \frac{1}{\theta})$, then for some $\omega \in (\eta\theta, 1)$, and

$$k_l \ge \frac{ln((\omega - \eta\theta)/(2\theta(1 + \eta)))}{ln((\theta - 1)/(\theta + 1))}, \quad l = 0, 1, 2, ...,$$

we have $\omega^{(l)} \le \omega, (l = 0, 1, 2, ...)$ and the sequence $\{[X^{(l)}, Y^{(l)}]^T\}_{l=0}^{\infty}$ converges to the exact solution $\{[X^{(*)}, Y^{(*)}]^T\}$ of linear equation system (1.1).

(d) For $\eta \in \left(0, \frac{\sqrt{(\theta + 1)^2 + 4\theta} - (\theta + 1)}{2\theta}\right)$ *and some*
$\hat{\omega} \in ((1 + \eta)\eta\theta/(1 - \eta), 1)$, *and*

$$k_l \ge \frac{ln((\hat{\omega}(1 - \eta) - \theta\eta(1 + \eta))/(2\theta(1 + \eta)^2))}{ln((\theta - 1)/(\theta + 1))}, \quad l = 0, 1, 2, ...,$$

we have $\hat{\omega}^{(l)} \le \hat{\omega}, l = 0, 1, 2, ...,$ and the residual sequence $\{R^{(l)}\}_{l=0}^{\infty}$ converges to zero.

Proof. By using lemma 2.2 and theorem 4.1, we can deduce that

$$\begin{aligned}
\|\mathcal{H}\begin{pmatrix} vec(X^{(l)}-X^*) \\ vec(Y^{(l)}-Y^*) \end{pmatrix}\|_2 &\leq \sqrt{\lambda_{max}(\mathcal{H})}\|\begin{pmatrix} vec(X^{(l)}-X^*) \\ vec(Y^{(l)}-Y^*) \end{pmatrix}\|_{\mathcal{H}} \\
&\leq \gamma^{(l)}\sqrt{\lambda_{max}(\mathcal{H})}\|\begin{pmatrix} vec(X^{(l-1)}-X^*) \\ vec(Y^{(l-1)}-Y^*) \end{pmatrix}\|_{\mathcal{H}} \\
&\leq \gamma^{(l)}\frac{\sqrt{\lambda_{max}(\mathcal{H})}}{\sqrt{\lambda_{min}(\mathcal{H})}}\|\mathcal{H}\begin{pmatrix} vec(X^{(l-1)}-X^*) \\ vec(Y^{(l-1)}-Y^*) \end{pmatrix}\|_2 \\
&\leq \gamma^{(l)}\sqrt{\kappa(\mathcal{H})}\|\mathcal{H}\begin{pmatrix} vec(X^{(l-1)}-X^*) \\ vec(Y^{(l-1)}-Y^*) \end{pmatrix}\|_2.
\end{aligned}$$

Therefore, we have

$$\|\mathcal{H}\begin{pmatrix} vec(X^{(l)}-X^*) \\ vec(Y^{(l)}-Y^*) \end{pmatrix}\|_2 \leq \gamma^{(l)}\sqrt{\kappa(\mathcal{H})}\|\mathcal{H}\begin{pmatrix} vec(X^{(l-1)}-X^*) \\ vec(Y^{(l-1)}-Y^*) \end{pmatrix}\|_2. \tag{4.9}$$

By (4.5), we obtain

$$\begin{aligned}
\gamma^{(l)} &= 2(\frac{\sqrt{\kappa(\mathcal{H})}-1}{\sqrt{\kappa(\mathcal{H})}+1})^{k_l}(1+\varrho)+\varrho \\
&\leq 2(\frac{\theta-1}{\theta+1})^{k_l}(1+\varrho)+\varrho. \qquad (4.10)
\end{aligned}$$

By (4.5) and (4.10), the inequality (4.9) can be rewritten equivalently as follow:

$$\|\mathcal{H}\begin{pmatrix} vec(X^{(l)}-X^*) \\ vec(Y^{(l)}-Y^*) \end{pmatrix}\|_2 \leq \omega^{(l)}\|\mathcal{H}\begin{pmatrix} vec(X^{(l-1)}-X^*) \\ vec(Y^{(l-1)}-Y^*) \end{pmatrix}\|_2, \tag{4.11}$$

where

$$\omega^{(l)} = (2(\frac{\theta-1}{\theta+1})^{k_l}(1+\eta)+\eta)\theta.$$

It is obvious that for $\eta \in (0, \frac{1}{\theta})$ and $\omega \in (\eta\theta, 1)$, if

$$k_l \geq \frac{ln(\frac{\omega-\eta\theta}{2\theta(1+\eta)})}{ln(\frac{\theta-1}{\theta+1})}, \quad l = 1, 2, 3, ...,$$

then $\omega^{(l)} \leq \omega$.
Therefore, from (4.11), we conclude

$$\|\mathcal{H}\begin{pmatrix} vec(X^{(l)}-X^*) \\ vec(Y^{(l)}-Y^*) \end{pmatrix}\|_2 \leq \omega^{l+1}\|\mathcal{H}\begin{pmatrix} vec(X^{(0)}-X^*) \\ vec(Y^{(0)}-Y^*) \end{pmatrix}\|_2.$$

Hence, the sequence $\{[X^{(l)}, Y^{(l)}]^T\}$ converges to the solution $\{[X^*, Y^*]^T\}$ of the coupled linear matrix equation (1.1). Similarly, by using Theorem 4.1, for $r^{(l)} = c - \mathcal{A}x^{(l)}$, we obtain

$$\|\mathcal{H}r^{(l)}\|_2 \leq \hat{\gamma}^{(l)} \sqrt{\kappa(\mathcal{H})}\|\mathcal{H}r^{(l-1)}\|_2, \tag{4.12}$$

where $\hat{\gamma}^{(l)} = \gamma^{(l)}\dfrac{1+\varrho}{1-\varrho}$, by (4.5) and $\varrho < \eta$ we have

$$\hat{\gamma}^{(l)} \leq \left(2\left(\frac{\theta-1}{\theta+1}\right)^{k_l}(1+\eta)+\eta\right)\frac{1+\eta}{1-\eta}\theta,$$

and thus, (4.12) can be rewritten as follow:

$$\|\mathcal{H}r^{(l)}\|_2 \leq \hat{\omega}^{(l)}\|\mathcal{H}r^{(l-1)}\|_2 \tag{4.13}$$

where

$$\hat{\omega}^{(l)} = \omega^{(l)}\frac{1+\eta}{1-\eta}.$$

It is obvious that for $\eta \in (0, \dfrac{\sqrt{(\theta+1)^2+4\theta}-(\theta+1)}{2\theta})$, we can obtain $0 < \dfrac{(1+\eta)\eta\theta}{(1-\eta)} < 1$. Thus for $\hat{\omega} \in (\dfrac{(1+\eta)\eta\theta}{(1-\eta)}, 1)$, if

$$k_l \geq \frac{ln((\hat{\omega}(1-\eta)-\theta\eta(1+\eta))/(2\theta(1+\eta)^2))}{ln((\theta-1)/(\theta+1))}, \quad l = 0, 1, 2, ...,$$

then $\hat{\omega}^{(l)} \leq \hat{\omega}$. Now, from (4.13), we deduce

$$\|\mathcal{H}r^{(l)}\|_2 \leq \hat{\omega}^{l+1}\|\mathcal{H}r^{(0)}\|_2.$$

Therefore, the sequences $\{r^{(l)}\}_{l=0}^{\infty}$ and $\{R^{(l)}\}_{l=0}^{\infty}$ converge to zero. This completes the proof. □

The following theorem shows that under what condition on eigenvalues of the matrices A, B, H and G, the symmetric part of matrix $\mathcal{A}$ is dominant in the skew-symmetric part.

Theorem 4.4. *Suppose* $\mathcal{H} = \mathcal{M}_1 + \mathcal{N}_1$ *and* $\mathcal{S} = \mathcal{M}_2 + \mathcal{N}_2$*, where* $\mathcal{M}_1, \mathcal{N}_1, \mathcal{M}_2$ *and* $\mathcal{N}_2$ *are defined as in Lemma 4.2. If*

$$\min_{i,j}\{\lambda_i(B)\lambda_j(A), \lambda_i(H)\lambda_j(G)\} \geq 0,$$

then $\|\mathcal{H}\|_2 > \|\mathcal{S}\|_2$.

Proof. In view of proof of Lemma 4.2, it can be seen that

$$\begin{aligned}\|\mathcal{H}\|_2 &= |\lambda_{max}(\mathcal{H})| \geq \lambda_{min}(\mathcal{M}_1 + \mathcal{N}_1) \geq \lambda(\mathcal{M}_1) + \lambda(\mathcal{N}_1)\\ &\geq \min_{i,j}\{\lambda_i(B)\lambda_j(A), \lambda_i(H)\lambda_j(G)\}\\ &+ \frac{1}{2}\min_{i,j}\{2\lambda_i(DF)\lambda_j(EC) + \sigma_i(F)\sigma_j(E) + \sigma_i(D)\sigma_j(C)]^{\frac{1}{2}},\end{aligned} \tag{4.14}$$

and

$$\|\mathcal{S}\|_2 \leq |\lambda_{max}(\mathcal{N}_2)| \leq \frac{1}{2}\max_{i,j}|2\lambda_i(DF)\lambda_j(EC) - \sigma_i(F)\sigma_j(E) - \sigma_i(D)\sigma_j(C)|^{\frac{1}{2}}. \tag{4.15}$$

Since the eigenvalues of a skew-symmetric matrix $\mathcal{N}_2$ are complex,it follows that

$$2\lambda_i(DF)\lambda_j(EC) - \sigma_i(F)\sigma_j(E) - \sigma_i(D)\sigma_j(C) \leq 0, \forall i, j.$$

In addition to this, it can be shown that

$$\begin{aligned}&\{2\lambda_i(DF)\lambda_j(EC) + \sigma_i(F)\sigma_j(E) + \sigma_i(D)\sigma_j(C)]^{\frac{1}{2}} \geq\\ &|2\lambda_i(DF)\lambda_j(EC) - \sigma_i(F)\sigma_j(E) - \sigma_i(D)\sigma_j(C)|^{\frac{1}{2}}, \forall i, j.\end{aligned} \tag{4.16}$$

Now, using the relations (4.14), (4.15) and (4.16) and hypothesis of theorem, it can be concluded that $\|\mathcal{H}\|_2 > \|\mathcal{S}\|_2$. □

This implies that the coefficient of matrix $\mathcal{A}$ has a dominant symmetric part.

5. Numerical examples

In this section, we present some numerical examples to illustrate the ability of the new algorithm for solving the generalized coupled Sylvester equation (1.1).
In the following, we compare and evaluate the implementation of the new method against the GLGMRES [13] and Bicgstab methods.
For all examples, we use $[X_0, Y_0] = [0_{n\times s}, 0_{n\times s}]$ as the initial guess. The inner and outer threshold are considered as $\eta = 0.01$ and $\epsilon = 10^{-6}$, respectively. Also, we use the largest maximum number of outer iterations $l_{max} = 2000$ and the inner iterations number $j_{max} = 5$. We compare these methods to four aspects: the number of iterations (iter), the CPUtime in seconds (referred to as CPU), the error term (Error), and the relative residual F-norm. The error term and the relative residual are defined by

$$Error = \|[X^{(l)}, Y^{(l)}] - [X^*, Y^*]\|_F,$$

$$res.norm = \frac{\sqrt{\|M - AX^{(l)}B - CY^{(l)}D\|_F^2 + \|N - EX^{(l)}F - GY^{(l)}H\|_F^2}}{\sqrt{\|M - AX^{(0)}B - CY^{(0)}D\|_F^2 + \|N - EX^{(0)}F - GY^{(0)}H\|_F^2}},$$

where $[X^*, Y^*]$ shows the exact solution. We used the conditions

$$\frac{\sqrt{\|\hat{R}_1^{(l,j+1)}\|_F^2 + \|\hat{R}_2^{(l,j+1)}\|_F^2}}{\sqrt{\|\hat{R}_1^{(l,0)}\|_F^2 + \|\hat{R}_2^{(l,0)}\|_F^2}} \leq \eta,$$

and

$$\frac{\sqrt{\|M - AX^{(l)}B - CY^{(l)}D\|_F^2 + \|N - EX^{(l)}F - GY^{(l)}H\|_F^2}}{\sqrt{\|M - AX^{(0)}B - CY^{(0)}D\|_F^2 + \|N - EX^{(0)}F - GY^{(0)}H\|_F^2}} \leq \epsilon,$$

as inner and outer stopping criterion, repectively. Also, the number of iterations of NSCG method in the Tables 1 and 2 indicate the number of outer iterations. All the numerical examples were executed in MATLAB 2015b on a PC-pentium (R), Core(TM) i5 CPU, 4.00 GB of RAM.

Example 5.1. For the first example, we consider the generalized coupled Sylvester equation

$$\begin{cases} AXB + CYD = M, \\ EXF + GYH = N, \end{cases}$$

where

$$A = \begin{pmatrix} 16 & -2 & & -2 \\ -2 & 16 & \ddots & \\ & \ddots & \ddots & -2 \\ -2 & & -2 & 16 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad B = \begin{pmatrix} 16 & -1 & & -1 \\ -1 & 16 & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & 16 \end{pmatrix} \in \mathbb{R}^{s\times s},$$

$$D = \begin{pmatrix} 16 & -4 & & -4 \\ -4 & 16 & \ddots & \\ & \ddots & \ddots & -4 \\ -4 & & -4 & 16 \end{pmatrix} \in \mathbb{R}^{s\times s}, \quad G = \begin{pmatrix} 4 & -1 & & -1 \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

$E = A$, $H = D$ and $F = C = I_n$. The matrices M and N are chosen such that $[X^* = tridiag(1, 1, 0), Y^* = tridiag(0, -1, 1)]$ is the exact solution of the generalized coupled matrix equation. In this example, the matrix $\mathcal{A}$ in (1.2) is a non-symmetric positive definite matrix.

From Table 1 and Figure 1, we can see that the NSCG method is more efficient than GLGMRES(3) and Bicgstab. But in general, we may not conclude that this method is better than GLGMRES and Bicgstab because NSCG method convergence is depend on the symmetric part and skew-symmetric part of the cofficient matrix $\mathcal{A}$. If the symmetric part of the coefficient matrix is dominant positive definite, then we can conclude that NSCG method is a good iteration method for solving (1.1).

Example 5.2. For the second test example, consider $A = nos5$, in which nos5 has been downloaded from [2]. The matrix nos5 is real symmetric of size 468×468 and with 5172 nonzero

Table 1: The number of iterations, CPU-time, the Frobenius norm of the residual and the Frobenius norm of the absolute error for Example 5.1.

(n, s)	Method	iter.	CPU	res.norm	Error
(1000, 1000)	NSCG	7	105.10	8.6884e-07	1.7153e-04
	GLGMRES(3)	15	187.59	8.5756e-07	4.6898e-04
	Bicgstab	22	148.78	5.0480e-07	3.2600e-04
(2000, 1000)	NSCG	7	404.89	8.3823e-07	1.6602e-04
	GLGMRES(3)	15	482.91	9.0155e-07	4.9267e-04
	Bicgstab	22	415.46	6.0497e-07	3.8220e-04
(3000, 1000)	NSCG	7	542.55	8.3823e-07	1.6602e-04
	GLGMRES(3)	15	555.92	9.0155e-07	4.9267e-04
	Bicgstab	22	682.85	6.0497e-07	3.8220e-04

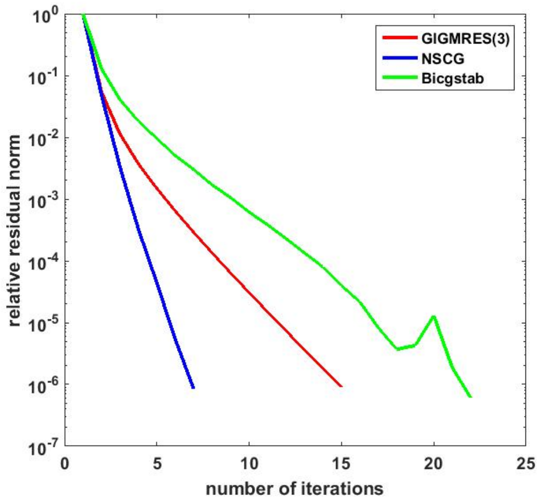

Figure 1: The convergence curves of NSCG, GL-GMRES(3) and Bicgstab methods for example 5.1 with n=3000 and s=1000.

entries. We consider $G = A, E = C, H = B$, where

$$C = \begin{pmatrix} 2 & 1+\frac{1}{n+1} & & \\ -1-\frac{1}{n+1} & 2 & \ddots & \\ & \ddots & \ddots & 1+\frac{1}{n+1} \\ & & -1-\frac{1}{n+1} & 2 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

$$B = \begin{pmatrix} 3 & 1+\frac{1}{s+1} & & \\ -1-\frac{1}{s+1} & 3 & \ddots & \\ & \ddots & \ddots & 1+\frac{1}{s+1} \\ & & -1-\frac{1}{s+1} & 3 \end{pmatrix} \in \mathbb{R}^{s\times s},$$

and

$$D = \begin{pmatrix} 16 & -1 & & \\ -1 & 16 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 16 \end{pmatrix} \in \mathbb{R}^{s\times s}, \quad F = \begin{pmatrix} 20 & -5 & & \\ -3 & 20 & \ddots & \\ & \ddots & \ddots & -5 \\ & & -3 & 20 \end{pmatrix} \in \mathbb{R}^{s\times s}.$$

Also, the matrices M and N are choosen such that $[X^*, Y^*]$ is the exact solution of (1.1), where

$$X^* = zeros(n, s), \quad Y^* = eye(n, s).$$

Finally, the Table 2 and Figure 2 show the convergence histories for the NSCG, GL-GMRES(3) and Bicgstab methods. When s increases, the Frobenius norm of the absolute error of GLGMRES(3) and Bicgstab increase. We observe that the convergence behavior of NSCG is better than GLGMRES(3) and Bicgstab.

6. Conclusion

We have described NSCG method for solving the generalized coupled Sylvester matrix equation (1.1) and it is compared with GL-GMRES(3) and Bicgstab. For large sparse problems, we can apply NSCG, GLGMRES and Bicgstab methods. In examples 5.1 and 5.2, we observe that the convergence of NSCG is better than the other approaches. But, in general case, we can not say NSCG method is the best iteration method for solving (1.1). Because The conditions of the problem must be met.

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Table 2: The number of iterations, CPU-time, the Frobenius norm of the residual and the Frobenius norm of the absolute error for Example 5.2.

(n, s)	Method	iter.	CPU	res.norm	Error
(468, 16)	NSCG	514	16.454	9.9991e-07	2.0766e-03
	GLGMRES(3)	2001	28.759	1.5396e-06	1.1745e-02
	Bicgstab	1091	7.354	6.4860e-07	3.9206e-03
(468, 32)	NSCG	584	40.072	9.9951e-07	3.6052e-03
	GLGMRES(3)	2001	52.116	2.6984e-06	2.9165e-02
	Bicgstab	1580	18.625	9.4374e-07	9.6811e-03
(468, 64)	NSCG	616	103.56	9.9734e-07	5.5404e-03
	GLGMRES(3)	2001	109.23	3.2368e-06	4.7600e-02
	Bicgstab	1344	35.788	8.0320e-07	1.4406e-02
(468, 234)	NSCG	718	587.33	9.9769e-07	1.3569e-03
	GLGMRES(3)	2001	694.42	2.5496e-06	1.0549e-01
	Bicgstab	1224	193.97	7.5594e-07	3.5302e-02

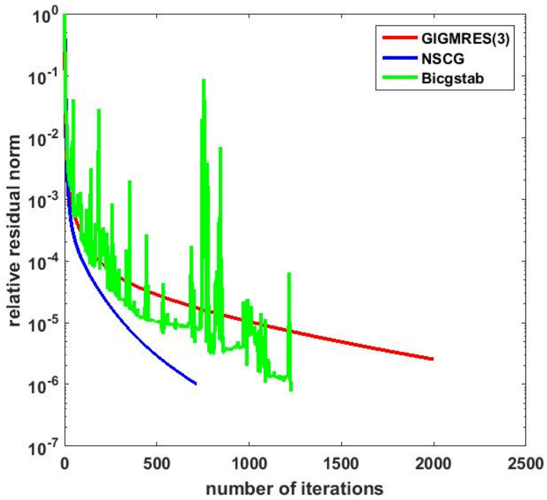

Figure 2: The Convergence curves of NSCG, GL-GMRES(3) and Bicgstab methods with n=468, s=234.

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