C*-Extreme Points and C*-Faces of the Epigraph of C*-Affine Maps in *-Rings

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Abstract

In this paper, we define the notion of C*-affine maps in the unital *-rings and we investigate the C*-extreme points of the graph and epigraph of such maps. We show that for a C*-convex map f defined on a unital *-ring R satisfying the positive square root axiom with an additional condition, the graph of f is a C*-face of the epigraph of f. Moreover, we prove some results about the C*-faces of C*-convex sets in *-rings.

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1. Introduction

One of the form of non-commutative convexity is C*-convexity. Formal study of C*-convexity was initiated by Loebl and Paulsen in [9]. Farenick and Morenz proved that each irreducible ele-
Calised the notion of discovered a right analog of linear extreme points, called structural elements to prove a generalised Definition 1.1. A subset the epigraph of \( f \) convex space \( S \) there exists Also, it is said that case \( y \) for some identity element. An element \( f \) (in particular in \( B \)) of \( \mathbb{R} \) is called proper if the condition \( x \leq y \) whenever \( x \in \mathbb{R} \). The involution of \( \mathbb{R} \) may be ordered by writing \( x \leq y \) in case \( y - x \geq 0 \). The involution of \( \mathbb{R} \) is called proper if \( x^*x = 0 \) implies that \( x = 0 \) for every \( x \in \mathbb{R} \). Also, it is said that \( \mathbb{R} \) satisfies the positive square root axiom if for every positive element \( x \in \mathbb{R} \), there exists \( y \in \{x\}'' \) such that \( y \geq 0 \), and \( x = y^2 \), where \( \{x\}'' \) denotes the double commutant of \( \{x\} \) in \( \mathbb{R} \). To study the \( \ast \)-rings and Baer \( \ast \)-rings, the reference [1] is essential.

Definition 1.1. A subset \( K \) of a unital \( \ast \)-ring \( \mathcal{R} \) is called \( C^\ast \)-convex, if
\[
\sum_{i=1}^{n} a_i^* x_i a_i \in K,
\]
whenever \( x_i \in K, a_i \in \mathcal{R} \) for all \( i \) and \( \sum_{i=1}^{n} a_i^* a_i = 1_\mathcal{R} \).

Definition 1.2. Let \( K \) be a \( C^\ast \)-convex subset of \( \mathcal{R} \). An element \( x \in K \) is called a \( C^\ast \)-extreme point of \( K \) if the condition
\[
x = \sum_{i=1}^{n} a_i^* x_i a_i, \quad \sum_{i=1}^{n} a_i^* a_i = 1_\mathcal{R}, \quad x_i \in K, a_i \text{ is invertible in } \mathcal{R}, n \in \mathbb{N}
\]
implies that all \( x_i \) are unitarily equivalent to \( x \) in \( \mathcal{R} \), that is, there exist unitaries \( u_i \in \mathcal{R} \) such that \( x_i = u_i^* x u_i \) for all \( i \).

The set of all \( C^\ast \)-extreme points of \( K \) is denoted by \( C^\ast\text{-ext}(K) \).

In addition, if condition (1) holds, then we say that \( x \) is a proper \( C^\ast \)-convex combination of \( x_1, \ldots, x_n \).

In [4] we defined the notion of \( C^\ast \)-convex maps as the following:

Definition 1.3. Let \( K \) be a \( C^\ast \)-convex subset of \( \mathcal{R} \). We say that a map \( f : K \to K \) is \( C^\ast \)-convex if
\[
f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i
\]
where \( n \in \mathbb{N}, x_i \in K, a_i \in \mathcal{R} \), and \( \sum_{i=1}^{n} a_i^* a_i = 1_\mathcal{R} \). If \( -f \) is \( C^\ast \)-convex, we say that \( f \) is \( C^\ast \)-concave.
In this paper we focus on the equality in the above definition and we call such maps the $C^*$-affine maps.

**Definition 1.4.** Let $K$ be a $C^*$-convex subset of $\mathcal{R}$. We say that a map $f : K \to K$ is $C^*$-affine if

$$f(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i f(x_i)$$

where $n \in \mathbb{N}$, $x_i \in K$, $a_i \in \mathcal{R}$, and $\sum_{i=1}^{n} a_i a_i = 1_{\mathcal{R}}$.

Note that if $\mathcal{R}$ is a $\ast$-algebra, then every $C^*$-affine map is affine in the classical sense.

**Example 1.5.** The following maps are $C^*$-affine on $\mathcal{R}$.

1. $f(x) = nx$ where $n \in \mathbb{N}$.
2. $f(x) = x^\ast$.
3. $f(x) = \alpha x$ where $\alpha \in \mathbb{C}$ and $\mathcal{R}$ is a $\ast$-algebra.
4. $f(x) = \alpha x + b$ where $\alpha \in \mathbb{C}$, $b \in \mathbb{Z}(\mathcal{R})$ (the center of $\mathcal{R}$) and $\mathcal{R}$ is a $\ast$-algebra.

**Remark 1.6.** The composition of two $C^*$-affine maps, is also a $C^*$-affine map.

**Definition 1.7.** The graph and epigraph of a map $f : \mathcal{R} \to \mathcal{R}$ that we will denote by $\text{graph}(f)$ and $\text{epi}(f)$ respectively, are defined as:

$$\text{graph}(f) = \{(x, y) : x \in \mathcal{R}, y = f(x)\} \subseteq \mathcal{R} \oplus \mathcal{R},$$

$$\text{epi}(f) = \{(x, y) : x \in \mathcal{R}, f(x) \leq y\} \subseteq \mathcal{R} \oplus \mathcal{R}.$$ 

**2. $C^*$-extreme points of the graph and epigraph of $C^*$-affine maps**

In this section the relation between the $C^*$-extreme points of the graph and epigraph of a $C^*$-affine map and the $C^*$-extreme points of its domain and image are investigated.

It is shown in [3] and [4] that for every $C^*$-affine map $f$ on a $C^*$-convex subset $K$ of the unital $\ast$-ring $\mathcal{R}$, $\text{graph}(f)$ and $\text{epi}(f)$ are $C^*$-convex subsets of $\mathcal{R} \oplus \mathcal{R}$ in the sense that

$$\sum_{i=1}^{n} (a_i, a_i)^\ast (x_i, f(x_i))(a_i, a_i) \in \text{graph}(f),$$

and

$$\sum_{i=1}^{n} (a_i, a_i)^\ast (x_i, y_i)(a_i, a_i) \in \text{epi}(f),$$

for every $(x_i, f(x_i)) \in \text{graph}(f)$ and $(x_i, y_i) \in \text{epi}(f)$ and $a_i \in \mathcal{R}$ where $\sum_{i=1}^{n} a_i a_i = 1_{\mathcal{R}}$. Also, the author has proved the following theorem in [3].

**Theorem 2.1.** Let $f$ be a $C^*$-affine map on a $C^*$-convex subset $K$ of the unital $\ast$-ring $\mathcal{R}$. Then the following conditions are equivalent,

1) $x \in C^* - \text{ext}(K),$

2) $(x, f(x)) \in C^* - \text{ext}(\text{graph}(f)).$
In this section we investigate some properties of $C^*$-affine maps preserve $C^*$-extreme points ([3], Theorem 2.3]). Indeed, the equivalence of the following assertions for every injective $C^*$-affine map $f$ on a $C^*$-convex set $K$ has been shown in [3].

1) $x \in C^* - \text{ext}(K)$,
2) $f(x) \in C^* - \text{ext}(f(K))$,
3) $(x, f(x)) \in C^* - \text{ext}(\text{graph}(f))$.

If $\mathcal{A}$ is a unital $*$-algebra, then every $C^*$-convex set $K$ in $\mathcal{A}$ is a convex set and every $C^*$-extreme point of $K$ is an extreme point in the usual sense. Since we conclude that if we replace the $C^*$-convexity and the concept of $C^*$-extreme point by $C^*$-convexity and the concept of extreme point in the above corollaries, respectively, we obtain the similar conclusions in the classical convexity immediately ([4], corollaries 2.5, 2.6]).

The author has proved the following theorem ([2], Theorem 2.3]) as a result about the $C^*$-extreme points of the epigraph of $C^*$-affine maps on $C^*$-convex sets.

**Theorem 2.2.** For every $C^*$-affine map $f$ on a $C^*$-convex set $K$ in the unital $*$-ring $\mathcal{R}$ with the following two conditions:

1) $(2.1)_{\mathcal{R}}^{-1/2}$ exists in $\mathcal{R}$,
2) $x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$

we have

$$C^* - \text{ext}(\text{epi}(f)) = \text{graph}(f|_{C^* - \text{ext}(K)}) = \{(x, f(x)) \mid x \in C^* - \text{ext}(K)\}.$$

### 3. $C^*$-faces

P. B. Morenz extended the notion of face from linear convexity to $C^*$-face of $C^*$-convex subsets of a $C^*$-algebra in [13]. The author and G. H. Esslamzadeh generalised this notion to $*$-rings [4]. In this section we investigate some properties of $C^*$-faces and specially we show that the graph of a $C^*$-affine map is a $C^*$-face of its epigraph provided that it satisfies some conditions.

**Definition 3.1.** A nonempty subset $F$ of a $C^*$-convex set $K \subseteq \mathcal{R}$ is called a $C^*$-face of $K$, if the condition $x \in F$ and $x = \sum_{i=1}^{n} a_i^* x_i a_i$ as a proper $C^*$-convex combination of elements $x_i \in K$, implies that $x_i \in F$ for all $i$.

**Example 3.2.** (1) Let $K$ be a $C^*$-convex subset of $\mathcal{R}$. Then $K$ is a $C^*$-face of $K$. Thus the set of $C^*$-faces of every $C^*$-convex set is nonempty.

(2) The set $C^*-\text{ext}(K)$ is a $C^*$-face of $K$.

**Theorem 3.3** ([4], Theorem 3.7). Suppose that $F_1$ and $F_2$ are $C^*$-faces of $C^*$-convex subsets $K_1$ and $K_2$ in $\mathcal{R}$ respectively. Then,

1) $F_1 \cap F_2$ is a $C^*$-face of $K_1 \cap K_2$ provided that $F_1 \cap F_2 \neq \emptyset$.
2) If $K_1 \subseteq K_2$, then $F_2 \cap K_1$ is a $C^*$-face of $K_1$ provided that it is nonempty.
3) If $F \subseteq F_1$ and $F$ is a $C^*$-face of $C^*$-co($F_1$), then $F$ is a $C^*$-face of $K_1$.
4) If $K_1 \subseteq K_2$, then $K_1 \cap C^*-\text{ext}(K_2) \subseteq C^*-\text{ext}(K_1)$.

**Theorem 3.4.** Let $\mathcal{R}$ be a unital $*$-ring satisfying the positive square root axiom and $x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$ and $f$ be a $C^*$-convex map on $\mathcal{R}$. Then graph$(f)$ is a $C^*$-face of epi$(f)$.
Proof. Suppose that \((x, f(x)) \in \text{graph}(f)\) and

\[(x, f(x)) = \sum_{i=1}^{n} (a_i, a_i^\ast)(x_i, y_i)(a_i, a_i)\]

is a proper \(C^*\)-convex combination of elements \((x_i, y_i) \in \text{epi}(f)\). We show that \((x_i, y_i) \in \text{graph}(f)\) for each \(i \ (1 \leq i \leq n)\). Since

\[(x, f(x)) = (\sum_{i=1}^{n} a_i^\ast x_i a_i, \sum_{i=1}^{n} a_i^\ast y_i a_i),\]

so,

\[x = \sum_{i=1}^{n} a_i^\ast x_i a_i, \quad f(x) = \sum_{i=1}^{n} a_i^\ast y_i a_i. \tag{3.1}\]

Since \(f\) is a \(C^*\)-convex map, so we have

\[f(x) = f\left(\sum_{i=1}^{n} a_i^\ast x_i a_i\right) \leq \sum_{i=1}^{n} a_i^\ast f(x_i) a_i. \tag{3.2}\]

We conclude from (3.1) and (3.2) that

\[\sum_{i=1}^{n} a_i^\ast y_i a_i \leq \sum_{i=1}^{n} a_i^\ast f(x_i) a_i\]

and hence

\[\sum_{i=1}^{n} a_i^\ast (y_i - f(x_i)) a_i \leq 0. \tag{3.3}\]

On the other hand, \((x_i, y_i) \in \text{epi}(f)\), so \(y_i \geq f(x_i)\), and hence \(y_i - f(x_i) \geq 0\). Thus

\[\sum_{i=1}^{n} a_i^\ast (y_i - f(x_i)) a_i \geq 0. \tag{3.4}\]

Using the fact that every positive element in \(\mathcal{R}\) has a positive square root, and the assumption \(x_1^\ast x_1 + x_2^\ast x_2 + \cdots + x_n^\ast x_n = 0\) implies that \(x_1 = x_2 = \cdots = x_n = 0\) for every \(x_i \in \mathcal{R}\) and \(n \in \mathbb{N}\), we conclude from (3.3) and (3.4) that

\[\sum_{i=1}^{n} a_i^\ast (y_i - f(x_i)) a_i = 0.\]

and hence for each \(i\),

\[a_i^\ast (y_i - f(x_i)) a_i = 0.\]

The invertibility of \(a_i\) for each \(i\), implies that \(y_i - f(x_i) = 0\). Therefore, \(y_i = f(x_i)\) and the proof is complete. \(\Box\)
Injectivity of $g_i$ for each $1 \leq i \leq n$.

**Proposition 3.5.** $F$ is a $C^*$-face of a $C^*$-convex set $K$ in the unital $*$-ring $\mathcal{R}$ if and only if $F^*$ is a $C^*$-face of $C^*$-convex set $K^*$.

**Proof.** First note that if $K$ is a $C^*$-convex subset of $\mathcal{R}$, then $K^*$ is also a $C^*$-convex subset of $\mathcal{R}$. Suppose that $F$ is a $C^*$-face of $K$. So $F^*$ is a nonempty subset of $K^*$. Let $y \in F^*$ and $y = \sum_{i=1}^{n} a_i^* y_i a_i$ be a proper $C^*$-convex combination of elements $y_i \in K^*$. Then there exist $x(y^*)$ in $F$ and $x_i \in K$ such that $y = x^*$ and $y_i = x_i^*$. So

$$x^* = \sum_{i=1}^{n} a_i^* x_i^* a_i = (\sum_{i=1}^{n} a_i^* x_i a_i)^*,$$

and hence $x = \sum_{i=1}^{n} a_i^* x_i a_i$ is a proper $C^*$-convex combination of elements $x_i \in K$ ($1 \leq i \leq n$). Thus, $x_i \in F$ and hence $g_i = x_i^* \in F^*$ for each $i$ ($1 \leq i \leq n$). Therefore, $F^*$ is a $C^*$-face of $K^*$. The converse is immediate by replacing $F^*$ and $K^*$ instead of $F$ and $K$ respectively.

The following theorem deals with the $C^*$-faces of invertible $C^*$-affine maps.

**Theorem 3.6.** Suppose that $K_1$ and $K_2$ are $C^*$-convex subsets of the unital $*$-ring $\mathcal{R}$ and $g : K_1 \rightarrow K_2$ is a bijective $C^*$-affine map. Then

1) $g^{-1} : K_2 \rightarrow K_1$ is a $C^*$-affine map.

2) $F$ is a $C^*$-face of $K_1$ if and only if $g(F)$ is a $C^*$-face of $K_2$.

3) $F$ is a $C^*$-face of $K_2$ if and only if $g^{-1}(F)$ is a $C^*$-face of $K_1$.

**Proof.**

1) Let $\sum_{i=1}^{n} a_i^* y_i a_i$ be a $C^*$-convex combination of elements $y_i \in K_2$ ($1 \leq i \leq n$). Then there exist $x_i \in K_1$ such that $g(x_i) = y_i$ for each $i$ ($1 \leq i \leq n$). So,

$$g^{-1}(\sum_{i=1}^{n} a_i^* y_i a_i) = g^{-1}(\sum_{i=1}^{n} a_i^* g(x_i) a_i) = g^{-1}(g(\sum_{i=1}^{n} a_i^* x_i a_i)) = \sum_{i=1}^{n} a_i^* x_i a_i = \sum_{i=1}^{n} a_i^* g^{-1}(g(x_i)) a_i = \sum_{i=1}^{n} a_i^* g^{-1}(y_i) a_i.$$

Therefore, $g^{-1}$ is a $C^*$-affine map.

2) Let $F$ be a $C^*$-face of $K_1$, $y \in g(F)$ and $y = \sum_{i=1}^{n} a_i^* y_i a_i$ a proper $C^*$-convex combination of elements $y_i \in K_2$ ($1 \leq i \leq n$). So there exist $x \in F$ and $x_i \in K_1$ such that $y = g(x)$ and $y_i = g(x_i)$ for each $i$ ($1 \leq i \leq n$). Thus,

$$g(x) = \sum_{i=1}^{n} a_i^* g(x_i) a_i = g(\sum_{i=1}^{n} a_i^* x_i a_i).$$

Injectivity of $g$ implies that

$$x = \sum_{i=1}^{n} a_i^* x_i a_i.$$
Since $F$ is a $C^*$-face of $K_1$, $x_i \in F$ ($1 \leq i \leq n)$. So $y_i = g(x_i) \in g(F)$ ($1 \leq i \leq n$) and $g(F)$ is a $C^*$-face of $K_2$.

Conversely, let $g(F)$ be a $C^*$-face of $K_2$, $x \in F$, and $x = \sum_{i=1}^{n} a_i^* x_i a_i$ a proper $C^*$-convex combination of elements $x_i \in K_1$ ($1 \leq i \leq n$). Then

$$g(x) = g\left(\sum_{i=1}^{n} a_i^* x_i a_i\right) = \sum_{i=1}^{n} a_i^* g(x_i) a_i,$$

where $g(x) \in g(F)$ and $g(x_i) \in K_2$ for each $i$ ($1 \leq i \leq n$). Thus, $g(x_i) \in g(F)$ and since $g$ is injective, $x_i \in F$ for each $i$ ($1 \leq i \leq n$). Therefore, $F$ is a $C^*$-face of $K_1$.

3) Since $g^{-1}$ is a bijective $C^*$-affine map, so part 3) comes from part 2).

**Theorem 3.7.** Suppose that $F_1$ and $F_2$ are $C^*$-faces of $C^*$-convex subsets $K_1$ and $K_2$ of $\mathcal{R}$ respectively. Then,

1) $K_1 \times K_2$ is a $C^*$-convex subset of $\mathcal{R} \times \mathcal{R}$ with the pointwise operations. 2) $F_1 \times F_2$ is a $C^*$-face of $K_1 \times K_2$.

**Proof.** 1) Suppose that $(x_i, y_i) \in K_1 \times K_2$ and $(a_i, b_i) \in \mathcal{R}$ for each $i$ ($1 \leq i \leq n$) such that

$$\sum_{i=1}^{n} (a_i, b_i)^* (a_i, b_i) = (1_{\mathcal{R}}, 1_{\mathcal{R}}).$$

Since $K_1$ and $K_2$ are $C^*$-convex subsets of $\mathcal{R}$ and

$$\sum_{i=1}^{n} a_i^* a_i = \sum_{i=1}^{n} b_i^* b_i = 1_{\mathcal{R}},$$

we have

$$\sum_{i=1}^{n} (a_i, b_i)^* (x_i, y_i) (a_i, b_i) = \sum_{i=1}^{n} (a_i^* x_i a_i, b_i^* y_i b_i) = \left( \sum_{i=1}^{n} a_i^* x_i a_i, \sum_{i=1}^{n} b_i^* y_i b_i \right) \in K_1 \times K_2.$$

2) Let $(x, y) \in F_1 \times F_2$ and

$$(x, y) = \sum_{i=1}^{n} (a_i, b_i)^* (x_i, y_i) (a_i, b_i) \quad (3.5)$$

be a proper $C^*$-convex combination of elements $(x_i, y_i) \in K_1 \times K_2$. We show that $(x_i, y_i) \in F_1 \times F_2$ for all $i$ ($1 \leq i \leq n$). The relation (3.3) implies that $x = \sum_{i=1}^{n} a_i^* x_i a_i$ and $y = \sum_{i=1}^{n} b_i^* y_i b_i$. Also,

$$\sum_{i=1}^{n} (a_i, b_i)^* (a_i, b_i) = 1_{\mathcal{R} \times \mathcal{R}} = (1_{\mathcal{R}}, 1_{\mathcal{R}})$$

implies that $\sum_{i=1}^{n} a_i^* a_i = \sum_{i=1}^{n} b_i^* b_i = 1_{\mathcal{R}}$. Moreover, since $(a_i, b_i)$ is invertible in $\mathcal{R} \times \mathcal{R}$, we have $a_i$ and $b_i$ are invertible in $\mathcal{R}$. Thus, $x = \sum_{i=1}^{n} a_i^* x_i a_i$ and $y = \sum_{i=1}^{n} b_i^* y_i b_i$ are proper $C^*$-convex combinations of elements $x_i \in K_1$ and $y_i \in K_2$ ($1 \leq i \leq n$) respectively. Since $F_1$ and $F_2$ are $C^*$-faces in $K_1$ and $K_2$ respectively, we conclude that $x_i \in F_1$ and $y_i \in F_2$ for each $i$ ($1 \leq i \leq n$). Therefore, $(x_i, y_i) \in F_1 \times F_2$ for each $i$ ($1 \leq i \leq n$) and hence $F_1 \times F_2$ is a $C^*$-face of $K_1 \times K_2$ in $\mathcal{R} \times \mathcal{R}$.

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