Wavelets and Linear Algebra 5(2) (2018) 21- 28

Vali-e-Asr University of Rafsanjan

Wavelets and Linear Algebra

http://wala.vru.ac.ir

C^*-Extreme Points and C^*-Faces of the Epigraph of C^*-Affine Maps in $*$-Rings

Ali Ebrahimi Meymand[a]

[a] *Assistant professor, Department of Mathematics, Faculty of Mathematical Sciences, Vali-e-asr university of Rafsanjan, Rafsanjan, Islamic Republic of Iran.*

Article Info

Article history:
Received 17 July 2018
Accepted 30 December 2018
Available online 12 January 2019
Communicated by Abbas Salemi

Keywords:
C^*-affine map,
C^*-convexity,
C^*-extreme point,
C^*-face.

2000 MSC:
52A01, 16W10, 46L89.

Abstract

In this paper, we define the notion of C^*-affine maps in the unital $*$-rings and we investigate the C^*-extreme points of the graph and epigraph of such maps. We show that for a C^*-convex map f defined on a unital $*$-ring $\mathcal{R}$ satisfying the positive square root axiom with an additional condition, the graph of f is a C^*-face of the epigraph of f. Moreover, we prove some results about the C^*-faces of C^*-convex sets in $*$-rings.

1. Introduction

One of the form of non-commutative convexity is C^*-convexity. Formal study of C^*-convexity was initiated by Loebl and Paulsen in [9]. Farenick and Morenz proved that each irreducible ele-

*Corresponding author
Email address: a.ebrahimi@vru.ac.ir (Ali Ebrahimi Meymand)

http://doi.org/10.22072/wala.2018.90202.1184

ment of the C^*-algebra M_n of complex $n \times n$ matrices, is a C^*-extreme point, the relative extreme point, of the C^*-convex set that it generates [6]. Also they studied the C^*-extreme points of the C^*-convex space $S_H(A)$ of unital completely positive linear maps from a C^*-algebra A to the algebra $B(H)$of continuous linear operators on a complex Hilbert space H [5]. Moreover, in [13] Morenz discovered a right analog of linear extreme points, called structural elements to prove a generalised Krein Milman theorem for C^*-convex subsets of M_n. The author and G. H. Esslamzadeh generalised the notion of C^*-convexity and the relative extreme points, C^*-extreme points, to $*$-rings [4], see also [14]. Magajna proved that each weak* compact C^*-convex set in a hyper finite factor (in particular in $B(H)$) is the weak* closure of the C^*-convex hull of its C^*-extreme points [12], see also [10] and [11]. It is known that a real-valued function $f : \mathbb{R} \to \mathbb{R}$ is convex if and only if the epigraph of f is a convex set in $\mathbb{R}^2$. M. Kian extended this result for operator convex functions and C^*-convex sets in [8].

Throughout this paper $\mathcal{R}$ is a unital $*$-ring, that is, a ring with an involution which has an identity element. An element x in $\mathcal{R}$ is called positive, written $x \geq 0$, if $x = y_1^* y_1 + y_2^* y_2 + \cdots + y_n^* y_n$ for some $y_1, y_2, \cdots, y_n$ in $\mathcal{R}$. The self-adjoint elements of $\mathcal{R}$ may be ordered by writing $x \leq y$ in case $y - x \geq 0$. The involution of $\mathcal{R}$ is called proper if $x^* x = 0$ implies that $x = 0$ for every $x \in \mathcal{R}$. Also, it is said that $\mathcal{R}$ satisfies the positive square root axiom if for every positive element $x \in \mathcal{R}$, there exists $y \in \{x\}''$ such that $y \geq 0$, and $x = y^2$, where $\{x\}''$ denotes the double commutant of $\{x\}$ in $\mathcal{R}$. To study the $*$-rings and Baer $*$-rings , the reference [1] is essential.

Definition 1.1. A subset K of a unital $*$-ring $\mathcal{R}$ is called C^*-convex, if

$$\sum_{i=1}^{n} a_i^* x_i a_i \in K,$$

whenever $x_i \in K$, $a_i \in \mathcal{R}$ for all i and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

Definition 1.2. Let K be a C^*-convex subset of $\mathcal{R}$. An element $x \in K$ is called a C^*-extreme point of K if the condition

$$x = \sum_{i=1}^{n} a_i^* x_i a_i, \ \sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}, x_i \in K, a_i \text{ is invertible in } \mathcal{R}, n \in \mathbb{N} \quad (1)$$

implies that all x_i are unitarily equivalent to x in $\mathcal{R}$, that is, there exist unitaries $u_i \in \mathcal{R}$ such that $x_i = u_i^* x u_i$ for all i.

The set of all C^*-extreme points of K is denoted by C^*-$ext(K)$.

In addition, if condition (1) holds, then we say that x is a proper C^*-convex combination of $x_1, \ldots, x_n$.

In [4] we defined the notion of C^*-convex maps as the following:

Definition 1.3. Let K be a C^*-convex subset of $\mathcal{R}$. We say that a map $f : K \to K$ is C^*-convex if

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i$$

where $n \in \mathbb{N}$, $x_i \in K$, $a_i \in \mathcal{R}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$. If $-f$ is C^*-convex, we say that f is C^*-concave.

In this paper we focuse on the equality in the above definition and we call such maps the C^*-affine maps.

Definition 1.4. Let K be a C^*-convex subset of $\mathcal{R}$. We say that a map $f : K \to K$ is C^*-affine if

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) = \sum_{i=1}^{n} a_i^* f(x_i) a_i,$$

where $n \in \mathbb{N}$, $x_i \in K$, $a_i \in \mathcal{R}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

Note that if $\mathcal{R}$ is a $*$-algebra, then every C^*-affine map is affine in the classical sense.

Example 1.5. The following maps are C^*-affine on $\mathcal{R}$.
(1) $f(x) = nx$ where $n \in \mathbb{N}$.
(2) $f(x) = x^*$.
(3) $f(x) = \alpha x$ where $\alpha \in \mathbb{C}$ and $\mathcal{R}$ is a $*$-algebra.
(4) $f(x) = \alpha x + b$ where $\alpha \in \mathbb{C}$, $b \in Z(\mathcal{R})$ (the center of $\mathcal{R}$) and $\mathcal{R}$ is a $*$-algebra.

Remark 1.6. The composition of two C^*-affine maps, is also a C^*-affine map.

Definition 1.7. The graph and epigraph of a map $f : \mathcal{R} \to \mathcal{R}$ that we will denote by $graph(f)$ and $epi(f)$ respectively, are defined as:

$$graph(f) = \{(x, y) : x \in \mathcal{R}, y = f(x)\} \subseteq \mathcal{R} \oplus \mathcal{R},$$

$$epi(f) = \{(x, y) : x \in \mathcal{R}, f(x) \leq y\} \subseteq \mathcal{R} \oplus \mathcal{R}.$$

2. C^*-extreme points of the graph and epigraph of C^*-affine maps

In this section the relatin between the C^*-extreme points of the graph and epigraph of a C^*-affine map and the C^*-extreme points of its domain and image are investigated.

It is shawn in [3] and [4] that for every C^*-affine map f on a C^*-convex subset K of the unital $*$-ring $\mathcal{R}$, $graph(f)$ and $epi(f)$ are C^*-convex subsets of $\mathcal{R} \oplus \mathcal{R}$ in the sense that

$$\sum_{i=1}^{n} (a_i, a_i)^* (x_i, f(x_i))(a_i, a_i) \in graph(f),$$

and

$$\sum_{i=1}^{n} (a_i, a_i)^* (x_i, y_i)(a_i, a_i) \in epi(f),$$

for every $(x_i, f(x_i)) \in graph(f)$ and $(x_i, y_i) \in epi(f)$ and $a_i \in \mathcal{R}$ where $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$. Also, the author has proved the following theorem in [3].

Theorem 2.1. *Let f ba a C^*-affine map on a C^*-convex subset K of the unital $*$-ring $\mathcal{R}$. Then the following conditions are equivalent,*
1) $x \in C^* - ext(K)$,
2) $(x, f(x)) \in C^* - ext(graph(f))$.

The image of a C^*-convex set under a C^*-affine map is a C^*-convex set. Also, injective C^*-affine maps preserve C^*-extreme points ([[3], Theorem 2.3]). Indeed, the equivalence of the following assertions for every injective C^*-affine map f on a C^*-convex set K has been shawn in [3].
1) $x \in C^* - ext(K)$,
2) $f(x) \in C^* - ext(f(K))$,
3) $(x, f(x)) \in C^* - ext(graph(f))$.

If $\mathcal{A}$ is a unital $*$-algebra, then every C^*-convex set K in $\mathcal{A}$ is a convex set and every C^*-extreme point of K is an extreme point in the usual sense. So we conclude that if we replace the C^*-convexity and the concept of C^*-extreme point by convexity and the concept of extreme point in the above corollaries, respectively, we obtain the similar conclusions in the classical convexity immediately ([[4], corollaries 2.5 , 2.6]).

The author has proved the following theorem ([[2], Theorem 2.3]) as a result about the C^*-extreme points of the epigraph of C^*-affine maps on C^*-convex sets.

Theorem 2.2. *For every C^*-affine map f on a C^*-convex set K in the unital $*$-ring $\mathcal{R}$ with the following two conditions:*
1) $(2.1_{\mathcal{R}})^{-1/2}$ exists in $\mathcal{R}$,
2) $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$ we have

$$C^* - ext(epi(f)) = graph(f|_{C^*-ext(K)}) = \{(x, f(x)) \mid x \in C^* - ext(K)\}.$$

3. C^*-faces

P. B. Morenz extended the notion of face from linear convexity to C^*-face of C^*-convex subsets of a C^*-algebra in [13]. The author and G. H. Esslamzadeh generalised this notion to $*$-rings [4]. In this section we investigate some properties of C^*-faces and specially we show that the graph of a C^*-affine map is a C^*-face of its epigraph provided that it satisfies some conditions.

Definition 3.1. A nonempty subset F of a C^*-convex set $K \subseteq \mathcal{R}$ is called a C^*-face of K, if the condition $x \in F$ and $x = \sum_{i=1}^{n} a_i^* x_i a_i$ as a proper C^*-convex combination of elements $x_i \in K$, implies that $x_i \in F$ for all i.

Example 3.2. (1) Let K be a C^*-convex subset of $\mathcal{R}$. Then K is a C^*-face of K. Thus the set of C^*-faces of every C^*-convex set is nonempty.

(2) The set C^*-ext(K) is a C^*-face of K.

Theorem 3.3 ([4], Theorem 3.7)**.** *Suppose that F_1 and F_2 are C^*-faces of C^*-convex subsets K_1 and K_2 in $\mathcal{R}$ respectively. Then,*

1) $F_1 \cap F_2$ is a C^*-face of $K_1 \cap K_2$ provided that $F_1 \cap F_2 \neq \emptyset$.
2) If $K_1 \subseteq K_2$, then $F_2 \cap K_1$ is a C^*-face of K_1 provided that it is nonempty.
3) If $F \subset F_1$ and F is a C^*-face of C^*-$Co(F_1)$, then F is a C^*-face of K_1.
4) If $K_1 \subseteq K_2$, then $K_1 \cap C^*$-$ext(K_2) \subseteq C^*$-$ext(K_1)$.

Theorem 3.4. *Let $\mathcal{R}$ be a unital $*$-ring satisfying the positive square root axiom and $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$ and f be a C^*-convex map on $\mathcal{R}$. Then $graph(f)$ is a C^*-face of $epi(f)$.*

Proof. Suppose that $(x, f(x)) \in graph(f)$ and

$$(x, f(x)) = \sum_{i=1}^{n}(a_i, a_i)^*(x_i, y_i)(a_i, a_i)$$

is a proper C^*-convex combination of elements $(x_i, y_i) \in epi(f)$. We show that $(x_i, y_i) \in graph(f)$ for each i $(1 \leq i \leq n)$. Since

$$(x, f(x)) = (\sum_{i=1}^{n} a_i^* x_i a_i, \sum_{i=1}^{n} a_i^* y_i a_i),$$

so,

$$x = \sum_{i=1}^{n} a_i^* x_i a_i, \quad f(x) = \sum_{i=1}^{n} a_i^* y_i a_i. \tag{3.1}$$

Since f is a C^*-convex map, so we have

$$f(x) = f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i. \tag{3.2}$$

We conclude from (3.1) and (3.2) that

$$\sum_{i=1}^{n} a_i^* y_i a_i \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i$$

and hence

$$\sum_{i=1}^{n} a_i^* (y_i - f(x_i)) a_i \leq 0. \tag{3.3}$$

On the other hand, $(x_i, y_i) \in epi(f)$, so $y_i \geq f(x_i)$, and hence $y_i - f(x_i) \geq 0$. Thus

$$\sum_{i=1}^{n} a_i^* (y_i - f(x_i)) a_i \geq 0. \tag{3.4}$$

Using the fact that every positive element in $\mathcal{R}$ has a positive square root, and the assumption $x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n = 0$ implies that $x_1 = x_2 = \cdots = x_n = 0$ for every $x_i \in \mathcal{R}$ and $n \in \mathbb{N}$, we conclude from (3.3) and (3.4) that

$$\sum_{i=1}^{n} a_i^* (y_i - f(x_i)) a_i = 0.$$

and hence for each i,

$$a_i^* (y_i - f(x_i)) a_i = 0.$$

The invertibility of a_i for each i, implies that $y_i - f(x_i) = 0$. Therefore, $y_i = f(x_i)$ and the proof is complete. □

Note that with the same assumptions, the above theorem holds for C^*-affine maps immediately.

Proposition 3.5. *F is a C^*-face of a C^*-convex set K in the unital $*$-ring $\mathcal{R}$ if and only if F^* is a C^*-face of C^*-convex set K^**

Proof. First note that if K is a C^*-convex subset of $\mathcal{R}$, then K^* is also a C^*-convex subset of $\mathcal{R}$. Suppose that F is a C^*-face of K. So F^* is a nonempty subset of K^*. Let $y \in F^*$ and $y = \sum_{i=1}^n a_i^* y_i a_i$ be a proper C^*-convex combination of elements $y_i \in K^*$. Then there exist $x(= y^*)$ in F and $x_i \in K$ such that $y = x^*$ and $y_i = x_i^*$. So

$$x^* = \sum_{i=1}^n a_i^* x_i^* a_i = (\sum_{i=1}^n a_i^* x_i a_i)^*,$$

and hence $x = \sum_{i=1}^n a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in K$ $(1 \le i \le n)$. Thus, $x_i \in F$ and hence $y_i = x_i^* \in F^*$ for each i $(1 \le i \le n)$. Therefore, F^* is a C^*-face of K^*. The converse is immediate by replacing F^* and K^* instead of F and K respectively. □

The following theorem deals with the C^*-faces of invertible C^*-affine maps.

Theorem 3.6. *Suppopse that K_1 and K_2 are C^*-convex subsets of the unital $*$-ring $\mathcal{R}$ and $g : K_1 \longrightarrow K_2$ is a bijective C^*-affine map. Then*
1) $g^{-1} : K_2 \longrightarrow K_1$ is a C^*-affine map.
2) F is a C^*-face of K_1 if and only if $g(F)$ is a C^*-face of K_2.
3) F is a C^*-face of K_2 if and only if $g^{-1}(F)$ is a C^*-face of K_1.

Proof. 1) Let $\sum_{i=1}^n a_i^* y_i a_i$ be a C^*-convex combination of elements $y_i \in K_2$ $(1 \le i \le n)$. Then there exist $x_i \in K_1$ such that $g(x_i) = y_i$ for each i $(1 \le i \le n)$. So,

$$g^{-1}(\sum_{i=1}^n a_i^* y_i a_i) = g^{-1}(\sum_{i=1}^n a_i^* g(x_i) a_i) = g^{-1}(g(\sum_{i=1}^n a_i^* x_i a_i))$$

$$= \sum_{i=1}^n a_i^* x_i a_i = \sum_{i=1}^n a_i^* g^{-1}(g(x_i)) a_i = \sum_{i=1}^n a_i^* g^{-1}(y_i) a_i.$$

Therefore, g^{-1} is a C^*-affine map.

2) Let F be a C^*-face of K_1, $y \in g(F)$ and $y = \sum_{i=1}^n a_i^* y_i a_i$ a proper C^*-convex combination of elements $y_i \in K_2$ $(1 \le i \le n)$. So there exist $x \in F$ and x_i in K_1 such that $y = g(x)$ and $y_i = g(x_i)$ for each i $(1 \le i \le n)$. Thus,

$$g(x) = \sum_{i=1}^n a_i^* g(x_i) a_i = g(\sum_{i=1}^n a_i^* x_i a_i).$$

Injectivity of g implies that

$$x = \sum_{i=1}^n a_i^* x_i a_i.$$

Since F is a C^*-face of K_1, $x_i \in F$ $(1 \le i \le n)$. So $y_i = g(x_i) \in g(F)$ $(1 \le i \le n)$ and $g(F)$ is a C^*-face of K_2.

Conversely, let $g(F)$ be a C^*-face of K_2, $x \in F$, and $x = \sum_{i=1}^n a_i^* x_i a_i$ a proper C^*-convex combination of elements $x_i \in K_1$ $(1 \le i \le n)$. Then

$$g(x) = g(\sum_{i=1}^n a_i^* x_i a_i) = \sum_{i=1}^n a_i^* g(x_i) a_i,$$

where $g(x) \in g(F)$ and $g(x_i) \in K_2$ for each i $(1 \le i \le n)$. Thus, $g(x_i) \in g(F)$ and since g is injective, $x_i \in F$ for each i $(1 \le i \le n)$. Therefore, F is a C^*-face of K_1.

3) Since g^{-1} is a bijective C^*-affine map, so part 3) comes from part 2). □

Theorem 3.7. *Suppose that F_1 and F_2 are C^*-faces of C^*-convex subsets K_1 and K_2 of $\mathcal{R}$ respectively. Then,*
1) $K_1 \times K_2$ is a C^*-convex subset of $\mathcal{R} \times \mathcal{R}$ with the pointwise operations. 2) $F_1 \times F_2$ is a C^*-face of $K_1 \times K_2$.

Proof. 1) Suppose that $(x_i, y_i) \in K_1 \times K_2$ and $(a_i, b_i) \in \mathcal{R}$ for each i $(1 \le i \le n)$ such that

$$\sum_{i=1}^n (a_i, b_i)^*(a_i, b_i) = (1_{\mathcal{R}}, 1_{\mathcal{R}}).$$

Since K_1 and K_2 are C^*-convex subsets of $\mathcal{R}$ and

$$\sum_{i=1}^n a_i^* a_i = \sum_{i=1}^n b_i^* b_i = 1_{\mathcal{R}},$$

we have

$$\sum_{i=1}^n (a_i, b_i)^*(x_i, y_i)(a_i, b_i) = \sum_{i=1}^n (a_i^* x_i a_i, b_i^* y_i b_i) = (\sum_{i=1}^n a_i^* x_i a_i, \sum_{i=1}^n b_i^* y_i b_i) \in K_1 \times K_2.$$

2) Let $(x, y) \in F_1 \times F_2$ and

$$(x, y) = \sum_{i=1}^n (a_i, b_i)^*(x_i, y_i)(a_i, b_i) \tag{3.5}$$

be a proper C^*-convex combination of elements $(x_i, y_i) \in K_1 \times K_2$. We show that $(x_i, y_i) \in F_1 \times F_2$ for all i $(1 \le i \le n)$. The relation (3.3) implies that $x = \sum_{i=1}^n a_i^* x_i a_i$ and $y = \sum_{i=1}^n b_i^* y_i b_i$. Also,

$$\sum_{i=1}^n (a_i, b_i)^*(a_i, b_i) = 1_{\mathcal{R}\times\mathcal{R}} = (1_{\mathcal{R}}, 1_{\mathcal{R}})$$

implies that $\sum_{i=1}^n a_i^* a_i = \sum_{i=1}^n b_i^* b_i = 1_{\mathcal{R}}$. Moreover, since (a_i, b_i) is invertible in $\mathcal{R} \times \mathcal{R}$, we have a_i and b_i are invertible in $\mathcal{R}$. Thus, $x = \sum_{i=1}^n a_i^* x_i a_i$ and $y = \sum_{i=1}^n b_i^* y_i b_i$ are proper C^*-convex combinations of elements $x_i \in K_1$ and $y_i \in K_2$ $(1 \le i \le n)$ respectively. Since F_1 and F_2 are C^*-faces in K_1 and K_2 respectively, we conclude that $x_i \in F_1$ and $y_i \in F_2$ for each i $(1 \le i \le n)$. Therefore, $(x_i, y_i) \in F_1 \times F_2$ for each i $(1 \le i \le n)$ and hence $F_1 \times F_2$ is a C^*-face of $K_1 \times K_2$ in $\mathcal{R} \times \mathcal{R}$. □

Acknowledgement. The author would like to thank Dr. I. Nikoufar for careful reading of the manuscript and useful comments.

References

[1] S.K. Berberian, *Baer $*$-Rings*, New York: Springer Verlag, 1972.

[2] A. Ebrahimi, On C^*-extreme points of the epigraph of C^*-affine maps, 9th National conf. on mathematics of payame noor univ., Kerman, Iran, 2017.

[3] A. Ebrahimi, On C^*-extreme points of the graph of C^*-affine maps, 48th Annu. Iranian math. conf., Hamedan, Iran, 2017.

[4] A. Ebrahimi Meymand and G.H. Esslamzadeh, C^*-convexity and C^*-faces in $*$-rings, **Turk. J. Math.**, **36** (2012), 131–145.

[5] D.R. Farenick and P.B. Morenz, C^*-extreme points in the generalised state spaces of a C^*-algebra, *Trans. Am. Math. Soc.*, **349**, (1997), 1725–1748.

[6] D.R. Farenick and P.B. Morenz, C^*-extreme points of some compact C^*-convex sets, *Proc. Am. Math. Soc.*, **118** (1993), 765–775.

[7] A. Hopenwasser, R.L. Moore, and V. I. Paulsen, C^*-extreme points, *Trans. Am. Math. Soc.*, **163** (1981), 291–307.

[8] M. Kian, Epigraph of operator functions, *Quaest. Math.*, **39**(5) (2016) 587–594.

[9] R. Loebl and V.I. Paulsen, Some remarks on C^*-convexity, *Linear Algebra Appl.*, **35** (1981), 63–78.

[10] B. Magajna, C^*-convex sets and completely bounded bimodule homomorphisms, *Proc. R. Soc. Edinb., Sect. A, Math.*, **130**(2) (2000), 375–387.

[11] B. Magajna, C^*-convexity and the numerical range, *Canad. Math. Bull.*, **43**(2), (2000), 193–207.

[12] B. Magajna, On C^*-extreme points, *Proc. Am. Math. Soc.*, **129** (2000), 771–780.

[13] P.B. Morenz, The structure of C^*-convex sets, *Canad. J. Math.*, **46** (1994), 1007–1026.

[14] I. Nikoufar, A note on non-unital homomorphisms on C^*-convex sets in $*$-rings, *Acta Univ. M. Belii Ser. Math.*, **39** (2016), 21–24.