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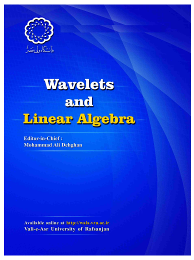

On the Remarkable Formula for Spectral Distance of Block Southeast Submatrix

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ABSTRACT

This paper presents a remarkable formula for spectral distance of a given block normal matrix $G_{D_0} = \begin{pmatrix} A & B \\ C & D_0 \end{pmatrix}$ to set of block normal matrix G_D (as same as G_{D_0} except block D which is replaced by block D_0), in which $A \in \mathbb{C}^{n\times n}$ is invertible, $B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$ with $\mathrm{Rank}\{\mathrm{G_D}\} < \mathrm{n} + \mathrm{m} - 1$ and given eigenvalues of matrix $\mathcal{M} = D - CA^{-1}B$ as $z_1, z_2, \ldots, z_m$ where $|z_1| \geq |z_2| \geq \cdots \geq |z_{m-1}| \geq |z_m|$. Finally, an explicit formula is proven for spectral distance G_D and G_{D_0} which is expressed by the two last eigenvalues of $\mathcal{M}$.

1. Introduction

Garcia and Valesco [1] introduced a block matrix G_{D_0} which is the closest matrix to a given block matrix $G_D = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that matrix $G_{D_0} = \begin{pmatrix} A & B \\ C & D_0 \end{pmatrix}$ has two zero eigenvalues (duplicate

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zero). Recently, Nazari and Nezami in [3] modified Gracia and Valesco's formula, when the matrix G_D is a block normal matrix. Ikramov and Nazari in [1] obtained an explicit formula for the spectral distance $\rho_2(A, L)$ in the case of a normal matrix A, where $\rho_2(A, L)$ is spectral distance from A to the set L of normal matrices with a multiple eigenvalue zero. If G_{D_0} is a given normal matrix, then we can obtain a formula similar to [1] for the distance $\rho_2(G_{D_0}, \mathcal{L})$, where $\mathcal{L}$ is set of block normal matrices G_D which has multiple eigenvalue zero.

2. Main results

Let G_{D_0} be a block matrix as $\begin{pmatrix} A & B \\ C & D_0 \end{pmatrix}$ and let G_D be defined similarly $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and the nearest matrix to G_{D_0} that having two eigenvalue zero. Define

$$\mathscr{M} = D - CA^{-1}B, \qquad \mathscr{N} = I_m + CA^{-2}B,$$

and assume that two matrices $\mathscr{M}$ and $\mathscr{N}$ are normal matrices and the singular values of matrix $\mathscr{M}$ denoted by

$$\sigma_1(\mathscr{M}) \geqslant \cdots \geqslant \sigma_m(\mathscr{M}).$$

If

$$\mathscr{M} = Q\Lambda_{\mathscr{M}}Q^{\star}, \quad \Lambda_{\mathscr{M}} = \operatorname{diag}(z_1, \ldots, z_m), \quad |z_1| \geqslant \cdots \geqslant |z_m|,$$

is spectral decomposition of $\mathscr{M}$, then we have

$$|z_i| = \sigma_i, \qquad i = 1, 2, \ldots, m.$$

Let

$$S(t) = \begin{pmatrix} \mathscr{M} & t\mathscr{N} \\ 0 & \mathscr{M} \end{pmatrix},$$

the matrix $R'(t)$ is defined as following

$$R'(t) := U'^{\star} S(t) U',$$

where $U' := Q \oplus Q$ then

$$R'(t) = \begin{pmatrix} Q^{\star} & 0 \\ 0 & Q^{\star} \end{pmatrix} \begin{pmatrix} \mathscr{M} & t\mathscr{N} \\ 0 & \mathscr{M} \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} \Lambda_{\mathscr{M}} & tQ^{\star}\mathscr{N}Q \\ 0 & \Lambda_{\mathscr{M}} \end{pmatrix}.$$

Whereas the matrix $\mathscr{N}$ is normal, then $Q^{\star}\mathscr{N}Q$ also is normal matrix. If

$$Q^{\star}\mathscr{N}Q = P\Lambda_{Q^{\star}\mathscr{N}Q}P^{\star}, \quad \Lambda_{Q^{\star}\mathscr{N}Q} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m), \quad |\lambda_1| \geqslant \cdots \geqslant |\lambda_m|,$$

is spectral decomposition of matrix $\mathscr{N}$ then

$$R(t) = U^{\star} R'(t) U,$$

where $U := P \oplus P$, also

$$R(t) = \begin{pmatrix} P^\star & 0 \\ 0 & P^\star \end{pmatrix} \begin{pmatrix} \Lambda_{\mathscr{M}} & tQ^\star \mathscr{N} Q \\ 0 & \Lambda_{\mathscr{M}} \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} \Lambda_{\mathscr{M}} & t\Lambda_{Q^\star \mathscr{N} Q} \\ 0 & \Lambda_{\mathscr{M}} \end{pmatrix}.$$

By symmetric permutation of columns and rows the matrix $R(t)$ reduce to the matrix $F(t)$ as following

$$F(t) := \Gamma_1 \oplus \cdots \oplus \Gamma_n,$$

where

$$\Gamma_i := \begin{pmatrix} z_i & t\lambda_i \\ 0 & z_i \end{pmatrix}, \qquad i = 1, 2, \ldots, m. \tag{2.1}$$

Since two matrix $S(t)$ and $R'(t)$ and also two matrices $R'(t)$ and $R(t)$ and $R(t)$ with $F(t)$ are unitarly similar, consequently the matrix $S(t)$ is unitarly similar to matrix $F(t)$. Assume that

$$\Gamma := \begin{pmatrix} a & tb \\ 0 & a \end{pmatrix}, \qquad a, b \in \mathbb{C},$$

is a function of variable $t \in \mathbb{R}$. Then

$$\Gamma^\star \Gamma = \begin{pmatrix} \bar{a}a & t\bar{a}b \\ t\bar{b}a & \bar{a}a + t^2 \bar{b}b \end{pmatrix}.$$

If μ_1 and μ_2 are two singular values of Γ where $\mu_1 \leqslant \mu_2$ and we define $\tau_1 = \mu_1^2$ and $\tau_2 = \mu_2^2$ then τ_1 and τ_2 are the roots of characteristic polynomial of matrix $\Gamma^\star\Gamma$, i.e. τ_1 and τ_2 are the root of following polynomial

$$\tau^2 - (2|a|^2 + |b|^2 t^2)\tau + |a|^4 = 0. \tag{2.2}$$

The roots of (2.2) can be written as

$$\tau = |a|^2 + \frac{t^2|b|^2}{2} \pm \sqrt{|a|^2|b|^2t^2 + \frac{t^4|b|^4}{4}}.$$

The greater root of (2.2) is increasing with increase t and $\mu_1 \to \infty$ as $t \to \infty$, the smaller root $\mu_2 \to 0$ as $t \to \infty$, also we have $\tau_1\tau_2 = |a|^4$ and $\mu_1\mu_2 = |a|^2$, therefore $\mu_1\mu_2$ has fixed value. Consequently we have $\mu_1 \geqslant |a|$ and $\mu_2 \leqslant |a|$. The following Figure shows the behavior $\mu_1(t)$ and $\mu_2(t)$.

Let for the function $F(t)$ we have $|z_{m-1}| = |z_m|$. If ξ_1 and ξ_2 are singular values of Γ_{m-1} and η_1 and η_2 are the singular values of Γ_m. When $t > 0$ we have $\eta_2 < |z_m|$ and $\xi_2 < |z_{m-1}|$, whereas $|z_m| = |z_{m-1}|$ and $\xi_2 < |z_m|$ and we have $\sigma_{2m-1}(F(t)) < |z_m|$, and since $\sigma_{2m-1}(F(0)) = |z_m|$ then the maximum value of $\sigma_{2m-1}(F(t)) = \sigma_{2m-1}(S(t))$ happens in $t = 0$. Now let $|z_{m-1}| > |z_m|$. Similarly to above result we can say that $\eta_2(t)$ is decreasing and $\eta_1(t)$ is increasing as $t \to \infty$ (FIGURE 2). On the other hand

$$\eta_1(0) = |z_m| < \xi_2(0) = |z_{m-1}|.$$

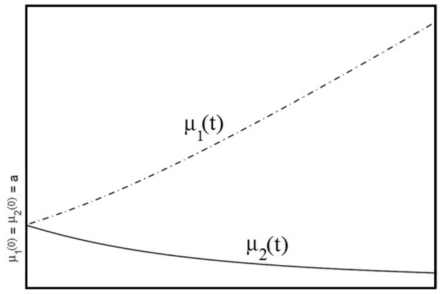

Figure 1: The behavior μ_1 and μ_2 for normal matrices

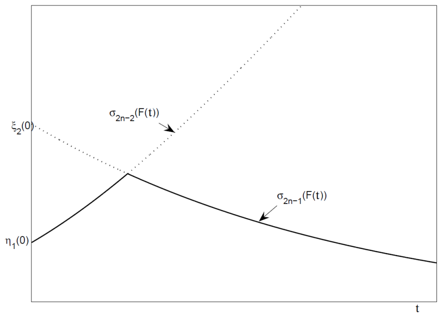

Figure 2: The behavior $\xi_2(t)$ and $\eta_1(t)$ for normal matrices.

Since $\eta_1(t)$ and $\xi_2(t)$ are continuous function, then there exist a unique $t_0 > 0$ such that $\eta_1(t_0) = \xi_2(t_0)$. Thus we have

$$\sigma_{2m-1}(S(t)) = \begin{cases} \eta_1(t), & 0 \leqslant t \leqslant t_0, \\ \xi_2(t), & t \geqslant t_0, \end{cases}$$

where the maximum of $\sigma_{2m-1}(S(t))$ happen in t_0, and

$$\rho_2(G_D, \mathcal{L}) = \sigma_{2(m)-1}(S(t_0)). \tag{2.3}$$

Thus, we have that,

$$|z_m| = \sigma_{2(m)-1}(S(0)) < \rho_2(G_D, \mathcal{L}) < |z_{m-1}| = \rho_2(G_D, \mathcal{K}).$$

Meanwhile, $\mathcal{K}$ is a subset of $\mathcal{L}$. We arrive at a remarkable conclusion:

Let $\mathcal{M}$ be a normal $m \times m$ matrix with the eigenvalues z_i. If

$$|z_{m-1}| > |z_m|,$$

then the matrix with a multiple zero eigenvalue closest (with respect to the 2-norm) to G_D has rank $m + n - 1$ and hence is non diagonalizable.

In fact, we are able to give an explicit formula for the distance $\rho_2(G_D, \mathcal{L})$ in case of a normal matrix G_D.

Theorem 2.1. *Let* $G_D = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a normal* $(n+m)\times(n+m)$ *matrix such that* $\mathrm{Rank} G_D < m+n+1$ *and matrix A invertible. Assume that* $\mathcal{M} = D - CA^{-1}B$ *has the eigenvalues* $z_1, z_2, \cdots, z_m$*, such that* $|z_1| > |z_2| > \cdots > |z_m|$*, then*

$$\rho_2(G_D, \mathcal{L}) = \left(\frac{|z_{m-1}|^2 + |z_m|^2}{2}\right)^{1/2}. \tag{2.4}$$

Proof. If $|z_{m-1}| = |z_m|$, then the equation (2.4) converges to $\max_{\gamma>0} F(\gamma) = |z_m|$, which corresponds to the analysis of mode $|z_{m-1}| = |z_m|$.

Suppose $|z_{m-1}| > |z_m|$. We calculate the value of $\sigma_{2m-1}(F(\gamma_*))$. Since

$$\eta_1(\gamma_*) = \xi_2(\gamma_*),$$

so

$$|z_m|^2 + \frac{\gamma_*^2}{2} + \left(|z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2} = |z_{m-1}|^2 + \frac{\gamma_*^2}{2} - \left(|z_{m-1}|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2}$$

then

$$|z_m|^2 - |z_{m-1}|^2 + \left(|z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2} = -\left(|z_{m-1}|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2}$$

and

$$\begin{aligned} |z_m|^4 + |z_{m-1}|^4 + |z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4} - 2|z_m|^2|z_{m-1}|^2 - 2|z_{m-1}|^2\left(|z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2} \\ + 2|z_m|^2\left(|z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2} = |z_{m-1}|^2\gamma_*^2 + \frac{\gamma_*^4}{4} \end{aligned}$$

so

$$(|z_{m-1}|^2 - |z_{m-1}|^2)^2 + (|z_{m-1}|^2 - |z_m|^2)\gamma_*^2 = 2(|z_{m-1}|^2 - |z_m|^2)\left(|z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2}$$

and

$$\left(|z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2} = \frac{|z_{m-1}|^2 - |z_m|^2}{2} - \frac{\gamma_*^2}{2}.$$

By substituting the above relation in $\xi_2(\gamma_*)$ or $\eta_1(\gamma_*)$, we have

$$\begin{aligned} \eta_1(\gamma_*) &= |z_m|^2 + \frac{\gamma_*^2}{2} + \left(|z_m|^2\gamma_*^2 + \frac{\gamma_*^4}{4}\right)^{1/2} \\ &= |z_m|^2 + \frac{\gamma_*^2}{2} + \frac{|z_{m-1}|^2 - |z_m|^2}{2} - \frac{\gamma_*^2}{2} \\ &= \left(\frac{|z_{m-1}|^2 + |z_m|^2}{2}\right)^{1/2}, \end{aligned}$$

then

$$\max_{\gamma>0} F(\gamma) = \left(\frac{|z_{m-1}|^2 + |z_m|^2}{2}\right)^{1/2}.$$

□

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