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$p-$Adic Shearlets

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Abstract

The field Q_p of p-adic numbers is defined as the completion of the field of the rational numbers Q with respect to the p-adic norm $|.|_p$. In this paper, we study the continuous and discrete $p-$adic shearlet systems on $L^2(Q_p^2)$. We also suggest discrete $p-$adic shearlet frames. Several examples are provided.

1. Introduction

p-adic numbers, Q_p, were introduced in 1897 by the German mathematician K. Hensel [7]. The field Q_p is the example of an ultrametric space. The wavelets was developed in the beginning

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of the 1990s. Y. Meyer and S. Mallat introduced the notion of multiresolution analysis(MRA). Scientists use this theory to cunstruct new wavelet systems. An analog of real Haar base in $p-$adic case is a compactly supported $p-$adic wavelet basis, which was found by S. Kozyrev in 2002, see [12]. These wavelets are eigenfunctions of some p-adic pseudo-differential operators. This property of wavelets may be used to solve p-adic pseudo-differential equations. In last few years, application of $p-$adic wavelets in solving $p-$adic pseudo-differential equations introduced by S. V. Kozyrev, see [11]. The simplest equation of this type is the $p-$adic pseudo-differential heat equation with time parameter $t \in (0, +\infty)$ and $x \in Q_p$,

$$\frac{\partial u(x,t)}{\partial t} + D_x^{\alpha} u(x,t) = 0,$$

which appears in modeling interbasin kinetics of macromolecules. Another example of these kind equations is the following Cauchy problem,

$$\frac{\partial u(x,t)}{\partial t} + D_x^{\alpha}(x,t) = f(x,t) \text{ with } u(x,0) = u^0(x),$$

where D_x^{α} is the fractional operator, introduced and studied by M. H. Taibleson. Again $t \in (0, +\infty)$ is the time parameter, x is in Q_p and $f(.,t)$ belongs to the $p-$adic Lizorkin space for any t and $f(x,.)$ is continuous for any x, see [17]. Recently, M. Skopina and her coauthors introduced $p-$adic refinable functions [16], MRA on $L^2(Q_2^2)$ [10] and $p-$adic MRA and wavelet transform [1, 2, 8].

In real case however, wavelets have been successful in characterizing the singular support of a function and in applications from Harmonic analysis, it has limitations in higher dimensions. So the shearlets system were introduced and developed by D. Labate, G. Kutyniok and many other researchers, see [3, 14, 15], also the references therein. The theory of shearlet systems generated with parabolic scaling, shearing and translation operators.

The main importamce of shearlets is their efficient in representation of multivariable functions with discontinuities in many directions such as edges of a natural image. In such situation one needs many 2-D wavelet coefficients to accurately of representing such images. But, a (discrete) shearlet system works better than two dimensional wavelets. Now we introduce shearlets in $p-$adic case. In this paper we introduce shearlets on $L^2(Q_p^2)$, which we call $p-$adic shearlets. In section 2 we will peresent the preliminaries and notations. In section 3 the $p-$adic shearlets group and the continuous $p-$adic shearlet transform on $L^2(Q_p^2)$ are discussed. In section 4 the discrete $p-$adic shearlet frame is presented. Several examples are provided.

2. Preliminaries and notations

Let p be a prime number, the field Q_p of p-adic numbers is defined as the completion of the field of the rational numbers Q with respect to the non-Archimedean p-adic norm $|.|_p$. This p-adic norm is defined as follows,

$$|x|_p = \begin{cases} 0, & \text{if } x = 0 \\ \\ p^{-\gamma}, & \text{if } x \neq 0 \text{ and } x = p^{\gamma}\frac{m}{n}, \end{cases} \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and the integers m, n are not divisible by p. The norm $|\cdot|_p$ satisfies the strong triangle inequality:

$$|x+y|_p \leq \max\left(|x|_p, |y|_p\right).$$

In the case $|x|_p \neq |y|_p$, we have equality: $|x+y|_p = \max(|x|_p, |y|_p)$, and for $p = 2$ we also have

$$|x+y|_2 \leq \frac{1}{2}|x|_2 \quad \text{if } |x|_2 = |y|_2.$$

The canonical form of any p-adic number $x \neq 0$ is

$$x = p^{\gamma} \sum_{j=0}^{\infty} x_j p^j, \tag{2.2}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, $x_j = 0, 1, ..., p-1, x_0 \neq 0$. If $x_0 \neq 0$, then the above representation is unique. Using the expansion (2.2) one can define a fractional part $\{x\}_p$ of a number $x \in Q_p$,

$$\{x\}_p = \begin{cases} 0, & \text{if } \gamma(x) \geq 0 \ \ or \ x = 0 \\ \\ p^{\gamma}(x_0 + x_1 p + ... + x_{|\gamma|-1} p^{|\gamma|-1}), & \text{if } \gamma(x) < 0, \end{cases} \tag{2.3}$$

and this leads to $p^{\gamma} \leq \{x\}_p \leq 1 - p^{\gamma}$ if $\gamma(x) < 0$.
Ordinary arithmetic operations: sum, subtraction, multiplication and division are defined and Q_p is a field.
Denote by $B_N(a) = \{x \in Q_p : |x-a|_p \leq p^N\}$ the disc of radius p^N with the center at a point $a \in Q_p$, $N \in \mathbb{Z}$. The set of integer $p-$adic numbers is defined by $Z_p = \{x \in Q_p| \ |x|_p \leq 1\}$. The set of natural numbers is dense in Z_p. Set

$$I_p = \left\{x \in Q_p : \{x\}_p = x\right\}.$$

One can show that Q_p has a decomposition to a union of mutually disjoint discs: $Q_p = \bigcup_{x \in I_p} B_0(x)$.

Theorem 2.1. *[18]*
1)The disc $B_N(a)$ is both open and close set in Q_p.

2)Any two discs in Q_p either disjoint or one is contained in another.

3)Disc $B_N(a)$ is compact.

4)The space Q_p is locally-compact under addition.

For more details and proof of theorem see [18].

2.1. Fourier transform

A comlex -valued function f is called locally-constant if for any $x \in Q_p$ there exists an integer $l(x) \in \mathbb{Z}$ such that $f(x+y) = f(x)$, $y \in B_{l(x)}(0)$. We will denote by $D(Q_p)$ (or D), the linear space of locally-constant compactly supported functions, also called test functions.
The field Q_p is a locally-compact commutative group under addition so in Q_p there exists the Haar measure, a positive measure dx which is invariant with respect to shifts, $d(x+a) = dx$. We normalized the measure dx by $\int_{\mathbb{Z}_p} dx = 1$.

For any $d \geq 1$, Q_p^d is a vector space over Q_p ,i.e. Q_p^d consists of the vectors $x = (x_1, ..., x_d)$, where $x_j \in Q_p, j = 1, ..., d$. The p-adic norm on Q_p^d is

$$\left|x\right|_p := \max_{1 \leq j \leq d} \left|x_j\right|_p,$$

where $|\cdot|_p$ is defined as in (2.1). The Fourier transform of $\varphi \in D(Q_p^d)$ is defined as

$$F(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{Q_p^d} \chi_p(\xi.x)\varphi(x)d^d x, \text{ for all } \quad \xi \in Q_p^d,$$

where $\chi_p(\xi.x) = \chi_p(\xi_1 x_1)...\chi_p(\xi_d x_d)$ and $\chi_p(\xi x) = e^{2\pi i\{\xi x\}_p}$ is the additive character for the field Q_p, and dx is the Haar measure in Q_p.

Theorem 2.2. *The Fourier transform maps $L^2(Q_p)$ onto $L^2(Q_p)$ one-to-one and continuous.*

Theorem 2.3. *(a) If $f \in L^2(Q_p)$, $0 \neq a \in Q_p$, $b \in Q_P$, then*

$$F[f(a.+b)](\xi) = |a|_p^{-1}\chi_p(-\frac{b}{a}\xi)F[f](\frac{\xi}{a}).$$

(b) The Plancherel equality holds:

$$\int_{Q_p} f(x)g(x)dx = \int_{Q_p} \widehat{f}(\xi)\widehat{g}(\xi)d\xi,$$

far all $f, g \in L^2(Q_p)$.

Details can be found in [18].

2.2. Square root

A non zero integer $a \in \mathbb{Z}$ is called quadratic residue modulo p if the equation $x^2 \equiv a(\text{mod } p)$ has a solution $x \in \mathbb{Z}$. In the following we need the **Legendre symbol**,

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is quadratic residue modulo } p \\ -1, & \text{if } a \text{ is quadratic non-residue modulo } p. \end{cases} \tag{2.4}$$

Remark 2.4. Let p be an odd prime, and let $a, b \in \mathbb{Z}$ with $(a, p) = 1$ and $(b, p) = 1$, then

$$\Big(\frac{ab}{p}\Big) = \Big(\frac{a}{p}\Big)\Big(\frac{b}{p}\Big). \tag{2.5}$$

The following lemma shows which p-adic numbers have square root [18].

Lemma 2.5. *[18] Equation*

$$x^2 = a, \quad a = p^{\gamma(a)}(a_0 + a_1 p + \cdots), \qquad 0 \le a_i < p, \quad a_0 \neq 0,$$

has a solution, in Q_p if and only if

1)$\gamma(a)$ is even,

2) for $p \neq 2$, $\Big(\frac{a_0}{p}\Big) = 1$, for $p = 2$, $a_1 = a_2 = 0$.

Note: The above lemma only shows that $a^{\frac{1}{2}}$ exists in Q_p or not, but does not give the canonical form of $a^{\frac{1}{2}}$ in Q_p. We define the square root of a in Q_p to be the unique $a^{\frac{1}{2}}$ defined above.

3. Continuous $p-$adic Shearlets on $L^2(Q_p^2)$

In this section, we introduce the continuous $p-$adic shearlet system and $p-$adic shearlet transform on $L^2(Q_p^2)$. We would like to obtain an inversion formula for the $p-$adic shearlet transform. For $f : Q_p^d \to \mathbb{C}$, $y \in Q_p^d$ and $A \in GL(Q_p, d)$, the set of $d \times d$ matrices with entries in Q_p, define the following operators,

$$T_y(f)(.) = f(. - y), \qquad D_A f(.) = |detA|_p^{\frac{1}{2}} f(A.).$$

We denote by G_p the set of all non zero elements in Q_p that have square root. Using (2.5) one can show easily that G_p is a group with respect to multiplication of $p-$adic numbers. For example $G_2 = \{a = 2^{\gamma(a)} \sum_{j\in\mathbb{Z}} a_j 2^j \in Q_2 : \gamma(a) \text{ is even}, a_0 = 1, a_1 = a_2 = 0\}$.

Set

$$A_a = \begin{pmatrix} a^{-1} & 0 \\ 0 & a^{-\frac{1}{2}} \end{pmatrix} \text{ and } S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

for $a \in G_p$ and $s \in Q_p$. Let $\psi \in L^2(Q_p^2)$, and for $a \in G_p$, $s \in Q_p$ and $t \in Q_p^2$ define $\psi_{a,s,t} \in L^2(Q_p^2)$ by

$$\psi_{a,s,t}(x) = T_t D_{A_a} D_{S_s} \psi(x) = |a|_p^{-\frac{3}{4}} \psi\Big(A_a S_s (x - t)\Big).$$

Then the $p-$**adic shearlet system** is defined by

$$SH(\psi) = \Big\{\psi_{a,s,t} : a \in G_p, s \in Q_p, t \in Q_p^2\Big\}.$$

3.1. p−adic Shearlet group

The theory of shearlets in R^2 is connected to the group theory. Based on this property the shearlet coorbit spaces are defined([4]). In this section we introduce the shearlet group in p-adic case and its Haar measure.

Theorem 3.1. *The set $\mathbb{S} = G_p \times Q_p \times Q_p^2$ equipped with multiplication given by*

$$(a, s, t)(a', s', t') = (aa', s + \sqrt{a}s', t + S_s^{-1}A_a^{-1}t'),$$

forms a group.

Proof. $1 \in G_p$ is the neutral element, so $(1, 0, 0)$ is the neutral element in $\mathbb{S}$. A simple calculation using multiplication formula implies,

$$(a, s, t) \cdot (\frac{1}{a}, -\frac{s}{\sqrt{a}}, -A_aS_st) = (1, 0, 0).$$

Also the fact, $S^{-1}_{-\frac{s}{\sqrt{a}}}A^{-1}_{\frac{1}{a}} = A_aS_s$ implies that:

$$(\frac{1}{a}, -\frac{s}{\sqrt{a}}, -A_aS_st) \cdot (a, s, t) = (1, 0, 0),$$

so $(a, s, t)^{-1} = (\frac{1}{a}, -\frac{s}{\sqrt{a}}, -A_aS_st)$. One can show easily that this multiplication is associative by using the fact that $S_s^{-1}A_a^{-1}S_{s'}^{-1}A_{a'}^{-1} = S^{-1}_{s+\sqrt{a}s'}A^{-1}_{aa'}$. □

The Group $\mathbb{S} = G_p \times Q_p \times Q_p^2$ equipped with multiplication given by

$$(a, s, t)(a', s', t') = (aa', s + a^{\frac{1}{2}}s', t + S_s^{-1}A_a^{-1}t'),$$

is called the $p-$**adic shearlet group**. The left Haar measure of this group is $\frac{da\,ds\,dt}{|a|_p^3}$, since

$$\frac{d(aa')d(s + a^{\frac{1}{2}}s')d(t + S_s^{-1}A_a^{-1}t')}{|aa'|_p^3} = \frac{da\,ds\,dt}{|a|_p^3}.$$

Lemma 3.2. *Define $\sigma : \mathbb{S} \to U(L^2(Q_p^2))$ by*

$$\sigma(a, s, t)\psi(x) = \psi_{a,s,t}(x) = |a|_p^{-\frac{3}{4}}\psi\Big(A_aS_s(x - t)\Big).$$

Then σ is a unitary representation of $\mathbb{S}$ on $L^2(Q_p^2)$.

Proof. Let $\psi \in L^2(Q_p^2)$ and $x \in Q_p^2$.

$$\begin{aligned}
\sigma(a,s,t)\Big(\sigma(a',s',t')\psi\Big)x &= \Big|a\Big|_p^{\frac{3}{4}}\sigma(a',s',t')\psi\Big(A_aS_s(x-t)\Big) \\
&= \Big|aa'\Big|_p^{\frac{3}{4}}\psi\Big(A_{a'}S_{s'}\Big(A_aS_s(x-t)-t'\Big)\Big) \\
&= \Big|aa'\Big|_p^{\frac{3}{4}}\psi\Big(A_{a'}S_{s'}A_aS_s\Big(x-(t+S_s^{-1}A_a^{-1}t')\Big)\Big) \\
&= \Big|aa'\Big|_p^{\frac{3}{4}}\psi\Big(A_{aa'}S_{s+s'\sqrt{a}}\Big(x-(t+S_s^{-1}A_a^{-1}t')\Big)\Big) \\
&= \Big(\sigma\Big((a,s,t).(a',s',t')\Big)\psi\Big)x.
\end{aligned}$$

□

The continuous $p-$adic shearlet transform of $f \in L^2(Q_p^2)$ is defined as follows,

$$f \mapsto SH_\psi f(a,s,t) =< f, \psi_{a,s,t} > \qquad (a,s,t) \in \mathbb{S}.$$

Definition 3.3. Let $\psi \in D(Q_p^2)$. ψ is called an admissible p-adic shearlet if

$$C_\psi = \int_{Q_p}\int_{Q_p} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{|\xi_1|_p^2} d\xi_2 d\xi_1 < \infty.$$

Theorem 3.4. *Let $\psi \in D(Q_p^2)$ be an admissible p-adic shearlet. Define*

$$C_\psi^1 = \int_{G_p}\int_{Q_p} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{|\xi_1|_p^2} d\xi_2 d\xi_1,$$

and

$$C_\psi^2 = \int_{Q_p\setminus G_p}\int_{Q_p} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{|\xi_1|_p^2} d\xi_2 d\xi_1.$$

If $C_\psi^1 = C_\psi^2 = C_\psi$, then

$$\int_{\mathbb{S}} \Big|SH_\psi f(a,s,t)\Big|^2 da\, ds\, dt \leq C_\psi \Big\|f\Big\|_2^2 \qquad \textit{for all} \quad f \in D(Q_p^2).$$

Proof. Note that the shearlet transform of a function $f \in L^2(Q_p^2)$ can be regarded as a convolution. In fact we have

$$SH_\psi f(a,s,t) = \Big\langle f, |a|_p^{-\frac{3}{4}}\ \psi\Big(A_aS_s(x-t)\Big)\Big\rangle = f * \psi_{a,s,0}^*(t), \tag{3.1}$$

where $\psi^*_{a,s,0}(x) = \overline{\psi_{a,s,0}(-x)}$ for all $x \in Q_p^2$. Furthermore we have

$$\widehat{\psi_{a,s,t}}(\omega) = \chi_p(t\omega)\, |a|_p^{\frac{3}{4}}\, \widehat{\psi}\Big((A_a^{-1})^T (S_s^{-1})^T \omega\Big) = \chi_p(t\omega)\, |a|_p^{\frac{3}{4}}\, \widehat{\psi}\Big(a\omega_1, a^{\frac{1}{2}}\omega_2 - a^{\frac{1}{2}} s\omega_1\Big). \tag{3.2}$$

Employing (3.1), (3.2) and the Plancherel theorem in $L^2(Q_p^2)$, for $f \in D(Q_p^2)$ we have

$$\begin{aligned}
\int_{\mathbb{S}} \Big|SH_\psi f(a,s,t)\Big|^2 \frac{da\,ds\,dt}{|a|_p^3} &= \int_{\mathbb{S}} \Big|f * \psi^*_{a,s,0}(t)\Big|^2 \frac{dt\,ds\,da}{|a|_p^3} \\
&= \int_{\mathbb{S}} \Big|\widehat{f}(\omega)\Big|^2 \Big|\widehat{\psi^*_{a,s,0}}(\omega)\Big|^2 \frac{d\omega\,ds\,da}{|a|_p^3} \\
&= \int_{G_p}\int_{Q_p}\int_{Q_p^2} \Big|\widehat{f}(\omega)\Big|^2 \Big|a\Big|_p^{\frac{3}{2}} \Big|\widehat{\psi}\Big((A_a^{-1})^T (S_s^{-1})^T \omega\Big)\Big|^2 \frac{d\omega\,ds\,da}{|a|_p^3} \\
&= \int_{G_p}\int_{Q_p^2}\int_{Q_p} \Big|\widehat{f}(\omega)\Big|^2 \Big|a\Big|_p^{\frac{3}{2}} \Big|\widehat{\psi}\Big(a\omega_1, a^{\frac{1}{2}}\omega_2 - a^{\frac{1}{2}} s\omega_1\Big)\Big|^2 \frac{ds\,d\omega\,da}{|a|_p^3} \\
&= \int_{Q_p}\int_{G_p}\int_{G_p}\int_{Q_p} \Big|\widehat{f}(\omega)\Big|^2 \Big|\omega_1\Big|_p^{-1} \Big|\widehat{\psi}\Big(a\omega_1, \xi_2\Big)\Big|^2 d\xi_2 \frac{da\,d\omega_1\,d\omega_2}{|a|_p^2} \\
&+ \int_{Q_p}\int_{Q_p\setminus G_p}\int_{G_p}\int_{Q_p} \Big|\widehat{f}(\omega)\Big|^2 \Big|\omega_1\Big|_p^{-1} \Big|\widehat{\psi}\Big(a\omega_1, \xi_2\Big)\Big|^2 d\xi_2 \frac{da\,d\omega_1\,d\omega_2}{|a|_p^2} \\
&= \int_{Q_p}\int_{G_p} \Big|\widehat{f}(\omega)\Big|^2 d\omega_1 d\omega_2 \int_{G_p}\int_{Q_p} \Big|\xi_1\Big|_p^{-2} \Big|\widehat{\psi}(\xi_1,\xi_2)\Big|^2 d\xi_2 d\xi_1 \\
&+ \int_{Q_p}\int_{Q_p\setminus G_p} \Big|\widehat{f}(\omega)\Big|^2 \Big(\int_A\int_{Q_p} \Big|\xi_1\Big|_p^{-2} \Big|\widehat{\psi}(\xi_1,\xi_2)\Big|^2 d\xi_2 d\xi_1\Big) d\omega_1 d\omega_2 \\
&= I,
\end{aligned}$$

where A is a subset of $(Q_p \setminus G_p)$ depending on ω_1, so we have

$$\int_A\int_{Q_p} \Big|\xi_1\Big|_p^{-2} \Big|\widehat{\psi}(\xi_1,\xi_2)\Big|^2 d\xi_2 d\xi_1 \le C_\psi^2.$$

Since $C_\psi^1 = C_\psi^2 = C_\psi$, hence

$$I \le C_\psi \int_{Q_p}\int_{G_p} \Big|\widehat{f}(\omega)\Big|^2 d\omega_1 d\omega_2 + C_\psi \int_{Q_p}\int_{Q_p\setminus G_p} \Big|\widehat{f}(\omega)\Big|^2 d\omega_1 d\omega_2 = C_\psi \Big\|f\Big\|^2.$$

Since $f, \psi \in D(Q_p^2)$ we can use the Fubini Theorem. □

Note that in the above theorem, we prove that

$$\int_{\mathbb{S}} \left|SH_\psi f(a,s,t)\right|^2 \frac{da\, ds\, dt}{|a|_p^3} \leq C_\psi \left\|f\right\|^2.$$

But in the real case, the equality is proved i.e

$$\int_{\mathbb{S}} \left|SH_\psi f(a,s,t)\right|^2 \frac{da\, ds\, dt}{a^3} = C_\psi \left\|f\right\|^2.$$

because the set of positive real numbers has same measure to the negative real numbers, But in the p-adic case the set of G_p is much smaller than the set of $(Q_p \setminus G_p)$.

Example 3.5. Consider the space $L^2(Q_2^2)$. Set $\psi_1(x) = \chi_2(2^{-1}x)\Omega(|x|_2)$ and $\psi_2(x) = 2\Omega(2|x|_2)$, where $\Omega(|x|_2)$ is a characteristic function on $B_0(0)$ and $\chi_2(\xi x) = e^{2\pi i\{\xi x\}_2}$ is the additive character for the field Q_2. By definition of the Fourier transform on Q_p we have $\widehat{\psi_1}(\xi) = \Omega(|2^{-1}+\xi|_2)$, and by (2.3) we have

$$F(\psi_2)(\xi) = 2F\Big(\Omega(2|.|_2)\Big)(\xi) = 2F\Big(\Omega(|\frac{1}{2}.|_2)\Big)(\xi) = 2\Big(|\frac{1}{2}|_2^{-1}\Big)\Omega\Big(|2\xi|_2\Big) = \Omega\Big(2^{-1}|\xi|_2\Big),$$

this means $\widehat{\psi_2}(\xi) = \Omega(2^{-1}|\xi|_2)$.

Let

$$\widehat{\psi}(\xi_1,\xi_2) = \widehat{\psi_1}(\xi_1)\widehat{\psi_2}(\frac{\xi_2}{\xi_1}) = \Omega\Big(|2^{-1}+\xi_1|_2\Big)\Omega\Big(2^{-1}|\frac{\xi_2}{\xi_1}|_2\Big).$$

For ψ defined as above we have:

$$\begin{aligned} C_\psi &= \int_{Q_2^2} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{|\xi_1|_2^2} d\xi_2 d\xi_1 = \int_{Q_2}\int_{Q_2} \frac{|\widehat{\psi_1}(\xi_1)|^2|\widehat{\psi_2}(\frac{\xi_2}{\xi_1})|^2}{|\xi_1|_2^2} d\xi_2 d\xi_1 \\ &= \int_{Q_2} \frac{|\widehat{\psi_1}(\xi_1)|^2}{|\xi_1|_2} d\xi_1 \int_{Q_2} \left|\widehat{\psi_2}(u)\right|^2 du = \frac{1}{2}\times 2 = 1. \end{aligned}$$

Example 3.6. Define $\theta(x) = \chi_2(2^{-2}x)\Omega(|x|_2)$ in $L^2(Q_2)$. Then the set

$$\Big\{\theta_{k,j,a} = 2^{-\frac{j}{2}}\theta\Big(k(2^jx-a)\Big) : j\in\mathbb{Z},\ k=1,3,\ a\in I_2^2\Big\},$$

is an orthonormal wavelet basis for $L^2(Q_2)$. For details see [8]. We have $\widehat{\theta}(\xi) = \Omega(|\xi+2^{-2}|_2)$ and $\mathrm{supp}\widehat{\theta} = \{\xi : |\xi|_2 = 4, \xi_1 \neq 0\}$, so

$$\int_{Q_2} \frac{|\widehat{\theta}(\xi)|^2}{|\xi|_2} d\xi = \int_{|\xi|_2=4,\xi_1\neq 0} \frac{1}{4} d\xi = \frac{1}{4}\Big(2^2(1-\frac{1}{2}) - 2(1-\frac{1}{2})\Big) = \frac{1}{4}.$$

Let $\widehat{\psi}(\xi) = \widehat{\psi}(\xi_1,\xi_2) = \widehat{\theta}(\xi_1)\widehat{\phi}(\frac{\xi_2}{\xi_1})$, where θ is defined as above and $\phi(x) = 4\Omega(4|x|_2)$. Now we compute C_ψ:

$$C_\psi = \int_{Q_2^2} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{|\xi_1|_2^2} d\xi_2 d\xi_1 = \int_{Q_2} \frac{|\widehat{\theta}(\xi_1)|^2}{|\xi_1|_2} d\xi_1 \int_{Q_2} \left|\widehat{\phi}(u)\right|^2 du = 1.$$

4. Discrete $p-$adic shearlet frame

Set $J \subseteq \mathbb{Z}$ and $c \in Q_p$ such that $c \neq 0$. Let Λ be a discrete subset of $\mathbb{S}$ of the form

$$\Lambda = \Big\{(a_j, s_{jd}, S_{s_{jd}}^{-1} A_{a_j}^{-1} cb) : j \in J,\ d \in I_p,\ b \in I_p^2\Big\}, \tag{4.1}$$

where $a_j \in G_p$ and $s_{jd} \in Q_p$.
Then the discrete $p-$adic shearlet system is defined as follows,

$$SH(\psi, \Lambda) = \Big\{T_{S_{s_{jd}}^{-1} A_{a_j}^{-1} cb} D_{A_{a_j}} D_{S_{s_{jd}}} \psi : (a_j, s_{jd}, S_{s_{jd}}^{-1} A_{a_j}^{-1} cb) \in \Lambda\Big\},$$

where $\psi \in L^2(Q_p^2)$.

Theorem 4.1. *Let $c \neq 0$ be fixed and Λ defined as in (4.1). Let $\psi \in D(Q_p^2)$ and set*

$$\phi(\omega) = ess\ sup_{\xi \in Q_p^2} \sum_{j,d} \Big|\widehat{\psi}\Big(A_{a_j}^{-1} S_{s_{jd}}^{-T} \xi\Big)\Big| \Big|\widehat{\psi}\Big(A_{a_j}^{-1} S_{s_{jd}}^{-T} \xi + \omega\Big)\Big| \quad a.e\ \omega \in Q_p^2. \tag{4.2}$$

If there exist $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq \sum_{j,d} \Big|\widehat{\psi}(A_{a_j}^{-1} S_{s_{jd}}^{-T} \xi)\Big|^2 \leq \beta \quad a.e \quad \xi \in Q_p^2,$$

and

$$\sum_{b \in I_p^2, b \neq 0} \Big(\phi(\frac{1}{c} b) \phi(-\frac{1}{c} b)\Big)^{\frac{1}{2}} =: \gamma < \alpha,$$

then $SH(\psi, \Lambda)$ is a frame for $L^2(Q_p^2)$ with frame bounds C, D satisfying

$$\frac{1}{|c|_p^2}(\alpha - \gamma) \leq C \leq D \leq \frac{1}{|c|_p^2}(\beta + \gamma).$$

Proof. Employing the Plancheral theorem, for $f \in D(Q_p^2)$ we have

$$\sum_{j,d,b} \Big|\Big\langle f, D_{A_{a_j}} D_{S_{s_{jd}}} T_{cb} \psi\Big\rangle\Big|^2 = \sum_{j,d,b} \Big| \int_{Q_p^2} \widehat{f}(\xi) \chi_p\Big(- S_{s_{jd}}^{-1} A_{a_j}^{-1} cb.\xi\Big) \widehat{\psi}\Big(A_{a_j}^{-1} S_{s_{jd}}^{-T} \xi\Big) d\xi\Big|^2,$$

by (4.2), We have for fixed j, d, $Q_p^2 = \bigcup_{i\in I_p^2} B_\gamma(\frac{1}{c}S^T_{s_{jd}}A_{a_j}i)$ where $p^\gamma = \frac{1}{|c|_p^2}|a_j|_p^{-\frac{3}{2}}$. So we obtain

$$\begin{aligned}
&\sum_{j,d,b} \left|\left\langle f, D_{A_{a_j}} D_{S_{s_{jd}}} T_{cb}\psi\right\rangle\right|^2 \\
&= \sum_{j,d,b}\Big| |a_j|_p^{\frac{3}{4}} \sum_{i\in I_p^2} \int_{B_\gamma(\frac{1}{c}S^T_{s_{jd}}A_{a_j}i)} \widehat{f}(\xi)\,\chi_p\Big(-S^{-1}_{s_{jd}}A^{-1}_{a_j}cb.\xi\Big)\,\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi\Big)d\xi\Big|^2 \\
&= \sum_{j,d,b}\Big| |a_j|_p^{\frac{3}{4}} \sum_{i\in I_p^2} \int_{B_\gamma(0)} \widehat{f}\Big(\xi+\frac{1}{c}S^T_{s_{jd}}A_{a_j}i\Big)\,\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi+\frac{1}{c}i\Big)\,\chi_p\Big(-b.cA^{-1}_{a_j}S^{-T}_{s_{jd}}\xi\Big)d\xi\Big|^2 \\
&= \frac{1}{|c|_p^2}\sum_{j,d}\int_{B_\gamma(0)}\Big|\sum_{i\in I_p^2}\widehat{f}\Big(\xi+\frac{1}{c}S^T_{s_{jd}}A_{a_j}i\Big)\,\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi+\frac{1}{c}i\Big)\Big|^2 d\xi \\
&= \frac{1}{|c|_p^2}\sum_{j,d,i}\int_{Q_p^2}\widehat{f}(\xi)\,\overline{\widehat{f}\Big(\xi+\frac{1}{c}S^T_{s_{jd}}A_{a_j}i\Big)}\,\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi\Big)\,\overline{\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi+\frac{1}{c}i\Big)}d\xi \\
&= \frac{1}{|c|_p^2}\int_{Q_p^2}\Big|\widehat{f}(\xi)\Big|^2\sum_{j,d}\Big|\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi\Big)\Big|^2 d\xi \\
&+ \frac{1}{|c|_p^2}\sum_{i\neq 0,i,j,d}\int_{Q_p^2}\widehat{f}(\xi)\,\overline{\widehat{f}\Big(\xi+\frac{1}{c}S^T_{s_{jd}}A_{a_j}i\Big)}\,\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi\Big)\,\overline{\widehat{\psi}\Big(A^{-1}_{a_j}S^{-T}_{s_{jd}}\xi+\frac{1}{c}i\Big)}d\xi.
\end{aligned}$$

We denote by $R(f)$ the second term in the last equality, and we use the cauchy-Schwarz inequality twice:

$$\begin{aligned}\left|R(f)\right| &\leq \frac{1}{|c|_p^2}\sum_{i\neq 0,i,j,d}\Big[\int_{Q_p^2}\left|\widehat{f}(\xi)\right|^2\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)\right|\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi+\frac{1}{c}i)\right|d\xi\Big]^{\frac{1}{2}}\\
&\cdot\Big[\int_{Q_p^2}\left|\widehat{f}(\xi+\frac{1}{c}S_{s_{jd}}^{T}A_{a_j}i)\right|^2\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)\right|\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi+\frac{1}{c}i)\right|d\xi\Big]^{\frac{1}{2}}\\
&= \frac{1}{|c|_p^2}\sum_{i\neq 0,i,j,d}\Big[\int_{Q_p^2}\left|\widehat{f}(\xi)\right|^2\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)\right|\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi+\frac{1}{c}i)\right|d\xi\Big]^{\frac{1}{2}}\\
&\cdot\Big[\int_{Q_p^2}\left|\widehat{f}(\xi)\right|^2\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi-\frac{1}{c}i)\right|\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)\right|d\xi\Big]^{\frac{1}{2}}\\
&\leq \frac{1}{|c|_p^2}\sum_{i\neq 0}\Big[\int_{Q_p^2}\left|\widehat{f}(\xi)\right|^2\sum_{j,d}\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)\right|\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi+\frac{1}{c}i)\right|d\xi\Big]^{\frac{1}{2}}\\
&\cdot\Big[\int_{Q_p^2}\left|\widehat{f}(\xi)\right|^2\sum_{j,d}\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi-\frac{1}{c}i)\right|\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)\right|d\xi\Big]^{\frac{1}{2}}.\end{aligned}$$

Similar to the Theorem 3.1 in [13] and using this estimate, we can complete the proof. □

Remark 4.2. We know that $Q_p^2=\bigcup_{i\in I_p^2}B_0(i)$. If $x\in Q_p^2$ then $c(S_{s_{jd}}^{T}A_{a_j})^{-1}x\in Q_p^2$, hence there exists $i\in I_p^2$ such that $|c(S_{s_{jd}}^{T}A_{a_j})^{-1}x-i|_p\leq 1$, this means $|x-\frac{1}{c}S_{s_{jd}}^{T}A_{a_j}i|_p\leq |det(\frac{1}{c}S_{s_{jd}}^{T}A_{a_j})|_p=\frac{1}{|c|_p^2}|a_j|_p^{-\frac{3}{2}}$.

Corollary 4.3. *Define $\Lambda\subseteq\mathbb{S}$ as in (4.1). Suppose $\psi\in D(Q_p^2)$ such that*

$$supp\widehat{\psi}\subseteq B_N((0,0))=B_N(0)\times B_N(0),$$

$|c|_p<p^{-N}$ and there exist $0<\alpha\leq\beta<\infty$ such that

$$\alpha\leq\sum_{j,d}\left|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)\right|^2\leq\beta \qquad a.e\ \xi\in Q_p^2.$$

Then the shearlet system $SH(\psi,\Lambda)$ is a frame for $L^2(Q_p^2)$ with frame bounds A, B satisfying

$$\frac{1}{|c|_p^2}(\alpha)\leq A\leq B\leq\frac{1}{|c|_p^2}(\beta).$$

Proof. Note that $|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi)|\neq 0$ if and only if

$$A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi\in B_N(0)\times B_N(0),$$

and $|\widehat{\psi}(A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi + \omega)| \neq 0$ if and only if

$$A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi \in B_N(-\omega).$$

So if $|\omega|_p > p^N$, then $|A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi|_p \leq p^N$ and $|A_{a_j}^{-1}S_{s_{jd}}^{-T}\xi + \omega|_p = |\omega|_p > p^N$. This means $\phi(\omega) = 0$, as defined in (4.2), when $|\omega|_p > p^N$ .Hence

$$\sum_{b\in I_p^2, b\neq 0} \left(\phi(\frac{1}{c}b)\phi(-\frac{1}{c}b)\right)^{\frac{1}{2}} = 0,$$

for all $|c|_p < p^{-N}$. Note that since $b \in I_p^2$, $|b|_p \geq 1$. The proof now follows from Theorem 4.1. □

Example 4.4. Let $p = 2$ and $c \neq 0$ such that $|c|_2 < 2^{-2}$. Since $Q_p = \bigcup_{d\in I_p} B_0(d)$, then

$$\sum_{d\in I_p} \left|\varphi(x-d)\right|^2 = 1 \quad a.e,$$

where $\varphi(x) = \Omega(|x|_p)$.
The discrete Λ is defined by

$$\Lambda = \left\{\left(2^{2j}(1+2^{3+j}), d2^j(1+2^{3+j})^{\frac{1}{2}}, S^{-1}_{d2^j(1+2^{3+j})^{\frac{1}{2}}} A^{-1}_{2^{2j}(1+2^{3+j})} cb\right) : j \in \mathbb{Z}^+,\ \ d \in I_p,\ b \in I_p^2\right\}.$$

Let $\psi \in D(Q_p^2)$ be defined by

$$\widehat{\psi}(\xi_1,\xi_2) = \widehat{\psi_1}(\xi_1 + \frac{\xi_2}{\xi_1})\left(\widehat{\psi_2}(\xi_1) + \widehat{\psi_2}(2\xi_1)\right),$$

where $\widehat{\psi_1}(\omega) = \Omega(|\omega|_2)$ and $\widehat{\psi_2}(\omega) = \delta(|\omega|_2 - 1)$. The function δ is defined as follows,

$$\delta\left(|\omega|_p - p^\gamma\right) = \begin{cases} 1, & \text{if } \ \omega \in S_\gamma(0), \\ 0, & o.w, \end{cases}$$

Where

$$S_\gamma(a) = \left\{x \in Q_p : |x-a|_p = p^\gamma\right\},$$

for $a \in Q_p$ and $\gamma \in \mathbb{Z}$.
Now we can obtain the supp$\widehat{\psi}$, $\widehat{\psi}(\xi_1,\xi_2) \neq 0$ if and if $|\xi_1 + \frac{\xi_2}{\xi_1}|_2 \leq 1$ and ($|\xi_1|_2 = 1$ or $|\xi_1|_2 = 2$). We have

$$\left|\xi_1 + \frac{\xi_2}{\xi_1}\right|_2 = \begin{cases} \max\left\{|\xi_1|_2, |\frac{\xi_2}{\xi_1}|_2\right\}, & \text{if } \ |\xi_1|_2 \neq |\frac{\xi_2}{\xi_1}|_2 \\ \leq \frac{1}{2}|\xi_1|_2, & \text{if } \ |\xi_1|_2 = |\frac{\xi_2}{\xi_1}|_2. \end{cases}$$

So $\widehat{\psi}(\xi_1,\xi_2) \neq 0$ if and only if

$$\Big(|\xi_1|_2 = 1 \text{ and } |\xi_2|_2 \leq 1\Big) \quad \text{or} \quad \Big(|\xi_1|_2 = 2 \quad \text{and} \quad |\xi_2|_2 = 2^2\Big)$$

therefore we have supp$\widehat{\psi} \subseteq B_2((0,0))$.

$$\sum_{j,d} \left|\widehat{\psi}\Big(A^{-1}_{2^{2j}(1+2^{3+j})} S^{-T}_{d2^j(1+2^{3+j})^{\frac{1}{2}}} \xi\Big)\right|^2 =$$

$$\sum_{j,d} \left|\widehat{\psi_1}\Big(2^{2j}(1+2^{3+j})\xi_1 + 2^{-j}(1+2^{3+j})^{-\frac{1}{2}}\frac{\xi_2}{\xi_1} - d\Big)\right|^2 \left|\Big(\widehat{\psi_2}\Big(2^{2j}(1+2^{3+j})\xi_1\Big) + \widehat{\psi_2}\Big(2^{2j+1}(1+2^{3+j})\xi_1\Big)\Big)\right|^2 =$$

$$\sum_{j} \left|\Big(\widehat{\psi_2}\Big(2^{2j}(1+2^{3+j})\xi_1\Big) + \widehat{\psi_2}\Big(2^{2j+1}(1+2^{3+j})\xi_1\Big)\Big)\right|^2 \sum_{d} \left|\widehat{\psi_1}\Big(2^{2j}(1+2^{3+j})\xi_1 + 2^{-j}(1+2^{3+j})^{-\frac{1}{2}}\frac{\xi_2}{\xi_1} - d\Big)\right|^2$$

$$= 1 \qquad a.e.$$

So Corollary 4.3 implies that for all $c \neq 0$ such that $|c|_2 < 2^{-2}$, the shearlet system $SH(\psi, \Lambda)$ forms a tight frame for $L^2(Q_2^2)$.

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