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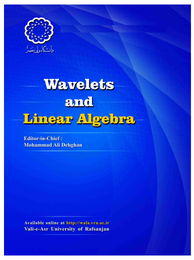

Characterizing sub-topical functions

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Abstract

In this paper, we first give a characterization of sub-topical functions with respect to their lower level sets and epigraph. Next, by using two different classes of elementary functions, we present a characterization of sub-topical functions with respect to their polar functions, and investigate the relation between polar functions and support sets of this class of functions. Finally, we obtain more results on the polar of sub-topical functions.

1. Introduction

The set of linear functions leads to the theory of usual convex sets and convex functions [12], and thus each class of elementary functions leads to a theory of abstract convex sets and abstract

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convex functions with respect to this class [9]. Abstract convexity and many special concepts have been investigated, such as support sets, subdifferentials, conjugacy, and the polarity of topical functions with respect to a certain class of elementary functions [7, 8, 10, 11]. In fact, functions of this type have emerged recently in various contexts [5] and the term "topical function" was used by Gunawardena and Keane [6]. Topical functions have been studied intensively (see [3, 4, 5] and the references therein) and they have many applications in various areas of applied mathematics, particularly in the modeling of discrete event systems [3, 4, 5, 6]. Topical functions are also interesting as a tool for studying the so-called downward sets. Downward sets arise in the study of some problems in mathematical economics and game theory, as well as in the study of inequality systems involving increasing functions [10].

In addition to topical functions, the much more general class of increasing and plus-sub-homogeneous functions, called sub-topical functions, has been studied [6, 10, 11]. In [2, 10, 11], it was shown that each sub-topical function is abstract convex with respect to a certain class of elementary functions. Recently, characterizations of the maximal elements of the support sets of sub-topical functions and also characterizations of global maximizers of the difference of two strictly sub-topical functions have been investigated in [1]. In this paper, we give a characterization of sub-topical functions with respect to their lower level sets and epigraph. Moreover, by using two different classes of elementary functions, we present a characterization of sub-topical functions with respect to their polar functions, and investigate the relation between polar functions and support sets of this class of functions.

The rest of this paper is organized as follows. In Section 2, we provide definitions, notations, and preliminary results related to plus-co-radiant sets and sub-topical functions. A characterization of sub-topical functions with respect to their lower level sets and epigraph is given in Section 3. In Section 4, we present a characterization of sub-topical functions with respect to their polar functions, and examine the relation between polar functions and support sets of this class of functions. Also, we obtain more related results.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real normed linear space. Assume that X is equipped with a closed convex cone $S \subset X$ such that $S \cap (-S) = \{0\}$. Moreover, we suppose that $S \neq \{0\}$ and $intS \neq \emptyset$. The increasing property of our functions will be understood to be with respect to the ordering $\leq$ induced on X by S :

$$x \leq y \iff y - x \in S, \ \forall\, x, y \in X.$$

For each $x, y \in X$, we say that $x < y$ if $y - x \in S \setminus \{0\}$. Let $\mathbf{1} \in intS$. Recall [2, 9] that an extended real valued function $f : X \longrightarrow [-\infty, +\infty]$ is called plus-sub-homogeneous if

$$f(x + \lambda\mathbf{1}) \leq f(x) + \lambda, \ \forall\, x \in X, \ \forall\, \lambda \geq 0.$$

The function f is called increasing if $(x \leq y \Longrightarrow f(x) \leq f(y))$.

The following definitions can be found in [2, 9].

(i) A function $f : X \longrightarrow \bar{\mathbb{R}} = [-\infty, +\infty]$ is called sub-topical if f is increasing and plus-sub-homogeneous.

(ii) A non-empty subset W of X is called downward if $(\hat{x} \in W,\ x \in X$ and $x \leq \hat{x}) \Longrightarrow x \in W$.

(iii) A non-empty subset V of X is called upward if $-V$ is downward.

(iv) A non-empty subset P of X is called plus-radiant if $(x \in P$ and $\mu \geq 0) \Longrightarrow x - \mu\mathbf{1} \in P$.

(v) A non-empty subset U of X is called plus-co-radiant if $(y \in U$ and $\mu \geq 0) \Longrightarrow y + \mu\mathbf{1} \in U$.

(vi) A non-empty subset D of $X \times X$ is called plus-radiant (plus-co-radiant) if $(x \in D$ and $\mu \geq 0)$ $\Longrightarrow x - \mu\mathbf{u} \in D$ $(x + \mu\mathbf{u} \in D)$, where $\mathbf{u}$ defined by

$$\mathbf{u} := (\mathbf{1}, \mathbf{1}) \in intS \times intS \subset X \times X. \tag{2.1}$$

(vii) Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function. The domain of f is defined by

$$domf := \Big\{x \in X : f(x) < +\infty\Big\},$$

the epigraph of f is defined by

$$epif := \Big\{(x, \lambda) \in X \times \mathbb{R} : f(x) \leq \lambda\Big\}, \tag{2.2}$$

and the lower level set of f at the height $\rho \in \mathbb{R}$ is defined by

$$lev_{\leq\rho}f := \{x \in X : f(x) \leq \rho\}. \tag{2.3}$$

The function f is called proper if $domf \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in X$.

In the sequel, recall the following definition from [9].

Definition 2.1. [9] Let X be a non-empty set, H be a non-empty set of functions $h : X \longrightarrow (-\infty, +\infty]$ and $f : X \longrightarrow (-\infty, +\infty]$ be a function.

(1) Define the lower support set of f (with respect to H) by

$$supp_l(f, H) := \Big\{h \in H : h(x) \leq f(x),\ \forall\ x \in X\Big\},$$

and the upper support set of f by

$$supp_u(f, H) := \Big\{h \in H : f(x) \leq h(x),\ \forall\ x \in X\Big\}.$$

(2) The function f is called abstract convex with respect to H (or, H-convex) if there exists a subset H_0 of H such that

$$f(x) = \sup_{h \in H_0} h(x),\ \forall\ x \in X.$$

In the following, we investigate the plus-co-radiant property for the sum, product, union and intersection of two sets.

Proposition 2.2. *Let $A, B \subseteq X$ be plus-co-radiant sets. Then the following assertions are true:*

(1) *$A + B := \{x + y : x \in A,\ y \in B\}$ is a plus-co-radiant subset of X.*

(2) *$A \times B := \{(v, u) : v \in A,\ u \in B\}$ is a plus-co-radiant subset of $X \times X$.*

(3) *$A \cup B$ and $A \cap B$ are plus-co-radiant subsets of X.*

Proof. (1) Let $z \in A + B$ and $\mu \geq 0$ be arbitrary. Since $z \in A + B$, there exist $x \in A$ and $y \in B$ such that $z = x + y$. Because B is a plus-co-radiant set, we conclude that $y + \mu\mathbf{1} \in B$. Thus,

$$\begin{aligned} z + \mu\mathbf{1} &= x + y + \mu\mathbf{1} \\ &= x + (y + \mu\mathbf{1}) \in A + B, \\ &\Longrightarrow z + \mu\mathbf{1} \in A + B. \end{aligned}$$

Hence, $A + B$ is a plus-co-radiant subset of X.

(2) Let $(v, u) \in A \times B$ and $\mu \geq 0$ be arbitrary. Therefore, $v \in A$ and $u \in B$. Since A and B are plus-co-radiant sets, it follows that $v + \mu\mathbf{1} \in A$ and $u + \mu\mathbf{1} \in B$, and hence, $(v + \mu\mathbf{1}, u + \mu\mathbf{1}) \in A \times B$. This together with (2.1) implies that $(v, u) + \mu\mathbf{u} \in A \times B$. So, $A \times B$ is a plus-co-radiant subset of $X \times X$.

(3) Let $x \in A \cup B$ and $\mu \geq 0$ be arbitrary. Then, $x \in A$ or $x \in B$, and hence, by the hypothesis we conclude that $x + \mu\mathbf{1} \in A$ or $x + \mu\mathbf{1} \in B$. This implies that $x + \mu\mathbf{1} \in A \cup B$. That is, $A \cup B$ is a plus-co-radiant subset of X. Similarly, $A \cap B$ is a plus-co-radiant subset of X.

□

Remark 2.3. Note that the parts (2) and (3) of Proposition 2.2 hold for an arbitrary collection of plus-co-radiant sets. Also, the part (1) holds if one of A or B is plus-co-radiant.

In the following, we give some topological properties of plus-co-radiant sets. For any subset W of a normed linear space X, we denote by $intW$ and clW the interior and the closure of W, respectively.

Proposition 2.4. *Let W be a plus-co-radiant subset of X. Then, $intW$ and clW are plus-co-radiant subsets of X.*

Proof. First, we show that $intW$ is a plus-co-radiant set. To this end, let $x \in intW$ and $\mu \geq 0$ be arbitrary. Since $x \in intW$, then there exists a neighborhood U of x such that $U \subset W$. Therefore, $V := U + \mu\mathbf{1}$ is a neighborhood of $x + \mu\mathbf{1}$, and so, $V \subset W + \mu\mathbf{1} \subset W$ because W is a plus-co-radiant set. Hence, $x + \mu\mathbf{1} \in intW$, that is, $intW$ is a plus-co-radiant subset of X.
Now, we show that clW is a plus-co-radiant set. Assume that $x \in clW$ and $\mu \geq 0$ are arbitrary. Suppose that U is any neighborhood of $x + \mu\mathbf{1}$. Thus, $V := U - \mu\mathbf{1}$ is a neighborhood of x, and

hence, $V \cap W \neq \emptyset$. Since W is a plus-co-radiant set, it follows that $U \cap (W + \mu\mathbf{1}) \subseteq U \cap W$. We claim that $U \cap (W + \mu\mathbf{1}) \neq \emptyset$, otherwise, $w + \mu\mathbf{1} \notin U$ for all $w \in W$. Therefore, $w \notin V$ for all $w \in W$, and so, $V \cap W = \emptyset$, which is a contradiction. Thus, $U \cap W \neq \emptyset$. This implies that $x + \mu\mathbf{1} \in clW$, and hence, clW is a plus-co-radiant subset of X. □

Now, consider the function $\eta : X \times X \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\eta(x, y, \alpha) := \max\left\{\lambda \in \mathbb{R} : \lambda \leq \alpha,\ \lambda\mathbf{1} \leq x + y\right\},\ \forall\, x, y\ \in X,\ \forall\, \alpha \in \mathbb{R}.$$

The function η was introduced and examined in [2]. The following properties of function η have been proved in [2, Proposition 3.1]. In fact, for every $x, y, x', y' \in X$, $\mu \geq 0$ and $\alpha, \alpha' \in \mathbb{R}$, one has

$$\eta(x + \mu\mathbf{1}, y, \alpha) \leq \eta(x, y, \alpha) + \mu, \tag{2.4}$$

$$\eta(x, y, \alpha) = \eta(y, x, \alpha), \tag{2.5}$$

$$x \leq x' \implies \eta(x, y, \alpha) \leq \eta(x', y, \alpha), \tag{2.6}$$

$$y \leq y' \implies \eta(x, y, \alpha) \leq \eta(x, y', \alpha), \tag{2.7}$$

$$\alpha \leq \alpha' \implies \eta(x, y, \alpha) \leq \eta(x, y, \alpha'), \tag{2.8}$$

$$\eta(x, -x + \alpha\mathbf{1}, \alpha) = \alpha, \tag{2.9}$$

$$\eta(x, y + t\mathbf{1}, \alpha) = \eta(x, y, \alpha - t) + t,\ \forall\, t \in \mathbb{R}. \tag{2.10}$$

We also introduce the function $\xi : X \times X \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\xi(x, y, \beta) := \min\left\{\lambda \in \mathbb{R} : \lambda \geq \beta,\ \lambda\mathbf{1} \geq x + y\right\},\ \forall\, x, y \in X,\ \forall\, \beta \in \mathbb{R}.$$

Also, the properties of the function ξ have been studied in [2, Proposition 3.2]. Indeed, for every $x, y, x', y' \in X$, $\mu \geq 0$ and $\beta, \beta' \in \mathbb{R}$, one has

$$\xi(x + \mu\mathbf{1}, y, \beta) \leq \xi(x, y, \beta) + \mu, \tag{2.11}$$

$$\xi(x, y, \beta) = \xi(y, x, \beta), \tag{2.12}$$

$$x \leq x' \implies \xi(x, y, \beta) \leq \xi(x', y, \beta), \tag{2.13}$$

$$y \leq y' \implies \xi(x, y, \beta) \leq \xi(x, y', \beta), \tag{2.14}$$

$$\beta \leq \beta' \implies \xi(x, y, \beta) \leq \xi(x, y, \beta'), \tag{2.15}$$

$$\xi(x, -x + \beta\mathbf{1}, \beta) = \beta, \tag{2.16}$$

$$\xi(x, y + t\mathbf{1}, \beta) = \xi(x, y, \beta - t) + t,\ \forall\, t \in \mathbb{R}. \tag{2.17}$$

Now, for each $y \in X$ and $\alpha \in \mathbb{R}$, consider the function $\eta_{y,\alpha} : X \longrightarrow \mathbb{R}$ defined by $\eta_{y,\alpha}(x) := \eta(x, y, \alpha)$ for all $x \in X$. Put

$$\Omega := \left\{\eta_{y,\alpha} : y \in X,\ \alpha \in \mathbb{R}\right\}, \tag{2.18}$$

which is called the set of elementary sub-topical functions. Moreover, for each $y \in X$ and $\beta \in \mathbb{R}$, consider the function $\xi_{y,\beta} : X \longrightarrow \mathbb{R}$ defined by $\xi_{y,\beta}(x) := \xi(x, y, \beta)$ for all $x \in X$. Let

$$\Sigma := \left\{\xi_{y,\beta} : y \in X,\ \beta \in \mathbb{R}\right\}, \tag{2.19}$$

which is called the set of elementary sub-topical functions.

Remark 2.5. In view of [1, Corollary 2.1], it should be noted that there exists a one-to-one corresponding between $X \times \mathbb{R}$ and Ω, and also between $X \times \mathbb{R}$ and Σ. Indeed, if we define the functions

$$\psi : X \times \mathbb{R} \longrightarrow \Omega \text{ by } \psi(y, \alpha) := \eta_{y,\alpha}, \ \forall\, y \in X, \ \forall\, \alpha \in \mathbb{R},$$

and

$$\phi : X \times \mathbb{R} \longrightarrow \Sigma \text{ by } \phi(y, \beta) := \xi_{y,\beta}, \ \forall\, y \in X, \ \forall\, \beta \in \mathbb{R},$$

then, ψ and ϕ are bijective (one-to-one and onto) functions.

3. Characterizing Sub-Topical Functions with Respect to Their Lower Level Sets and Epigraph

In this section, we give a characterization of sub-topical functions with respect to their lower level sets and epigraph. We start with the following results which have been proved in [1, 2]. For an easy reference, we state them without proof.

Remark 3.1. [1, 2, 10] It is easy to see that for a sub-topical function $f : X \longrightarrow \bar{\mathbb{R}}$, the following assertions are true.

(i) If $x \in X$ exists such that $f(x) = +\infty$, then, $f \equiv +\infty$.

(ii) If $x \in X$ exists such that $f(x) = -\infty$, then, $f \equiv -\infty$.

Hence, for any sub-topical function $f : X \longrightarrow \bar{\mathbb{R}}$, either $\operatorname{dom} f = X$, or $f \equiv +\infty$.

Note that we prove our results in Section 3 and Section 4 regarding Remark 3.1.

Theorem 3.2. *[1] Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function. Then the following assertions are equivalent:*

(1) f is sub-topical.

(2) $f(-y + \alpha \mathbf{1}) + \eta_{y,\alpha}(x) \le f(x) + \alpha, \ \forall\, x, y \in X, \ \forall\, \alpha \in \mathbb{R}$.

(3) $f(-y + \beta \mathbf{1}) + \xi_{y,\beta}(x) \ge f(x) + \beta, \ \forall\, x, y \in X, \ \forall\, \beta \in \mathbb{R}$.

Proposition 3.3. *[2] Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a sub-topical function. Then,*

$$supp_l(f, \Omega) = \Big\{\eta_{y,\alpha} \in \Omega : f(-y + \alpha \mathbf{1}) \ge \alpha\Big\}.$$

Proposition 3.4. *Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a sub-topical function. Then,*

$$supp_u(f, \Sigma) = \Big\{\xi_{y,\beta} \in \Sigma : f(-y + \beta \mathbf{1}) \le \beta\Big\}.$$

Proof. The proof is similar to that of Proposition 3.3, for more details see [2]. □

In the following, we give a characterization of sub-topical functions with respect to their lower level sets and epigraph.

Theorem 3.5. *Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function. Then, f is sub-topical if and only if its lower level sets are downward subsets of X and its epigraph is a plus-co-radiant subset of $X \times \mathbb{R}$.*

Proof. First, we show that f is increasing if and only if $lev_{\leq \rho} f$ is a downward subset of X for each $\rho \in \mathbb{R}$. Suppose that f is an increasing function. Let $\rho \in \mathbb{R}$ be arbitrary, and let $\hat{x} \in lev_{\leq \rho} f$ and $x \in X$ be such that $x \leq \hat{x}$. Since f is increasing, $f(x) \leq f(\hat{x})$. This together with (2.3) and the fact that $\hat{x} \in lev_{\leq \rho} f$ implies that $f(x) \leq \rho$. So, $x \in lev_{\leq \rho} f$, and hence, $lev_{\leq \rho} f$ is a downward set.
Conversely, let $x, y \in X$ be such that $y \leq x$. Since $x \in lev_{\leq f(x)} f$ and, by the hypothesis, $lev_{\leq f(x)} f$ is a downward subset of X, it follows that $y \in lev_{\leq f(x)} f$. Hence, $f(y) \leq f(x)$. Now, we show that f is plus-sub-homogeneous if and only if $epi f$ is plus-co-radiant. Assume that f is a plus-sub-homogeneous function. Let $(x, \rho) \in epi f$ and $\mu \geq 0$ be arbitrary. It follows from (2.2) that $f(x) \leq \rho$. Since f is plus-sub-homogeneous, we conclude that

$$\begin{aligned} f(x + \mu \mathbf{1}) &\leq f(x) + \mu \\ &\leq \rho + \mu, \\ &\implies (x + \mu \mathbf{1}, \rho + \mu) \in epi f, \\ &\implies (x, \rho) + \mu(\mathbf{1}, 1) \in epi f, \\ &\implies (x, \rho) + \mu \mathbf{u} \in epi f. \end{aligned}$$

Hence, $epi f$ is a plus-co-radiant subset of $X \times \mathbb{R}$. Note that $\mathbf{u} := (\mathbf{1}, 1) \in int S \times int\, \mathbb{R}_+ \subset X \times \mathbb{R}$ (also, see (2.1)).
Conversely, assume that $epi f$ is a plus-co-radiant subset of $X \times \mathbb{R}$. Let $x \in X$ and $\mu \geq 0$ be arbitrary. Since $(x, f(x)) \in epi f$ and $epi f$ is a plus-co-radiant subset of $X \times \mathbb{R}$, then,

$$\begin{aligned} &(x, f(x)) + \mu \mathbf{u} \in epi f, \\ &\implies (x + \mu \mathbf{1}, f(x) + \mu) \in epi f, \\ &\implies f(x + \mu \mathbf{1}) \leq f(x) + \mu, \end{aligned}$$

where, $\mathbf{u} := (\mathbf{1}, 1) \in int S \times int\, \mathbb{R}_+ \subset X \times \mathbb{R}$ defined by (2.1). Thus, f is a plus-sub-homogeneous function, which completes the proof. □

The indicator function of a subset A of X is $\delta_A : X \longrightarrow \bar{\mathbb{R}}$ defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{if } x \notin A, \end{cases} \quad \forall\, x \in X. \tag{3.1}$$

Proposition 3.6. *Let A be a subset of X. Then, A is plus-co-radiant if and only if δ_A is a plus-sub-homogeneous function.*

Proof. Suppose that A is a plus-co-radiant set. Let $x \in X$ and $\mu \geq 0$ be arbitrary.
Case (i) : If $x \in A$, then, by the hypothesis, $x + \mu \mathbf{1} \in A$. Therefore,

$$\delta_A(x + \mu \mathbf{1}) = 0 \leq \delta_A(x) + \mu.$$

Case (ii) : If $x \notin A$, then,

$$\delta_A(x) = +\infty, \implies \delta_A(x) + \mu = +\infty \geq \delta_A(x + \mu\mathbf{1}).$$

Hence, δ_A is a plus-sub-homogeneous function.
Conversely, assume that δ_A is a plus-sub-homogeneous function. We show that A is a plus-co-radiant subset of X. Assume if possible that there exist $x \in A$ and $\mu \geq 0$ such that $x + \mu\mathbf{1} \notin A$. So, $\delta_A(x + \mu\mathbf{1}) = +\infty$. Since $x \in A$ and δ_A is a plus-sub-homogeneous function, then, $+\infty = \delta_A(x + \mu\mathbf{1}) \leq \delta_A(x) + \mu = \mu$, which is a contradiction. Therefore, $x + \mu\mathbf{1} \in A$ for all $x \in A$ and all $\mu \geq 0$. That is, A is a plus-co-radiant subset of X. □

4. Polar of Sub-Topical Functions

In this section, we investigate properties of the upper and lower polar of sub-topical functions. Moreover, by using two different classes of elementary functions, we present a characterization of sub-topical functions with respect to their polar functions, and examine the relation between polar functions and support sets of this class of functions. We start with the following definition.

Definition 4.1. Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function. The upper polar function of f is the function $f_u^\circ : X \times \mathbb{R} \longrightarrow \bar{\mathbb{R}}$ defined by

$$f_u^\circ(y,\beta) := \inf_{x\in X} \left\{\xi_{y,\beta}(x) - f(x)\right\}, \ \forall\ (y,\beta) \in X \times \mathbb{R}. \tag{4.1}$$

For more details, see Corollary 2.5.

In the following, we give a characterization of sub-topical functions with respect to their upper polar functions.

Theorem 4.2. *Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function. Then,*

$$f_u^\circ(y,\beta) \leq \beta - f(-y + \beta\mathbf{1}), \ \forall\ (y,\beta) \in X \times \mathbb{R}. \tag{4.2}$$

Moreover, f is a sub-topical function if and only if

$$f_u^\circ(y,\beta) = \beta - f(-y + \beta\mathbf{1}), \ \forall\ (y,\beta) \in X \times \mathbb{R}. \tag{4.3}$$

Proof. Let $(y,\beta) \in X \times \mathbb{R}$ be arbitrary. In view of Definition 4.1, we have

$$f_u^\circ(y,\beta) \leq \xi_{y,\beta}(x) - f(x), \ \forall\ x \in X. \tag{4.4}$$

Put $x := -y + \beta\mathbf{1}$ in (4.4), we obtain $f_u^\circ(y,\beta) \leq \xi_{y,\beta}(-y + \beta\mathbf{1}) - f(-y + \beta\mathbf{1})$. Therefore, by (2.12) and (2.16), we conclude that $f_u^\circ(y,\beta) \leq \beta - f(-y + \beta\mathbf{1})$. Hence, (4.2) holds. Now, let $f : X \longrightarrow \bar{\mathbb{R}}$ be a sub-topical function and $(y,\beta) \in X \times \mathbb{R}$ be arbitrary. Then, by Theorem 3.2 (the implication (1) $\implies$ (3)), one has $f(-y+\beta\mathbf{1})+\xi_{y,\beta}(x) \geq \beta+f(x)$ for all $x \in X$. So, $\xi_{y,\beta}(x)-f(x) \geq \beta-f(-y+\beta\mathbf{1})$ for all $x \in X$. This implies that

$$\begin{aligned} f_u^\circ(y,\beta) &= \inf_{x\in X} \left\{\xi_{y,\beta}(x) - f(x)\right\} \\ &\geq \beta - f(-y + \beta\mathbf{1}). \end{aligned}$$

This together with (4.2) implies that $f_u^\circ(y,\beta) = \beta - f(-y+\beta\mathbf{1})$.
Conversely, let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function satisfying (4.3). Therefore, we conclude from Definition 4.1 that

$$\xi_{y,\beta}(x) - f(x) \geq \beta - f(-y+\beta\mathbf{1}), \ \forall\, x \in X, \ \forall\, (y,\beta) \in X \times \mathbb{R}. \tag{4.5}$$

It follows that $f(-y+\beta\mathbf{1}) + \xi_{y,\beta}(x) \geq \beta + f(x)$ for all $x, y \in X$ and all $\beta \in \mathbb{R}$. Hence, in view of Theorem 3.2 (the implication (3) $\Longrightarrow$ (1)), we deduce that f is a sub-topical function. □

Now, we give a characterization of the upper support set of a sub-topical function $f : X \longrightarrow \bar{\mathbb{R}}$ with respect to the upper polar of f.

Corollary 4.3. *Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a sub-topical function. Then*

$$supp_u(f,\Sigma) = \left\{\xi_{y,\beta} \in \Sigma : f_u^\circ(y,\beta) \geq 0\right\}.$$

Proof. This is an immediate consequence of Theorem 4.2 and Proposition 3.4. □

Definition 4.4. Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function. The lower polar function of f is the function $f_l^\circ : X \times \mathbb{R} \longrightarrow \bar{\mathbb{R}}$ defined by

$$f_l^\circ(y,\alpha) := \sup_{x\in X}\left\{\eta_{y,\alpha}(x) - f(x)\right\}, \ \forall\, (y,\alpha) \in X \times \mathbb{R}.$$

For more details, see Corollary 2.5.

In the following, we give a characterization of sub-topical functions with respect to their lower polar functions.

Theorem 4.5. *Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a function. Then,*

$$f_l^\circ(y,\alpha) \geq \alpha - f(-y+\alpha\mathbf{1}), \ \forall\, (y,\alpha) \in X \times \mathbb{R}.$$

Moreover, f is a sub-topical function if and only if

$$f_l^\circ(y,\alpha) = \alpha - f(-y+\alpha\mathbf{1}), \ \forall\, (y,\alpha) \in X \times \mathbb{R}.$$

Proof. By using Theorem 3.2 (the implication (1) $\Longleftrightarrow$ (2)) and a similar argument as the proof of Theorem 4.2 the result follows. □

Now, we present a characterization of the lower support set of a sub-topical function $f : X \longrightarrow \bar{\mathbb{R}}$ with respect to the lower polar of f.

Corollary 4.6. *Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a sub-topical function. Then,*

$$supp_l(f,\Omega) = \left\{\eta_{y,\alpha} \in \Omega : f_l^\circ(y,\alpha) \leq 0\right\}.$$

Proof. This is an immediate consequence of Theorem 4.5 and Proposition 3.3. □

In the sequel, for a subset W of X, we define the $(X \times \mathbb{R})$-support function and the $(X \times \mathbb{R})$-polar set of W (note that there exists a one-to-one corresponding between $X \times \mathbb{R}$ and Ω, for more details, see Corollary 2.5).

Definition 4.7. Let W be a subset of X. Define the $(X \times \mathbb{R})$-support function of W, $\sigma_W : X \times \mathbb{R} \longrightarrow \bar{\mathbb{R}}$, by

$$\sigma_W(y, \alpha) := \sup_{x \in W} \eta_{y,\alpha}(x), \ \forall\, (y, \alpha) \in X \times \mathbb{R}. \tag{4.6}$$

Lemma 4.8. *Let W be a subset of X. Then,*

$$(\delta_W)_l^\circ(y, \alpha) = \sigma_W(y, \alpha), \ \forall\, (y, \alpha) \in X \times \mathbb{R}.$$

Proof. By using Definition 4.4, we have

$$\begin{aligned}(\delta_W)_l^\circ(y, \alpha) &= \sup_{x \in X} \left\{\eta_{y,\alpha}(x) - \delta_W(x)\right\} \\ &= \sup_{x \in W}\{\eta_{y,\alpha}(x)\} \\ &= \sigma_W(y, \alpha), \ \forall\, (y, \alpha) \in X \times \mathbb{R},\end{aligned}$$

which completes the proof. □

Definition 4.9. Let W be a subset of X. Define the $(X \times \mathbb{R})$-polar set of W (W°) by

$$W^\circ := \left\{(y, \alpha) \in X \times \mathbb{R} : \eta_{y,\alpha}(x) \leq 0, \ \forall\, x \in W\right\}.$$

Remark 4.10. Let W be a subset of X.

(1) It is easy to see that the $(X \times \mathbb{R})$-polar set of W is the 0-level set of the $(X \times \mathbb{R})$-support function of W, i.e.,

$$W^\circ = \left\{(y, \alpha) \in X \times \mathbb{R} : \sigma_W(y, \alpha) \leq 0\right\}.$$

(2) It is easy to check that W° is a convex cone and downward set.

Lemma 4.11. *Let W_1 and W_2 be non-empty subsets of X. Then the following assertions are true:*

(1) *If $W_1 \subseteq W_2$, then, $W_2^\circ \subseteq W_1^\circ$.*

(2) $(W_1 \cup W_2)^\circ = W_1^\circ \cap W_2^\circ$.

(3) $W_1^\circ \cup W_2^\circ \subset (W_1 \cap W_2)^\circ$.

Proof. (1) Let $(y, \alpha) \in W_2^\circ$ be arbitrary. Then, $\eta_{y,\alpha}(x) \leq 0$ for all $x \in W_2$. Since $W_1 \subset W_2$, we have $\eta_{y,\alpha}(x) \leq 0$ for all $x \in W_1$, and so, $(y, \alpha) \in W_1^\circ$.

(2) Since $W_1 \subseteq W_1 \cup W_2$ and $W_2 \subseteq W_1 \cup W_2$, it follows from (1) that $(W_1 \cup W_2)^\circ \subseteq W_1^\circ \cap W_2^\circ$. Conversely, suppose that $(y, \alpha) \in W_1^\circ \cap W_2^\circ$. Therefore,

$$\eta_{y,\alpha}(x) \leq 0, \ \forall \ x \in W_1 \text{ and } \eta_{y,\alpha}(x) \leq 0, \ \forall \ x \in W_2. \tag{4.7}$$

Now, let $x \in W_1 \cup W_2$ be arbitrary. Thus, in view of (4.7), we conclude that $\eta_{y,\alpha}(x) \leq 0$. So, $(y, \alpha) \in (W_1 \cup W_2)^\circ$. Hence, $W_1^\circ \cap W_2^\circ \subseteq (W_1 \cup W_2)^\circ$.

(3) This is an immediate consequence of (1).

□

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