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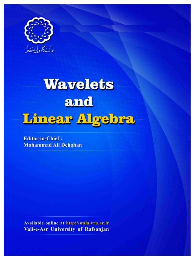


# Linear preservers of Miranda-Thompson majorization on $\mathbf{M}_{m,n}$

**Ahmad Mohammadhasani[a,*], Asma Ilkhanizadeh Manesh[b]**

[a]*Department of Mathematics, Sirjan University of technology, Sirjan, Islamic Republic of Iran.*
[b]*Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O. Box: 7713936417, Rafsanjan, Islamic Republic of Iran.*

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## Abstract

Miranda-Thompson majorization is a group-induced cone ordering on $\mathbb{R}^n$ induced by the group of generalized permutation with determinants equal to 1. In this paper, we generalize Miranda-Thompson majorization on the matrices. For $X$, $Y \in \mathbf{M}_{m,n}$, $X$ is said to be Miranda-Thompson majorized by $Y$ (denoted by $X \prec_{mt} Y$) if there exists some $D \in \mathrm{Conv(G)}$ such that $X = DY$. Also, we characterize linear preservers of this concept on $\mathbf{M}_{m,n}$.



*Corresponding author
*Email addresses:* a.mohammadhasani53@gmail.com (Ahmad Mohammadhasani ), a.ilkhani@vru.ac.ir (Asma Ilkhanizadeh Manesh)

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## 1. Introduction

In matrix theory, majorization plays a significant role. The best general reference on this subject is *Inequalities: Theory of majorization and its applications* by A. W. Marshall, I. Olkin, and B. C. Arnold [8]. Some kinds of majorization with their linear preservers can be found in [1]-[2] and [4]-[6].

Let $V$ be a finite-dimensional real vector space, O(V) the orthogonal group acting on $V$ and $G$ a closed subgroup of O(V). The group $G$ induces an equivalence relation on $V$, defined by $x \approx y$ if and only if $y = gx$ for some $g \in G$. The equivalence classes of this relation are called the orbits of $G$. For each $y \in V$ the orbit of $y$ is $\mathrm{O_G(y)} = \{\mathrm{gy} \mid \mathrm{g} \in \mathrm{G}\}$. A vector $x$ is G-majorized by $y$, denoted by $x \prec_G y$, if $x \in \mathrm{Conv(O_G(y))}$, where the notation Conv(A) is the convex hull of a set $A$. The group majorization $\prec_G$ generates an equivalence relation $\sim_G$ on $V$ defined as follows. $x \sim_G y$ if and only if $x \prec_G y \prec_G x$. In [3], A. Giovagnoli and H. P. Wynn observed that $x \sim_G y$ if and only if $y = gx$ for some $g \in G$. The classical majorization is a vector pre-ordering on $\mathbb{R}^n$ induced by the permutation group. In [9], M. Soleymani and A. Armandnejad introduced the concept of even majorization, and they characterized the linear preservers and strong linear preservers of this concept on $\mathbf{M}_{m,n}$.

In the peresent paper, let $V = \mathbb{R}^n$ with the standard inner product and $G = \{PC \mid P \in \mathbb{P}_n, C \in \mathbb{C}_n, \det(\mathrm{C}) = 1\}$, where $\mathbb{P}_n$ is the group of $n$-by-$n$ permutation matrices and $\mathbb{C}_n$ is the group of $n$-by-$n$ diagonal orthogonal matrices. Notice that $\mathbb{C}_n = \{diag(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \{1, -1\}, 1 \leq i \leq n\}$. $\prec_G$ will display with $\prec_{mt}$. In [7, Section 6], M. Niezgoda proved for any $x, y \in \mathbb{R}^n$ we have $x \prec_{mt} y$ if and only if

$$\sum_{k=1}^{i} |x|_{[k]} \leq \sum_{k=1}^{i} |y|_{[k]}, \text{ for all } i \ (1 \leq i \leq n-2),$$

$$\sum_{k=1}^{n-1} |x|_{[k]} + sing(x)|x|_{[n]} \leq \sum_{k=1}^{n-1} |y|_{[k]} + sing(y)|y|_{[n]},$$

$$\sum_{k=1}^{n-1} |x|_{[k]} - sing(x)|x|_{[n]} \leq \sum_{k=1}^{n-1} |y|_{[k]} - sing(y)|y|_{[n]},$$

where $|x| = (|x_1|, |x_2|, \ldots, |x_n|)$, $sign(x) = sgn(\prod_{i=1}^{n} x_i)$ and by $(|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]})^t$ we denote the entries of a vector $x = (x_1, x_2, \ldots, x_n)^t \in \mathbb{R}^n$ arranged in decreasing order.

Now, we generalize this concept on matrices.

**Definition 1.1.** For $X, Y \in \mathbf{M}_{m,n}$, $X$ is said to be Miranda-Thompson majorized by $Y$ (denoted by $X \prec_{mt} Y$) if there exists some $D \in \mathrm{Conv(G)}$ such that $X = DY$. That is, $X = \sum_{i=1}^{k} \lambda_i P_i C_i Y$, where $P_i \in \mathbb{P}_m$, $C_i \in \mathbb{C}_m$, $\det(\mathrm{C_i}) = 1$, $\lambda_i \geq 0$, for each $i$ $(1 \leq i \leq k)$, and $\sum_{i=1}^{k} \lambda_i = 1$.

The present paper continues in three further sections. Section 2 presents the structure of all linear preservers of $\prec_{mt}$ on $\mathbb{R}^2$. The third section contains the structure of all linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving Miranda-Thompson majorization. Section 4 states all linear preservers of $\prec_{mt}$ from $\mathbf{M}_{m,n}$ to $\mathbf{M}_{m,n}$.

## 2. Miranda-Thompson majorization on $\mathbb{R}^2$ and its linear preservers

This section studies facts of Miranda-Thompson majorization that are necessary for studying the linear preservers of this concept. Also, we characterize the structure of all linear preservers of $\prec_{mt}$ on $\mathbb{R}^2$.

**Lemma 2.1.** *Let $P \in \mathbb{P}_n$ and $C \in \mathbb{C}_n$. Then there exists some $C_* \in \mathbb{C}_n$ such that $PC = C_*P$.*

*Proof.* Let $C = [C_{ij}]$ and $\delta$ be the corresponding permutation with $P$. Put $J = \{1 \leq j \leq n \mid C_{jj} = -1\}$ and $J_* = \{\delta_j \mid j \in J\}$. Now, we define $C_* := diag(d_{11}, \ldots, d_{nn}) \in \mathbb{C}_n$, where $d_{ii} = \begin{cases} +1 & \text{if } i \notin J_* \\ -1 & \text{if } i \in J_*. \end{cases}$ We observe that $PC = C_*P$, as desired. □

In the following lemma, we express linear preservers of Miranda-Thompson majorization $T : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that $e_i$ is the $i^{th}$ unit vector.

**Lemma 2.2.** *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function. Then $T$ preserves $\prec_{mt}$ if and only if there exist $P \in \mathbb{P}_2$, $\boldsymbol{a} \in \mathbb{R}^2$, and $s \in \{1, -1\}$ such that $T\binom{x}{y} = x\boldsymbol{a} + syP\boldsymbol{a}$ for all $\binom{x}{y} \in \mathbb{R}^2$.*

*Proof.* It is obvious the proof of sufficient condition.

For the converse, we prove the necessity of the condition. Assume that $T$ preserves $\prec_{mt}$. Let $[T] = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}$, where $[T]$ is the matrix representation of $T$ on the standard basis $\{e_1, e_2\}$ on $\mathbb{R}^2$. For each $\binom{z}{t}, \binom{x}{y} \in \mathbb{R}^2$, we have $\binom{z}{t} \sim_{mt} \binom{x}{y}$ if and only if $\binom{z}{t} \in \{\binom{x}{y}, -\binom{x}{y}, \binom{y}{x}, -\binom{y}{x}\}$. We conclude from $e_2 \sim_{mt} e_1$ that $Te_2 \sim_{mt} Te_1$, hence $\binom{a_3}{a_4} \sim_{mt} \binom{a_1}{a_2}$, and finally that $\binom{a_3}{a_4} \in \{\binom{a_1}{a_2}, -\binom{a_1}{a_2}, \binom{a_2}{a_1}, -\binom{a_2}{a_1}\}$. If $\mathbf{a} = \binom{a_1}{a_2}$, then for some $2 \times 2$ permutation matrix $P$ $T\binom{x}{y} = x\binom{a_1}{a_2} + yPC\binom{a_1}{a_2} = x\mathbf{a} + yPC\mathbf{a} = \begin{cases} x\mathbf{a} + yP\mathbf{a} & \text{if } C = +I \\ x\mathbf{a} - yP\mathbf{a} & \text{if } C = -I \end{cases}$, and the proof is complete. □

**Lemma 2.3.** *Let $T_1$ and $T_2$ be two linear preservers of $\prec_{mt}$ on $\mathbb{R}^2$. If $T_1 + T_2$ preserves $\prec_{mt}$, then there exist some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$, $P \in \mathbb{P}_2$, and $s \in \{1, -1\}$ such that $T_1\binom{x}{y} = x\boldsymbol{a} + syP\boldsymbol{a}$ and $T_2\binom{x}{y} = x\boldsymbol{b} + syP\boldsymbol{b}$ for all $\binom{x}{y} \in \mathbb{R}^2$.*

*Proof.* As $T_1$, $T_2$, and $T_1 + T_2$ preserve $\prec_{mt}$ on $\mathbb{R}^2$, Lemma 2.2 ensures that for each $\binom{x}{y} \in \mathbb{R}^2$ we have $T_1\binom{x}{y} = x\mathbf{a} + s_1yP_1\mathbf{a}$, $T_2\binom{x}{y} = x\mathbf{b} + s_2yP_2\mathbf{b}$, and $(T_1 + T_2)\binom{x}{y} = x\mathbf{c} + syP\mathbf{c}$, for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$, $P_1, P_2, P \in \mathbb{P}_2$, and $s_1, s_2, s \in \{-1, 1\}$. It suffices to show that we can choose equal $s_1$ and $s_2$ also equal the permutations $P_1$ and $P_2$. We observe that $x(\mathbf{a} + \mathbf{b}) + y(s_1P_1\mathbf{a} + s_2P_2\mathbf{b}) = x\mathbf{c} + y(sP\mathbf{c})$, for all $x, y \in \mathbb{R}$. If $x = 1$ and $y = 0$, then $\mathbf{a} + \mathbf{b} = \mathbf{c}$. Choose $x = 0$ and $y = 1$. It follows that $ss_1(PP_1)\mathbf{a} + ss_2(PP_2)\mathbf{b} = \mathbf{a} + \mathbf{b}$. Put $\mathbf{a} = \binom{a_1}{a_2}$, $\mathbf{b} = \binom{b_1}{b_2}$, $t_i = ss_i$, and $Q_i = PP_i$, for $i = 1, 2$. So we prove that if $t_1Q_1\mathbf{a} + t_2Q_2\mathbf{b} = \mathbf{a} + \mathbf{b}$, where $t_1, t_2 \in \{-1, 1\}$ and $Q_1, Q_2 \in \mathbb{P}_2$, then we can choose equal $t_1$ and $t_2$ also equal $Q_1$ and $Q_2$. Given that $t_1 \neq t_2$ or $Q_1 \neq Q_2$, we consider three cases.

Case(1). $Q_1 = Q_2 = I$ and $t_1 \neq t_2$. If $t_1 = -t_2 = 1$, then $\mathbf{b} = \mathbf{0}$, and so we can select $t_1 = t_2 = 1$. In the same way, if $t_1 = -t_2 = -1$, then $\mathbf{a} = \mathbf{0}$, and choose $t_1 = t_2 = 1$.

Case(2). $Q_1 = Q_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $t_1 \neq t_2$. If $t_1 = -t_2 = 1$, then $\mathbf{a} = \binom{a_1}{a_1}$ and $\mathbf{b} = \binom{b_1}{-b_1}$. If $t_1 = -t_2 = -1$, then $\mathbf{a} = \binom{a_1}{-a_1}$ and $\mathbf{b} = \binom{b_1}{b_1}$. We can choose $Q_1 = Q_2 = I$ and $t_1 = t_2 = 1$.

Case(3). $Q_1 \neq Q_2$. Without loss of generality, assume that $Q_1 = I$ and $Q_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $t_1 = -t_2 = 1$, then $\mathbf{b} = \binom{b_1}{-b_1}$, and hence $t_2Q_2b = t_1Q_1b$. It implies that we can choose $Q_1 = Q_2 = I$ and $t_1 = t_2 = 1$. If $t_1 = -t_2 = -1$, then $\mathbf{a} = \binom{a_1}{-a_1}$. So we have $t_1Q_1\mathbf{a} = t_2Q_2\mathbf{a}$. Thus, we can select $Q_1 = Q_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $t_1 = t_2 = 1$. If $t_1 = t_2 = 1$, then $\mathbf{b} = \binom{b_1}{b_2}$, and it shows that we can choose $Q_1 = Q_2 = I$. If $t_1 = t_2 = -1$, then $\mathbf{a} = \binom{a_1}{a_1}$, and so we can choose $Q_1 = Q_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. □

Let $T$ preserve $\prec_{mt}$ on $\mathbb{R}^2$. That is, $[T] = [\mathbf{a} \mid sP\mathbf{a}]$, where $\mathbf{a} = \binom{a_1}{a_2} \in \mathbb{R}^2$, $s \in \{-1, 1\}$, and $P \in \mathbb{P}_2$.

(1) If $|a_1| \neq |a_2|$; We say $T$ is of the first type. In this case, $s$ and $P$ are unique. So in the previous lemma if $s$ and $P$ are related to $T_1$, then $s$ and $P$ are related to $T_2$, too.

Notice that if $|a_1| = |a_2|$, then $T(e_1) = T(e_2)$ or $T(e_1) = -T(e_2)$.

(2) If $a_1 = -a_2 \neq 0$; We say $T$ is of the second type. Then $[T] = a_1 \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ or $[T] = a_1 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. $s$ and $P$ are not unique. Because we can replace $-s$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P$ with $s$ and $P$.

(3) If $a_1 = a_2 \neq 0$; We say $T$ is of the third type, and $[T] = a_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ or $[T] = a_1 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Only $s$ is unique.

## 3. Miranda-Thompson majorization on $\mathbb{R}^n$ ($n \geq 3$) and its linear preservers

This section contains all linear preservers of $\prec_{mt}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Let $\| \, . \, \|$ be the Euclidean norm. In the following theorem the structure of linear functions $T : \mathbb{R}^n \to \mathbb{R}^n$ preserving Miranda-Thompson majorization will be characterized.

**Theorem 3.1.** *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear function whenever $n \geq 3$. Then $T$ preserves $\prec_{mt}$ if and only if there exist $a \in \mathbb{R}$, $P \in \mathbb{P}_n$, and $C \in \mathbb{C}_n$ such that $[T] = aPC$.*

*Proof.* First, assume that $[T] = aPC$, for some $a \in \mathbb{R}$, $P \in \mathbb{P}_n$, and $C \in \mathbb{C}_n$. Let $x, y \in \mathbb{R}^n$ such that $y \sim_{mt} x$. It implies that there exist some $Q \in \mathbb{P}_n$ and $D \in \mathbb{C}_n$, where $\det(D) = 1$ such that $y = QDx$. We observe that $Ty = (aPC)y = (PCQDCP^t)(aPCx)$. Lemma 2.1 ensures that $CQ = QC_1$ for some $C_1 \in \mathbb{C}_n$. So $Ty = (PQC_1DCP^t)Tx$. As $C_1DC \in \mathbb{C}_n$, there is some $C_2 \in \mathbb{C}_n$ such that $(C_1DC)P^t = P^tC_2$. Then $Ty = (PQP^tC_2)Tx$. Since $PQP^t \in \mathbb{P}_n$ and $C_2 \in \mathbb{C}_n$, we deduce that $Ty \sim_{mt} Tx$. Therefore, $T$ preserves $\sim_{mt}$. Now, let $x, y \in \mathbb{R}^n$ such that $y \prec_{mt} x$. So there exist some $P_i \in \mathbb{P}_n$, $C_i \in \mathbb{C}_n$, $\det(C_i) = 1$, $\lambda_i \geq 0$, for each $i$ ($1 \leq i \leq k$), and $\sum_{i=1}^{k} \lambda_i = 1$ such that $y = \sum_{i=1}^{k} \lambda_i P_i C_i x$. We have $Ty = T(\sum_{i=1}^{k} \lambda_i P_i C_i x) = \sum_{i=1}^{k} \lambda_i T(P_i C_i x)$. We proved that for each $i$

$(1 \leq i \leq k)$ there exist some $Q_i \in \mathbb{P}_n$, and $D_i \in \mathbb{C}_n$, $\det(D_i) = 1$ such that $T(P_iC_ix) = Q_iD_iTx$. It follows that $Ty = \sum_{i=1}^{k} \lambda_i Q_i D_i T x$, and hence $Ty \prec_{mt} Tx$.

Next, assume that $T$ preserves $\prec_{mt}$. If $T = 0$, then there is no thing to prove. Let $T \neq 0$, and $A = [T] = [A_1/A_2/\dots/A_n]$. So $Tx = Ax$ for all $x \in \mathbb{R}^n$. Suppose that $A_{i_0}$ is a row of $A$ which has the maximum Euclidean norm. That is, $\| A_i \| \leq \| A_{i_0} \|$ for each $i$ $(1 \leq i \leq n)$. Let $P \in \mathbb{P}_n$, $C \in \mathbb{C}_n$, and $\det(C) = 1$. Consider $x = A_{i_0}^t$ and $y = PCA_{i_0}^t$. As $x \sim_{mt} y$, we see that $Tx \sim_{mt} Ty$. It shows that $Ty = QDTx$ for some $Q \in \mathbb{P}_n$, $D \in \mathbb{C}_n$, where $\det(D) = 1$. So $Ty = QD[A_1/A_2/\dots/A_n]A_{i_0}^t = QD(\lambda_1 < A_1, A_{i_0} >, \dots, \lambda_n < A_n, A_{i_0} >)^t$, where $\lambda_j \in \{-1, 1\}$ for all $j = 1, \dots, n$. Selecting positive or negative depends on $D$. On the other hand, we have

$$Ty = [A_1/A_2/\dots/A_n]PCA_{i_0}^t = (< A_1, A_{i_0}C^tP^t >, \dots, < A_n, A_{i_0}C^tP^t >)^t. \tag{3.1}$$

Let $\delta$ be the corresponding permutation with $Q$. If $\delta_r = i_0$; By calculating $r^{th}$ component, we observe that $< A_r, A_{i_0}C^tP^t >= + < A_{\delta_r}, A_{i_0} >= + < A_{i_0}, A_{i_0} >= + \| A_{i_0} \|^2$ or $< A_r, A_{i_0}C^tP^t >= - < A_{\delta_r}, A_{i_0} >= - < A_{i_0}, A_{i_0} >= - \| A_{i_0} \|^2$ . The Cauchy-Schwarz inequality states that $\| A_{i_0} \|^2 = | < A_r, A_{i_0}C^tP^t > | \leq \| A_r \| \; \| A_{i_0}C^tP^t \| = \| A_r \| \; \| A_{i_0} \| \leq \| A_{i_0} \| \; \| A_{i_0} \| = \| A_{i_0} \|^2$. So

$$| < A_r, A_{i_0}C^tP^t > | = \| A_r \| \; \| A_{i_0}C^tP^t \| . \tag{3.2}$$

Then there exists some $\lambda \in \mathbb{R} \setminus \{0\}$ such that $PCA_{i_0}^t = \lambda A_r^t$. By putting this relation in 3.2 we conclude that $\lambda = 1$ or $\lambda = -1$. Thus $GA_{i_0}^t = \{QDA_{i_0}^t \mid Q \in \mathbb{P}_n, D \in \mathbb{C}_n, \det(D) = 1\} \subseteq \{\lambda_1 A_1^t, \dots, \lambda_n A_n^t \mid |\lambda_i| = 1, 1 \leq i \leq n\}$. Right-hand set has at most 2n elements, and so $GA_{i_0}^t$ has at most 2n elements, too. So $A_{i_0}$ has exactly a non-zero component, namely $a$. It follows that $GA_{i_0}^t = \{\lambda_1 A_1^t, \dots, \lambda_n A_n^t \mid |\lambda_i| = 1, 1 \leq i \leq n\}$, and hence $A = aPC$, for some $P \in \mathbb{P}_n$, $C \in \mathbb{C}_n$. □

**Lemma 3.2.** *Let $T_1$ and $T_2$ be two linear preservers of $\prec_{mt}$ on $\mathbb{R}^n$ where $n \geq 3$. If $T_1 + T_2$ preserve $\prec_{mt}$, then there exist some $\alpha_1, \alpha_2 \in \mathbb{R}$, $P \in \mathbb{P}_n$, $C \in \mathbb{C}_n$ such that $[T_1] = \alpha_1 PC$ and $[T_2] = \alpha_2 PC$.*

*Proof.* As $T_1, T_2$, and $T_1 + T_2$ preserve $\prec_{mt}$, Theorem 3.1 ensures that there exist some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, $P_1, P_2, P_3 \in \mathbb{P}_n$, $C_1, C_2, C_3 \in \mathbb{C}_n$ such that $[T_1] = \alpha_1 P_1 C_1$, $[T_2] = \alpha_2 P_2 C_2$, and $[T_1 + T_2] = \alpha_3 P_3 C_3$. We want to show that $P_1 = P_2$ and we can choose $C_1 = C_2$. If $\alpha_1 = 0$ or $\alpha_2 = 0$, there is nothing to prove. Assume that $\alpha_1$ and $\alpha_2$ are nonzero. For all $i = 1, 2, 3$, let $C_i = diag(c_{i1}, \dots, c_{in})$, where $c_{ij} \in \{-1, 1\}$, for each $j$ $(1 \leq j \leq n)$. We have

$$\alpha_1 P_1 C_1 x + \alpha_2 P_2 C_2 x = \alpha_3 P_3 C_3 x, \tag{3.3}$$

for all $x \in \mathbb{R}^n$.

If $P_1 \neq P_2$; Then there exist some $r, s, k, l$ $(1 \leq r, s, k, l \leq n)$, $k \neq l$ such that $P_1 e_r = e_k \neq e_l = P_2 e_r$ and $P_3 e_r = e_s$. By putting $x = e_r$ in the relation 3.3 we have $c_{1r}\alpha_1 e_k + c_{2r}\alpha_2 e_l = c_{3r}\alpha_3 e_s$. Since $k \neq l$, the vector $c_{1r}\alpha_1 e_k + c_{2r}\alpha_2 e_l$ has two non-zero components. On the other hand, the vector $c_{3r}\alpha_3 e_s$ has at most a non-zero component, which is a contradiction. It means that $P_1 = P_2$, and we have

$$\alpha_1 C_1 + \alpha_2 C_2 = \alpha_3 C_3, \tag{3.4}$$

and so $\alpha_1 c_{1j} + \alpha_2 c_{2j} = \alpha_3 c_{3j}$, for each $j$ $(1 \leq j \leq n)$. It follows that $\alpha_1 + (c_{1j}c_{2j})\alpha_2 = \alpha_3(c_{1j}c_{3j})$, and hence $|\alpha_1 + (c_{1j}c_{2j})\alpha_2| = |\alpha_3|$, for each $j$ $(1 \leq j \leq n)$. We observe that

$$|\alpha_1 + (c_{1j}c_{2j})\alpha_2| = |\alpha_1 + (c_{11}c_{21})\alpha_2|, \tag{3.5}$$

for each $j$ $(1 \leq j \leq n)$. Now, if $c_{11}c_{21} = 1$, as $\alpha_1, \alpha_2 \neq 0$, then $|\alpha_1 + \alpha_2| \neq |\alpha_1 - \alpha_2|$, and hence 3.5 ensures that $c_{1j}c_{2j} = c_{11}c_{21}$, for each $j$ $(1 \leq j \leq n)$. Thus, $c_{1j}c_{2j} = 1$ or $c_{1j}c_{2j} = -1$, for each $j$ $(1 \leq j \leq n)$. In the first case we have $C_1 = C_2$, as desired, and in the second case we see that $C_1 = -C_2$, and so $T_2x = \alpha_2 P_2 C_2 x = (-\alpha_2)P_1C_1x$. By changing $\alpha_2$ to $-\alpha_2$ we can assume that $C_1 = C_2$, and the proof is complete. □

## 4. Miranda-Thompson majorization on $\mathbf{M}_{m,n}$ and its linear preservers

In this section, we characterize the linear preservers of Miranda-Thompson majorization on $\mathbf{M}_{m,n}$.

The following sense is useful for finding the structure of linear preservers of Miranda-Thompson majorization.

For each $i, j$ $(1 \leq i, j \leq n)$, consider the embedding $E_j : \mathbb{R}^m \to \mathbf{M}_{m,n}$ and the projection $E^i : \mathbf{M}_{m,n} \to \mathbb{R}^m$, where $E_j(x) = xe_j^t$ and $E^i(X) = Xe_i$. It is easy to show that for every linear function $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$, $TX = T[X_1 \mid X_2 \mid \ldots \mid X_n] = [\sum_{j=1}^n T_{1j}X_j \mid \sum_{j=1}^n T_{2j}X_j \mid \ldots \mid \sum_{j=1}^n T_{nj}X_j]$, where $T_{ij} = E^iTE_j : \mathbb{R}^m \to \mathbb{R}^m$.

We claim that for each $i, j$ $(1 \leq i, j \leq n)$ $E^i$ and $E_j$ preserve $\prec_{mt}$. Let $x \in \mathbb{R}^m$, $X \in \mathbf{M}_{m,n}$, and $D \in \text{Conv(G)}$. We see

$$E_jDx = Dxe_j^t = DE_jx$$

and

$$E^iDX = DXe_i = DE^iX.$$

Then $E^i$ and $E_j$ preserve $\prec_{mt}$.

Now, suppose that $T$ preserves $\prec_{mt}$. So $TDE_jx = D'TE_jx$, for some $D' \in \text{Conv(G)}$. Then

$$T_{ij}Dx = E^iTE_jDx = E^iTDE_jx = E^iD'TE_jx = D'E^iTE_jx = D'T_{ij}x,$$

and hence $T_{ij}$ preserves $\prec_{mt}$.

The following lemma characterizes linear preservers of $\prec_{mt}$ on $\mathbf{M}_{2,n}$.

**Theorem 4.1.** *Let $T : \mathbf{M}_{2,n} \to \mathbf{M}_{2,n}$ be a linear function. Then $T$ preserves $\prec_{mt}$ if and only if there exist $A_1, \ldots, A_n \in \mathbf{M}_{2,n}$, $s \in \{-1, 1\}$, and $P \in \mathbb{P}_2$ such that $TX = \sum_{j=1}^n [x_{1j}A_j + sx_{2j}PA_j]$ for all $X = [x_{ij}] \in \mathbf{M}_{2,n}$.*

*Proof.* If: It is easy to see.

Only if: Let $T$ preserve $\prec_{mt}$. We consider two steps.

Step 1. At least one of $T_{ij}$ is of the first type. Suppose that $T_{pq}$ is of the first type and $[T_{pq}] = [\mathbf{a} \mid s P\mathbf{a}]$, where $s \in \{1, -1\}$, $P \in \mathbb{P}_2$, $\mathbf{a} \in \mathbb{R}^2$. We claim that for each $i, j$ $(1 \leq i, j \leq n)$ there exist some $a_{ij} \in \mathbb{R}^2$ such that

$$[T_{ij}] = [a_{ij} \mid sPa_{ij}]. \tag{4.1}$$

For each $i$ $(1 \leq i \leq n)$, since $(T_{pq} + T_{pi})x = (Tx(e_q^t + e_i^t))e_p$, we deduce that $T_{pq} + T_{pi}$ preserves $\prec_{mt}$. Similarly, $T_{pq} + T_{iq}$ preserves $\prec_{mt}$. So Lemma 3.2 ensures that $T_{pi}$ and $T_{iq}$ satisfy in 4.1. Moreover, we prove that if $k \neq p$ and $l \neq q$, then $T_{kl}$ satisfies in 4.1, too.

If one of $T_{pl}$ or $T_{kq}$ is of the first type, then $T_{kl}$ satisfies in 4.1, because of the uniqueness of $s$ and $P$.
If both $T_{pl}$ and $T_{kq}$ are not of the first type; As $T_{pq}$ is of the first type, then at least one of the mappings $T_{pq}+T_{kq}+T_{pl}$ or $-T_{pq}-T_{kq}+T_{pl}$ is of the first type. We have $[T_{pq}+T_{kq}+T_{pl}] = [a_{pq}+a_{kq}+a_{pl} \mid sP(a_{pq}+a_{kq}+a_{pl})]$ and $[-(T_{pq}+T_{kq})+T_{pl}] = [-(a_{pq}+a_{kq})+a_{pl} \mid -sP(a_{pq}+a_{kq}+a_{pl})]$. As $(T_{pq}+T_{kq}+T_{pl}+T_{kl})x = (T(xe_q^t+xe_l^t))(e_p+e_k)$ and $(-(T_{pq}+T_{kq})+T_{pl}+T_{kl})x = (T(-xe_q^t+xe_l^t))(e_p+e_k)$ for all $x \in \mathbb{R}^m$. So both mappings $(T_{pq}+T_{kq}+T_{pl})+T_{kl}$ and $(-T_{pq}-T_{kq}+T_{pl})+T_{kl}$ preserve $\prec_{mt}$. It implies that $T_{kl}$ satisfies in 4.1.
Step 2. None of $T_{ij}$ are not of the first type. So if $T_{ij} \neq 0$, then $T_{ij}$ is of the second type or the third type. Hence $T_{ij}(e_2) = T_{ij}(e_1)$ or $T_{ij}(e_2) = -T_{ij}(e_1)$. If both mapping $T_{pq}$ and $T_{kl}$ are of a type, then it is easy to show that $T_{pq}(e_1) = T_{pq}(e_2)$ if and only if $T_{kl}(e_1) = T_{kl}(e_2)$. Now, we choose $s \in \{-1, 1\}$ such that if $T_{ij}$ is of the third type, then $T_{ij}(e_2) = sT_{ij}(e_1)$. After selecting $s$, we choose the permutation $P \in \mathbb{P}^2$ such that if $T_{ij}$ is of the second type, then $T_{ij}(e_2) = sPT_{ij}(e_1)$. Put $a_{ij} = T_{ij}(e_1)$, for each $i, j$ $(1 \leq i, j \leq n)$. So $[T_{ij}] = [a_{ij} \mid sPa_{ij}]$, for each $i, j$ $(1 \leq i, j \leq n)$.
Now, we have $TX = T[X_1 \mid \dots \mid X_n] = [\sum_{j=1}^n T_{1j}X_j \mid \dots \mid \sum_{j=1}^n T_{nj}X_j] = [\sum_{j=1}^n [a_{1j} \mid sPa_{1j}]X_j \mid \dots \mid \sum_{j=1}^n [a_{nj} \mid sPa_{nj}]X_j] = [\sum_{j=1}^n (x_{1j}a_{1j} + sx_{2j}Pa_{1j}) \mid \dots \mid \sum_{j=1}^n (x_{1j}a_{nj} + sx_{2j}Pa_{nj})] = \sum_{j=1}^n (x_{1j}A_j + sx_{2j}PA_j)$, where $A_j = [a_{1j} \mid \dots \mid a_{nj}] \in \mathbf{M}_{2,n}$, for each $j$ $(1 \leq j \leq n)$. □

In the following theorem, we characterize linear preservers of Miranda-Thompson majorization $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ whenever $m \geq 3$. Note that the case $m = 1$ for every linear function holds.

**Theorem 4.2.** *Let $T : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ be a linear function whenever $m \geq 3$. Then $T$ preserves $\prec_{mt}$ if and only if there exist $A \in \mathbf{M}_n$, $P \in \mathbb{P}_m$, and $C \in \mathbb{C}_m$ such that $TX = PCXA$ for all $X \in \mathbf{M}_{m,n}$.*

*Proof.* Suppose that $TX = PCXA$, for some $A \in \mathbf{M}_n$, $P \in \mathbb{P}_m$, and $C \in \mathbb{C}_m$. It is a simple matter to prove that $T$ preserves $\prec_{mt}$.

Assume that $T$ preserves $\prec_{mt}$ and $m \geq 3$. For $T = 0$, it is clear. Let $T \neq 0$. Since $T$ preserves $\prec_{mt}$, we see that $T_{ij}$ preserves $\prec_{mt}$ for all $i, j$ $(1 \leq i, j \leq n)$. As $T \neq 0$, there exist some $r, s$ $(1 \leq r, s \leq n)$ such that $T_{rs} \neq 0$. Lemma 3.1 ensures that there exist some $a_{rs} \in \mathbb{R}$, $P \in \mathbb{P}_m$, and $C \in \mathbb{C}_m$ such that $T_{rs}x = a_{rs}PCx$, for all $x \in \mathbb{R}^m$. We claim that for each $k, l$ $(1 \leq k, l \leq n)$ there is some $a_{kl} \in \mathbb{R}$ such that $T_{kl}x = a_{kl}PCx$, for all $x \in \mathbb{R}^m$.
We divide the proof into three stages.
Step 1. $k = r$ and $l \neq s$. For each $x \in \mathbb{R}^m$, let $X = [X_1 \mid \dots \mid X_n]$, as follows. For each $j$ $(1 \leq j \leq n)$ $X_j = \begin{cases} x & \text{if } j = s, l \\ 0 & \text{if o.w.} \end{cases}$. We have $T_{rs}x + T_{rl}x = (T[X_1 \mid \dots \mid X_n])e_r$ for all $x \in \mathbb{R}^m$. So $T_{rs} + T_{rl}$ preserves $\prec_{mt}$, and hence Lemma 3.2 ensures that there is some $a_{rl} \in \mathbb{R}$ such that $T_{rl}x = a_{rl}PCx$, for all $x \in \mathbb{R}^m$.
Step 2. $k \neq r$ and $l = s$. Let $x \in \mathbb{R}^m$, and $X_j = \begin{cases} x & \text{if } j = s \\ 0 & \text{if o.w.} \end{cases}$, for each $j$ $(1 \leq j \leq n)$. Consider $X = [X_1 \mid \dots \mid X_n]$. We observe that $T_{rs}x + T_{ks}x = (T[X_1 \mid \dots \mid X_n])(e_r + e_k)$ for all $x \in \mathbb{R}^m$. For each $x, y \in \mathbb{R}^m$, if $x \prec_{mt} y$, then $[X_1 \mid \dots \mid X_n] \prec_{mt} [Y_1 \mid \dots \mid Y_n]$. As $T$ preserves $\prec_{mt}$, $T[X_1 \mid \dots \mid X_n] \prec_{mt} T[Y_1 \mid \dots \mid Y_n]$, and hence $T_{rs} + T_{ks}$ preserves $\prec_{mt}$. It implies that there is some $a_{ks} \in \mathbb{R}$ such that $T_{ks}x = a_{ks}PCx$, for all $x \in \mathbb{R}^m$, because of Lemma 3.2.

Step 3. $k \neq r$ and $l \neq s$. From the previous steps, there are some $a_{rl}, a_{ks} \in \mathbb{R}$ such that $T_{rl}x = a_{rl}PCx$ and $T_{ks}x = a_{ks}PCx$ for all $x \in \mathbb{R}^m$. If $T_{rl} \neq 0$ (or $T_{ks} \neq 0$), then step 2 (step 1) ensures that there is some $a_{kl} \in \mathbb{R}$ such that $T_{kl}x = a_{kl}PCx$ for all $x \in \mathbb{R}^m$, by choosing $l$ instead of $s$ ($k$ instead of $r$). If $T_{rl} = T_{ks} = 0$; For each $x \in \mathbb{R}^m$ define $X = [X_1 \mid \ldots \mid X_n] \in \mathbf{M}_{m,n}$, $X_j = \begin{cases} x & \text{if j = s,l} \\ 0 & \text{if o.w.} \end{cases}$, for each $j$ ($1 \leq j \leq n$). We see that $T_{rs}x + T_{kl}x = (T[X_1 \mid \ldots \mid X_n])(e_r + e_k)$ for all $x \in \mathbb{R}^m$, and then $T_{rs} + T_{kl}$ preserves $\prec_{mt}$. Since $T_{rs} \neq 0$, Lemma 3.2 ensures that there is some $a_{kl} \in \mathbb{R}$ such that $T_{kl}x = a_{kl}PCx$ for all $x \in \mathbb{R}^m$.
So $TX = T[X_1 \mid \ldots \mid X_n] = [\sum_{j=1}^n T_{1j}X_j \mid \ldots \mid \sum_{j=1}^n T_{nj}X_j] = [\sum_{j=1}^n a_{1j}PCX_j \mid \ldots \mid \sum_{j=1}^n a_{nj}PCX_j] = PC[\sum_{j=1}^n a_{1j}X_j \mid \ldots \mid \sum_{j=1}^n a_{nj}X_j] = PCXA$, where $A = [a_{ij}] \in \mathbf{M}_n$. □

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