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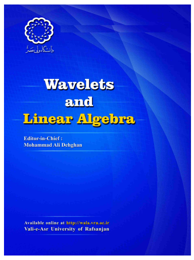

Wilson wavelets for solving nonlinear stochastic integral equations

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Abstract

A new computational method based on Wilson wavelets is proposed for solving a class of nonlinear stochastic Itô-Volterra integral equations. To do this a new stochastic operational matrix of Itô integration for Wilson wavelets is obtained. Block pulse functions (BPFs) and collocation method are used to generate a process to forming this matrix. Using these basis functions and their operational matrices of integration and stochastic integration, the problem under study is transformed to a system of nonlinear algebraic equations which can be simply solved to obtain an approximate solution for the main problem. Moreover, a new technique for computing nonlinear terms in such problems is presented. Furthermore, convergence of Wilson wavelets expansion is investigated. Several examples are presented to show the efficiency and accuracy of the proposed method.

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1. Introduction

Stochastic integral equations are used in modeling several phenomena in physics, science and engineering e.g. [30, 27, 23, 33]. In recent years, demanding on investigation of the behavior of complicated dynamical systems in physical, medical, engineering and finance is increased, for some cases see [24, 14, 16, 32, 28, 29, 8, 35, 15]. In these systems usually there is a noise source that is managed by probability laws. Modeling such phenomena include the use of various stochastic differential equations and stochastic integral equations or stochastic integro-differential equations [25, 26, 10, 34, 21, 1, 4]. Solving such problems is often difficult. So, it is necessary to obtain their approximate solutions by using numerical methods e.g. [27, 30, 24, 14, 28, 33, 25, 29]. Recently, orthogonal functions and wavelets basis functions have been used to obtain approximate solutions for some types of functional equations e.g. [3, 36, 9, 2, 17]. Approximation by orthogonal basis functions has been widely used in science and engineering. The main idea of using an orthogonal set of basis functions is that the solution of the problem under consideration is transferred to solution of a system of linear or nonlinear algebraic equations [18]. This can be done by truncated series of orthogonal basis functions for the solution of the problem and using the operational matrices.

Wavelets theory is a relatively new subject in mathematical research and has been applied in a wide range of engineering disciplines. Wavelets are localized functions, which are the basis for energy-bounded functions and in particular for $L^2(\mathbb{R})$, so that localized pulse problems can be simply approached and analyzed. However, wavelets are useful basis functions which offer considerable advantages over alternative basis ones and allow us to attack problems which are not accessible with conventional numerical methods. Wilson suggested a system of basis functions which these functions are localized around the positive and negative frequency [5]. We introduce a kind of wavelet called Wilson wavelets and using them as a specific type of orthonormal wavelets we will obtain the numerical solution of the following nonlinear stochastic Itô-Volterra integral equation

$$X(t) = f(t) + \int_0^t k_1(s,t)\mu(X(s))ds + \int_0^t k_2(s,t)\sigma(X(s))dB(s), \qquad t \in [0,1), \tag{1.1}$$

where $f(t), k_1(s,t)$ and $k_2(s,t)$ for $s,t \in [0,1)$ are known stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and $X(t)$ is an unknown stochastic process which should be computed. The functions μ and σ are analytic functions on $\mathbb{R}$. The second integral in Eq. (1.1) is the Itô integral. In order to obtain an approximate solution for Eq. (1.1), based on Wilson wavelets we derive a new operational matrix of Itô stochastic integration and reduce our problem to solving a system of nonlinear algebraic equations. Moreover, a new technique for computation of the nonlinear terms in such equations is presented. Furthermore, convergence analysis of Wilson wavelets is investigated. Few examples are presented to show the efficiency of the method.

The reminder of this paper is organized as follows. In section 3, the BPFs are given. In section 4, Wilson wavelets and their properties are described. In section 5, the proposed method

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is described for solving nonlinear stochastic Itô -Volterra integral equations (1.1). In section 7, the proposed method is applied for some numerical examples. Finally a conclusion is drawn in section 8.

2. Stochastic calculus

In this section we state some definitions in stochastic calculus. For more details see [34]. Let I be an index set, a collection of random variables $\{X(t), t \in I\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is called a stochastic process. A stochastic process $B(p)$, $p \in [0, T]$ satisfying the following conditions

(i) $B(0) = 0$ (with the probability 1),

(ii) For $0 \leq p < q \leq T$ the random variable given by the increment $B(p) - B(q)$ is normally distributed with mean zero and variance $p - q$,

(iii) For $0 \leq p < q < u < v \leq T$ the increments $B(p) - B(q)$ and $B(v) - B(u)$ are independent,

(iv) For $p \geq 0$, $B(p)$ is a continous function of p,

is called Brownian motion process.
A process $g(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{M}_t$-adapted if for each $t \geq 0$, the function $\omega \to g(t, \omega)$ is $\mathcal{M}_t$-measurable which $\{\mathcal{M}_t\}_{t\geq 0}$ is an increasing family of σ-algebras of subsets of Ω.
Now let $\mathcal{W} = \mathcal{W}(S, T)$ be a class of functions $g(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}$ such that

(a) The function $(t, \omega) \to g(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes Borel σ- algebra on $[0, \infty)$ and $\mathcal{F}$ is a σ-algebra on Ω.

(b) The random variable $g(t, .)$ is $\mathcal{F}_t$- measurable for $t \in [S, T]$ i.e the process g is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ is a σ-algebra generated by the random variable $B(s)$, $s \leq t$.

(c) $E\left[\int_S^T g^2(t, \omega)dt\right] < \infty$ where $E[X]$ denotes expected value of X.

Definition 2.1. ([34])(The Itô integral) Let $g \in \mathcal{W}(S, T)$, then the Itô integral of g is defined by

$$\int_S^T g(t, \omega)dB_t(\omega) = \lim_{n\to\infty} \int_S^T \lambda_n(t, \omega)dB_t(\omega),$$

where λ_n is a sequence of elementary functions such that

$$E\left[\int_S^T (g(t, \omega) - \lambda_n(t, \omega))^2 \, dt\right] \to 0, \quad as \ n \to \infty.$$

Definition 2.2. ([34])(The Itô isometry) Let $f \in \mathcal{W}(S, T)$, then

$$E\left[\left(\int_S^T g(t, \omega)dB_t(\omega)\right)^2\right] = E\left[\int_S^T g^2(t, \omega)dt\right]. \tag{2.1}$$

3. Block pulse functions (BPFs)

Here, we briefly introduce the BPFs and present some of their properties.
An $\hat{m}$-set of these basis functions is usually defined on the interval [0, 1] by [19]

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{\hat{m}} \leq t \leq \frac{i}{\hat{m}}, \\ 0, & o.w, \end{cases} \tag{3.1}$$

where $i = 1, 2, \ldots, \hat{m}$, with a positive integer value for $\hat{m}$.
The set of the BPFs is disjoint and orthogonal. Using the orthogonality property of the BPFs, we can approximate any function $f(t) \in L^2[0, 1]$ in terms of the BPFs as

$$f(t) \simeq \sum_{i=1}^{\hat{m}} f_i b_i(t) \triangleq F^T \Phi(t), \tag{3.2}$$

where

$$F = [f_1, f_2, \ldots, f_{\hat{m}}]^T, \quad \Phi(t) = [b_1(t), b_2(t), \ldots, b_{\hat{m}}(t)]^T, \tag{3.3}$$

and in which

$$f_i = \hat{m} \int_{\frac{i-1}{\hat{m}}}^{\frac{i}{\hat{m}}} f(t) dt, \quad i = 1, 2, \ldots, \hat{m}. \tag{3.4}$$

Lemma 3.1. *([19]). Let $\Phi(t)$ be the BPFs vector defined in Eq. (3.3), then we have*

$$\Phi(t)\Phi(t)^T = diag\,(b_1(t), b_2(t), \ldots, b_{\hat{m}}(t)) = diag\,(\Phi(t))\,, \tag{3.5}$$

where $diag(\Phi(t))$ is an $\hat{m} \times \hat{m}$ diagonal matrix.

Remark 3.2. [20] Let μ be an analytic function on $\mathbb{R}$ and $F^T\Phi(t)$ be the expansion of $f(t)$ in terms of the BPFs. Then we have

$$\mu(f(t)) \simeq \mu\left(F^T\right)\Phi(t), \tag{3.6}$$

where $\mu\left(F^T\right) = [\mu(f_1), \mu(f_2), \ldots, \mu(f_{\hat{m}})]^T$.

Theorem 3.3. *([24]) The integral of the vector $\Phi(t)$ defined in Eq. (3.3), can be expressed as*

$$\int_0^t \Phi(s)ds \simeq \hat{P}\Phi(t), \tag{3.7}$$

where $\hat{P}$ is called the operational matrix of integration for the BPFs and its elements are given by

$$\hat{P} = \frac{1}{2\hat{m}} \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{\hat{m}\times\hat{m}}. \tag{3.8}$$

Theorem 3.4. *([24]). The Itô stochastic integral of the vector $\Phi(t)$ defined in Eq. (3.3), can be expressed as*

$$\int_0^t \Phi(s)dB(s) \simeq \hat{P}_s\Phi(t), \tag{3.9}$$

where $\hat{P}_s$ is called the stochastic operational matrix for the BPFs and is given by

$$\hat{P}_s = \begin{pmatrix} B(\frac{1}{2\hat{m}}) & B(\frac{1}{\hat{m}}) & \dots & B(\frac{1}{\hat{m}}) \\ 0 & B(\frac{3}{2\hat{m}})-B(\frac{1}{\hat{m}}) & \dots & B(\frac{2}{\hat{m}})-B(\frac{1}{\hat{m}}) \\ 0 & 0 & \dots & B(\frac{3}{\hat{m}})-B(\frac{2}{\hat{m}}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & B(\frac{2\hat{m}-1}{2\hat{m}})-B(\frac{\hat{m}-1}{\hat{m}}) \end{pmatrix}_{\hat{m}\times\hat{m}}. \tag{3.10}$$

4. Wilson wavelets and their properties

A family of functions

$$\psi_{ab}(t) = |a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, \; a \neq 0, \tag{4.1}$$

constructed from the dilation and the translation of a single function ψ is called wavelets. The function ψ is called the mother wavelet. A continuous wavelet is the familly of Eq. (4.1) where the dilation parameter a and the translation parameter b vary continuously. If we choose the discrete values $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, where $a_0 > 1$, $b_0 > 0$ are fixed, then we have the following discrete wavelets

$$\psi_{kn}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0), \quad k, n \in \mathbb{Z}. \tag{4.2}$$

Wilson expressed a system of basis functions which these functions are localized and given by

$$\psi_{nm}(t) = \begin{cases} \epsilon_n \cos(2n\pi t)\omega(t-\frac{m}{2}), & m \text{ is even}, \\ \sqrt{2}\sin(2(n+1)\pi t)\omega(t-\frac{m+1}{2}), & m \text{ is odd}, \end{cases} \tag{4.3}$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ \sqrt{2}, & n \in \mathbb{N}, \end{cases}$$

with a smooth well-localized window function ω [11, 22, 6, 7, 13].
Using this system, Daubechies constructed an orthonormal system and called it as "Wilson bases"[12].
We will consider $\omega = \chi_{[0,1)}$ in Eq. (4.3), i.e.

$$\psi_{nm}(t) = \begin{cases} \epsilon_n \cos(2n\pi t)\chi_{[0,1)}(t-\frac{m}{2}), & m \text{ is even}, \\ \sqrt{2}\sin(2(n+1)\pi t)\chi_{[0,1)}(t-\frac{m+1}{2}), & m \text{ is odd}. \end{cases} \tag{4.4}$$

The set $\{\psi_{nm}(t) | m \in \mathbb{Z},\ n \in \mathbb{N} \cup \{0\}\}$ is a tight frame for $L^2(\mathbb{R})$ with bound 1 [22]. We recall that a sequence $\{f_n\}$ in a Hilbert space $\mathcal{H}$ (with inner product $\langle ., . \rangle$) is said to be a frame for $\mathcal{H}$, if there exist positive constants A, B such that

$$A\|f\|^2 \le \sum_n |\langle f, f_n \rangle|^2 \le B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{4.5}$$

A frame $\{f_n\}$ is said to be tight if $A = B$.

Now, we can show that the set $\{\psi_{nm}(t) | m \in \{-1, 0\}, n \in \mathbb{N} \cup \{0\}\}$ in Eq. (4.4) is an orthonormal basis for $L^2[0, 1)$, which we called Wilson wavelets.

Any square integrable function $f(t)$ defined over $[0, 1)$ can be expanded in terms of Wilson wavelets as

$$f(t) = \sum_{n=0}^{\infty} c_{n,0}\psi_{n,0}(t) + \sum_{n=0}^{\infty} c_{n,-1}\psi_{n,-1}(t), \tag{4.6}$$

where $c_{nm} = \langle f(t), \psi_{nm}(t) \rangle$, $m \in \{-1, 0\}$ and $\langle ., . \rangle$ denotes the inner product on $L^2[0, 1)$. If the infinite series in Eq. (4.6) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=-1}^{0} c_{nm}\psi_{nm}(t) \triangleq C^T\Psi(t), \tag{4.7}$$

where C and $\Psi(t)$ are $\hat{m} = 2^{k+1}$ column vectors. For simplicity, Eq. (4.7) can be written as

$$f(t) \simeq \sum_{i=1}^{\hat{m}} c_i\psi_i(t) \triangleq C^T\Psi(t), \tag{4.8}$$

where $c_i = c_{nm}$ and $\psi_i(t) = \psi_{nm}(x)$, and the index i is determined by the relation $i = 2n + m + 2$. Thus we have

$$C \triangleq [c_1, c_2, \ldots, c_{\hat{m}}]^T, \quad \Psi(t) \triangleq [\psi_1(t), \psi_2(t), \ldots, \psi_{\hat{m}}(t)]^T. \tag{4.9}$$

Similarly, an arbitrary function of two variables $k(s, t)$ defined over $L^2([0, 1) \times [0, 1))$, may be expanded by Wilson wavelets as

$$k(s, t) \simeq \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} K_{ij}\psi_i(x)\psi_j(t) \triangleq \Psi^T(s)K\Psi(t),$$

where $K = [k_{ij}]$ and $k_{ij} = \Big\langle \psi_i(s), \Big\langle k(s, t), \psi_j(t) \Big\rangle \Big\rangle$.

4.1. Convergence of Wilson wavelets

Here, we investigate the convergence of the Wilson wavelets.

Theorem 4.1. *If Wilson wavelets expansion of a continuous function $f(t)$ is uniformly convergence, then it converges to the function $f(t)$.*

Proof. For a complete proof see [Theorem 2.1, [33]] □

Theorem 4.2. *Let $f(t)$ be a function defined over $[0,1)$ with bounded second derivative, i.e. $|f''(t)| \leq M$, then it can be expanded in an infinite series of Wilson wavelets and the series converges uniformly to $f(t)$, i.e.*

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=-1}^{0} c_{nm}\psi_{nm}(t),$$

moreover, if $C^T\Psi(t)$ is Wilson wavelets expansion of $f(t)$, we have

$$\|C^T\Psi(t) - f(t)\|_\infty \leq \frac{M}{2\pi^2} \sum_{n=2^k+1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2}\right).$$

Proof. For a complete proof see [Theorem 2.2, [33]]

□

4.2. Connection between the BPFs and Wilson wavelets

Here, we investigate the relation between the BPFs and Wilson wavelets. With a mild perturbation in proof of Theorem 3.1 in [31] one can easily prove the following theorem.

Theorem 4.3. *Let $\Phi(t)$ and $\Psi(t)$ be the BPFs and Wilson wavelets vectors defined in Eqs. (3.3) and (4.9), respectively. Then the vector $\Psi(t)$ can be expanded by the BPFs vector $\Phi(t)$ as*

$$\Psi(t) \simeq Q\Phi(t), \tag{4.10}$$

where the $\hat{m} \times \hat{m}$ matrix Q is called Wilson wavelets matrix and is given by

$$q_{ij} = \psi_i\left(\frac{2j-1}{2\hat{m}}\right), \qquad i, j = 1, 2, \ldots, \hat{m}. \tag{4.11}$$

Lemma 4.4. *Let $\Psi(t)$ be Wilson wavelets vector defined in Eq. (4.9), and C be an arbitrary $\hat{m}$-column vector. Then we have*

$$\Psi(t)\Psi(t)^T C \simeq \widetilde{C}\Psi(t), \tag{4.12}$$

where $\widetilde{C}$ is an $\hat{m} \times \hat{m}$ as $\widetilde{C} = Q\bar{C}Q^{-1}$, and $\bar{C} = diag\left(Q^T C\right)$. Moreover, for any arbitrary $\hat{m} \times \hat{m}$ matrix B, we have

$$\Psi(t)^T B\Psi(t) \simeq \hat{B}^T\Psi(t), \tag{4.13}$$

where $\hat{B}^T = V^T Q^{-1}$ and $V = diag(Q^T BQ)$ is an $\hat{m}$-column vector.

Proof. By Lemma 3.1 and Theorem 4.3, the proof will be straightforward.

□

Theorem 4.5. *Let μ be an analytic function over $\mathbb{R}$ and $C^T\Psi(t)$ be the expansion of $f(t)$ by Wilson wavelets. Then we have*

$$\mu(f(t)) \simeq \mu\left(\widetilde{C}^T\right) Q^{-1}\Psi(t), \tag{4.14}$$

where $\widetilde{C}^T = C^T Q$, Q is Wilson wavelets matrix defined in Theorem 4.3 and the vector $\mu\left(\widetilde{C}^T\right)$ is defined in Remark 3.2.

Proof. Using Theorem 4.3 and Remark 3.2, we have

$$\mu(f(t)) \simeq \mu\left(C^T\Psi(t)\right) \simeq \mu\left(C^T Q\Phi(t)\right) = \mu\left(\widetilde{C}^T\Phi(t)\right) \simeq \mu\left(\widetilde{C}^T\right)\Phi(t) \simeq \mu\left(\widetilde{C}^T\right) Q^{-1}\Psi(t),$$

which completes the proof. □

4.3. Operational matrices

In this part, we derive a new operational matrix of Itô stochastic integration for Wilson wavelets. Moreover, the same process is used to obtain a new operational matrix of integration for these basis functions.

Theorem 4.6. *(Stochastic operational matrix). Suppose $\Psi(t)$ is Wilson wavelets vector defined in Eq. (4.9), the Itô stochastic integration of this vector can be expressed as*

$$\int_0^t \Psi(s)dB(s) \simeq \left(Q\hat{P}_s Q^{-1}\right)\Psi(t) \triangleq P_s\Psi(t), \tag{4.15}$$

where the $\hat{m}\times\hat{m}$ matrix P_s is called stochastic operational matrix for Wilson wavelets, Q is Wilson wavelets matrix introduced in Theorem 4.3 and $\hat{P}_s$ is the stochastic operational matrix of Itô stochastic integration for the BPFs, which is defined in Theorem 3.4.

Proof. Let $\Psi(t)$ be Wilson wavelets vector, by considering Theorems 3.4 and 4.3, we have

$$\begin{aligned}\int_0^t \Psi(s)dB(s) &\simeq \int_0^t Q\Phi(s)dB(s) = Q\int_0^t \Phi(s)dB(s) \simeq Q\hat{P}_s\Phi(t)\\ &\simeq \left(Q\hat{P}_s Q^{-1}\right)\Psi(t) \triangleq P_s\Psi(t),\end{aligned}$$

which completes the proof. □

Theorem 4.7. *(Operational matrix of integration). Suppose $\Psi(t)$ is Wilson wavelets vector defined in Eq. (4.9), the integration of this vector can be expressed as*

$$\int_0^t \Psi(s)ds \simeq \left(Q\hat{P}Q^{-1}\right)\Psi(t) \triangleq P\Psi(t), \tag{4.16}$$

where the $\hat{m}\times\hat{m}$ matrix P is called the operational matrix of integration for Wilson wavelets, Q is Wilson wavelets matrix introduced in Theorem 4.3 and $\hat{P}$ is the operational matrix of integration for the BPFs defined in Theorem 3.3.

Proof. Let $\Psi(t)$ be Wilson wavelets vector, by considering Theorems 3.3 and 4.3, we have

$$\int_0^t \Psi(s)ds \simeq \int_0^t Q\Phi(s)ds = Q\int_0^t \Phi(s)ds \simeq Q\hat{P}\Phi(t) \simeq \left(Q\hat{P}Q^{-1}\right)\Psi(t) \triangleq P\Psi(t),$$

which completes the proof. □

5. Description of the proposed method

To solve the nonlinear stochastic Itô-Volterra integral equation introduced in Eq. (1.1), we expand $X(t)$, $g(t)$, $k_1(s,t)$ and $k_2(s,t)$ in terms of Wilson wavelets as follows

$$\begin{aligned}&X(t) \simeq G^T\Psi(t) = \Psi^T(t)G, f(t) \simeq F^T\Psi(t) = \Psi^T(t)F,\\ &k_1(s,t) \simeq \Psi(s)^T K_1\Psi(t) = \Psi(t)^T K_1^T\Psi(s),\\ &k_2(s,t) \simeq \Psi(s)^T K_2\Psi(t) = \Psi(t)^T K_2^T\Psi(s),\end{aligned} \tag{5.1}$$

where G is an unknown vector, which should be computed, F is known Wilson wavelets coefficients vector for $f(t)$, and K_1 and K_2 are the known Wilson wavelets coefficient matrices for $k_1(s,t)$ and $k_2(s,t)$, respectively. By substituting Eqs.(5.1) into Eq. (1.1), we have

$$G^T\Psi(t) \simeq F^T\Psi(t) + \Psi(t)^T K_1 \left(\int_0^t \Psi(s)\mu\left(G^T\Psi(s)\right)ds\right) + \Psi(t)^T K_2 \left(\int_0^t \Psi(s)\sigma\left(G^T\Psi(s)\right)dB(s)\right). \tag{5.2}$$

Now, by considering Theorem 4.5, we can rewrite Eq. (5.2) in the following form

$$G^T\Psi(t) \simeq F^T\Psi(t) + \Psi(t)^T K_1 \left(\int_0^t \Psi(s)\Psi(s)^T Y_1 ds\right) + \Psi(t)^T K_2 \left(\int_0^t \Psi(s)\Psi(s)^T Y_2 dB(s)\right), \tag{5.3}$$

where $Y_1^T = \mu\left(\widetilde{G^T}\right)Q^{-1}$, $Y_2^T = \sigma\left(\widetilde{G^T}\right)Q^{-1}$ and $\widetilde{G^T} = G^T Q$.
Also by Lemma 4.4 and Eq. (5.3), we get

$$G^T\Psi(t) \simeq F^T\Psi(t) + \Psi(t)^T K_1 \left(\int_0^t \widetilde{Y_1}\Psi(s)ds\right) + \Psi(t)^T K_2 \left(\int_0^t \widetilde{Y_2}\Psi(s)dB(s)\right), \tag{5.4}$$

where $\widetilde{Y_1}$ and $\widetilde{Y_2}$ are $\hat{m}\times\hat{m}$ matrices described in Lemma 4.4.
Moreover, using Eq. (5.4) and employing the operational matrices of integration and Itô stochastic integration of Wilson wavelets which are mentioned in Theorems 4.7 and 4.6, respectively, we obtain

$$G^T\Psi(t) \simeq F^T\Psi(t) + \Psi(t)^T K_1\widetilde{Y_1}P\Psi(t) + \Psi(t)^T K_2\widetilde{Y_2}P_s\Psi(t). \tag{5.5}$$

By putting $\Lambda_1 = K_1\widetilde{Y_1}P$, $\Lambda_2 = K_2\widetilde{Y_2}P_s$ and using Lemma 4.4, we have

$$G^T\Psi(t) - \hat{\Lambda}_1\Psi(t) - \hat{\Lambda}_2\Psi(t) \simeq F^T\Psi(t), \tag{5.6}$$

where $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ are $\hat{m}$-row vectors, which their entries are nonlinear combinations of elements of G. This equation is hold for $t \in [0,1)$, so by replacing $\simeq$ by $=$, we obtain the following system of nonlinear algebraic equations $G^T - \hat{\Lambda}_1 - \hat{\Lambda}_2 = F^T$. Finally by solving the above system and determining G, we obtain an approximate solution for the stochastic problem (1.1) by substituting G into $X(t) \simeq G^T\Psi(t)$.

6. Convergence analysis

Here, the convergence of the proposed method in section 5 for solving Eq. (1.1) is investigated. The functions μ and σ in Eq. (1.1) are analytic functions and so they are Lipschitz functions. Thus for every X and Y in their domain the following inequalities hold

$$|\mu(X) - \mu(Y)| \le r_1|X - Y| \text{ and } |\sigma(X) - \sigma(Y)| \le r_2|X - Y|, \tag{6.1}$$

which r_1, r_2 are positive constants.

Theorem 6.1. *Let $X(t)$ and $X_N(t)$ be the exact and approximate solutions of Eq. (1.1), respectively. Also assume that*

(i) $\|X\| < \infty$,

(ii) $\|k_i\| \leq \mathcal{K}_i$ *for some* $\mathcal{K}_i \in \mathbb{R}$ *and* $r_1^2\|k_1\|^2 + r_2^2\|k_2\|^2 \neq \frac{1}{3}$,

then, $\|X - X_N\| \to 0$, *where* $\|X\|^2 = E(|X|^2)$.

Proof. Let $e_N(t) = X(t) - X_N(t)$ be the error function and $Res_N(t)$ be the residual error. Then using Eq. (1.1) we have

$$\begin{aligned} e_N(t) = &\int_0^t k_1(s,t)\,(\mu(X(s)) - \mu(X_N(s)))\,ds \\ &+ \int_0^t k_2(s,t)(\sigma(X(s)) - \sigma(X_N(s)))dB(s) + Res_N(t). \end{aligned} \tag{6.2}$$

Now using Eq. (6.1) and Eq. (6.2) we have

$$\begin{aligned} |e_N(t)| \leq &\int_0^t r_1|k_1(s,t)||e_N(s)|ds + \int_0^t r_2|k_2(s,t)||e_N(s)|dB(s) \\ &+ |Res_N(t)|, \qquad t \in [0,1), \end{aligned} \tag{6.3}$$

A simple calculation shows that for all $a, b, c \in \mathcal{R}^+$, $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, so we have

$$\begin{aligned} |e_N(t)|^2 \leq 3\Big(&\left|\int_0^t r_1|k_1(s,t)||e_N(s)|ds\right|^2 + \left|\int_0^t r_2|k_2(s,t)||e_N(s)|dB(s)\right|^2 \\ &+ |Res_N(t)|^2\Big). \end{aligned} \tag{6.4}$$

Now Holder's inequality implies that for $0 \leq s \leq t < 1$,

$$\begin{aligned} \left|\int_0^t r_1|k_1(s,t)||e_N(s)|ds\right|^2 &\leq \left|\int_0^1 r_1|k_1(s,t)||e_N(s)|ds\right|^2 \\ &\leq r_1^2\left(\int_0^1 |k_1(s,t)|^2 ds\right)\left(\int_0^1 |e_N(s)|^2 ds\right). \end{aligned} \tag{6.5}$$

Let $J_1(t) = r_1\int_0^t |k_1(s,t)||e_N(s)|ds$, then taking expectations from Eq. (6.5) implies

$$E\left(\left|\int_0^t r_1|k_1(s,t)||e_N(s)|ds\right|^2\right) \leq r_1^2 E\left(\int_0^1 |k_1(s,t)|^2 ds\right) E\left(\int_0^1 |e_N(s)|^2 ds\right), \tag{6.6}$$

So we have

$$\|J_1\|^2 \leq r_1^2\|k_1\|^2\|e_N\|^2. \tag{6.7}$$

From $0 \le s \le t < 1$, we have

$$\left| \int_0^t r_2 |k_2(s,t)||e_N(s)| dB(s) \right|^2 \le \left| r_2 \int_0^1 |k_2(s,t)||e_N(s)| dB(s) \right|^2. \tag{6.8}$$

Let $J_2(t) = r_2 \int_0^t |k_2(s,t)||e_N(s)| dB(s)$, then taking expectations from Eq. (6.8) and using Definition 2.2 we have

$$E\left(|J_2(t)|^2\right) \le E\left(|r_2 \int_0^1 |k_2(s,t)||e_N(s)| dB(s)|^2\right) \tag{6.9}$$
$$= E\left(r_2^2 \int_0^1 |k_2(s,t)e_N(s)|^2 ds\right).$$

So we have

$$||J_2||^2 \le r_2^2 ||k_2||^2 ||e_N||^2. \tag{6.10}$$

Let $J_3(t) = Res_N(t)$, so $E\left(|J_3(t)|^2\right) = E\left(|Res_N(t)|^2\right)$ so we have

$$||J_3||^2 = ||Res_N||^2. \tag{6.11}$$

Using Eq. (6.4) and taking expectations we have

$$||e_N||^2 \le 3\left(||J_1||^2 + ||J_2||^2 + ||J_3||^2\right). \tag{6.12}$$

Then using Eqs. (6.7), (6.10), (6.11) and (6.12) we have

$$||e_N||^2 \le 3\left(\left(r_1^2 ||k_1||^2 + r_2^2 ||k_2||^2\right) ||e_N||^2 + ||Res_N||^2\right) \tag{6.13}$$

Therefore we obtain

$$||e_N||^2 \le \frac{3||Res_N||^2}{1 - 3\left(r_1^2 ||k_1||^2 + r_2^2 ||k_2||^2\right)}. \tag{6.14}$$

So if Res_N tends to zero then $||e_N|| = ||X - X_N|| \to 0$. □

7. Numerical examples

In this section, we consider some numerical examples to illustrate the efficiency and reliability of the proposed method. For computational purposes, we consider discretized Brownian motion, where $B(t)$ is specified at t discrete values and employed an spline interpolation to construct $B(t)$. We thus set $\Delta t = \frac{1}{N}$ for some positive integer N and let B_i denote $B(t_i)$ with $t_i = i\Delta t$. Condition (i) in Section 2 says that $B_0 = 0$ with the probability 1, and conditions (ii) and (iii) tell us that $B_i = B_{i-1} + dB_i$, $i = 1, 2, \ldots, N$, where each dB_i is an independent random variable of the form $\sqrt{\Delta t}\mathcal{N}(0,1)$. In the following examples $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $B(t)$ is a Brownian motion process.

Example 7.1. ([26]) Consider the following problem

$$X(t) = X_0 + \int_0^t (aX(s) + bX(s)^2)ds + \int_0^t cX(s)dB(s).$$

The exact solution of this problem is $X(t) = \frac{U(t)}{\frac{1}{X_0} - b\int_0^t U(s)ds}$, where $U(t) = \exp((a - \frac{c^2}{2})t + cB(t))$, and a, b and c are constants. We solved it by the proposed method for $a = 1/100$, $b = 1/32$ and $c = 1/8$ and $X_0 = 1/20$. The graphs of the exact and approximate solutions (left side) and the absolute error (right side) for $\hat{m} = 64$ with $N = 60$ are shown in Fig. 1. The absolute errors of the approximate solutions at some different points $t \in [0, 1]$ for $\hat{m} = 8$, 16, 32 and 64 are shown in Table 1.

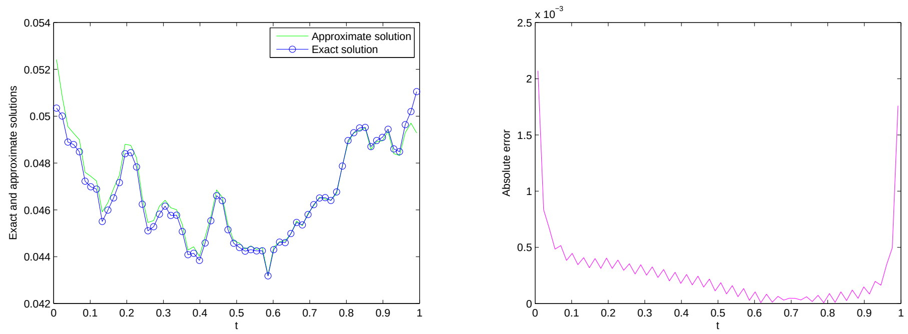

Figure 1: The graphs of the exact and approximate solutions (left side) and absolute error (right side) for Example 7.1.

t	$\hat{m} = 8$	$\hat{m} = 16$	$\hat{m} = 32$	$\hat{m} = 64$
0.1	2.4793E-3	1.8087E-4	7.9405E-4	3.6535E-4
0.3	1.9006E-4	8.5607E-4	1.3475E-4	8.4373E-5
0.5	8.7831E-4	1.0392E-3	6.8633E-4	5.7495E-5
0.7	2.5024E-4	3.5516E-5	7.3585E-6	4.6601E-6
0.9	6.5414E-4	7.2921E-5	2.5172E-4	1.1086E-4

Table 1: The absolute errors of the approximate solutions for Example 7.1.

From Figure 1 and Table 1, one can see that the proposed method provides a good approximate solution for this problem. Figure 1 shows that the error grows exponentially at boundary points.

Example 7.2. ([26]) Consider the following problem

$$X(t) = X_0 - a^2 \int_0^t \sin(X(s)) \cos^3(X(s))ds + a \int_0^t \cos^2(X(s))dB(s).$$

The exact solution of this problem is $X(t) = \arctan(aB(t) + \tan(X_0))$. We have solved this problem for $a = 1/4$ and $X_0 = 1/20$ by the proposed method. The graphs of the exact and approximate solutions (left side) and absolute error (right side) for $\hat{m} = 64$ with $N = 90$ are shown in Fig. 2. The absolute errors of the approximate solutions at some different points $t \in [0, 1]$ for $\hat{m} = 8, 16, 32$ and 64 are shown in Table 2.

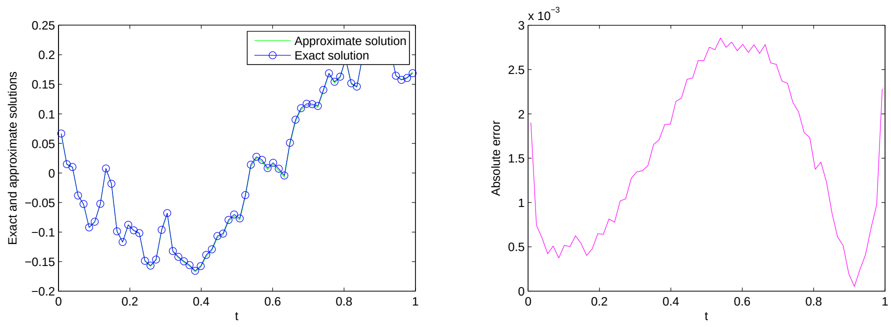

Figure 2: The graphs of the exact and approximate solutions (left side) and absolute error (right side) for Example 7.2.

t	$\hat{m} = 8$	$\hat{m} = 16$	$\hat{m} = 32$	$\hat{m} = 64$
0.1	1.2066E-2	9.5413E-3	4.4314E-3	3.2810E-4
0.3	3.2856E-2	7.5314E-2	1.5415E-2	7.7551E-3
0.5	1.7849E-2	3.0177E-2	3.5659E-3	4.7182E-3
0.7	1.8083E-2	3.5750E-2	4.5422E-3	9.3706E-3
0.9	6.6334E-3	1.1927E-2	3.7532E-3	2.2990E-3

Table 2: The absolute errors of the approximate solutions for Example 7.2.

Figure 2 and Table 2 show the accuracy of Wilson wavelets method for this problem.

Example 7.3. ([26]) Consider the following problem

$$X(t) = X_0 + a^2 \int_0^t \cos(X(s)) \sin^3(X(s))ds - a \int_0^t \sin^2(X(s))dB(s).$$

The exact solution of this problem is $X(t) = arccot(aB(t) + \cot(X_0))$. This problem is now solved by the proposed method for $a = 1/8$ and $X_0 = \pi/32$. The graphs of the exact and approximate solutions (left side) and absolute error for $\hat{m} = 64$ with $N = 35$ are shown in Fig. 3. The absolute errors of the approximate solutions at some different points $t \in [0, 1]$ for $\hat{m} = 8, 16, 32$ and 64 are shown in Table 3.

As the numerical results show, the proposed method is very efficient and accurate for solving this problem.

Example 7.4. ([26]) Consider the following problem

$$X(t) = X_0 - \frac{a^2}{2} \int_0^t \tanh(X(s))sech^2(X(s))ds + a \int_0^t sech(X(s))dB(s).$$

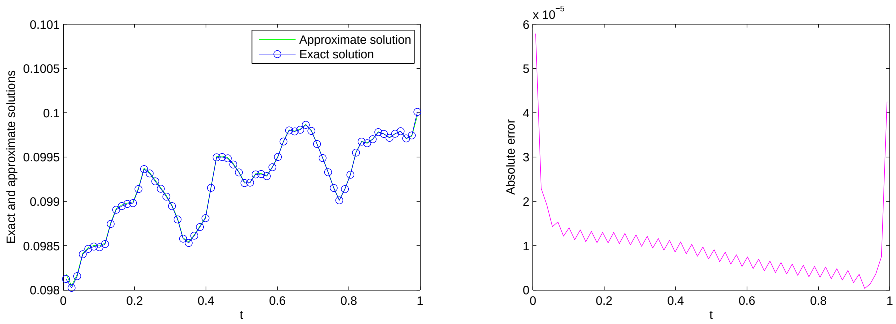

Figure 3: The graphs of the exact and approximate solutions (left side) and absolute error (right side) for Example 7.3.

t	$\hat{m} = 8$	$\hat{m} = 16$	$\hat{m} = 32$	$\hat{m} = 64$
0.1	3.8391E-5	1.1728E-4	7.6331E-5	3.0436E-5
0.3	8.0451E-5	1.0939E-4	9.9513E-6	4.8714E-6
0.5	1.3571E-3	8.9574E-5	1.0568E-5	6.3387E-6
0.7	1.0995E-3	2.6945E-5	2.0265E-6	1.3070E-5
0.9	6.9730E-4	4.1400E-5	1.1808E-4	1.4346E-5

Table 3: The absolute errors of the approximate solutions for Example 7.3.

The exact solution of this problem is $X(t) = arcsinh(aB(t) + \sinh(X_0))$. We have also solved this problem by the proposed method for $a = 1/20$ and $X_0 = 1/10$. The graphs of the exact and approximate solutions (left side) and absolute error (right side) for $\hat{m} = 64$ with $N = 35$ are shown in Fig. 4. The absolute errors of the approximate solutions at some different points $t \in [0, 1]$ for $\hat{m} = 8, 16, 32$ and 64 are shown in Table 4.

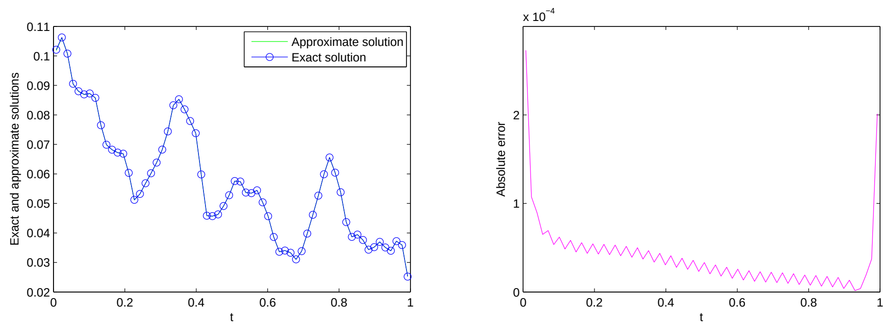

Figure 4: The graphs of the exact and approximate solutions (left side) and absolute error (right side) for Example 7.4.

Figure 4 and Table 4 show the accuracy of the proposed method for solving this problem.

t	$\hat{m} = 8$	$\hat{m} = 16$	$\hat{m} = 32$	$\hat{m} = 64$
0.1	2.1689E-3	4.4806E-3	3.6889E-3	8.1200E-4
0.3	3.7821E-3	4.9834E-3	4.0323E-5	2.1100E-4
0.5	4.9097E-3	3.3663E-3	7.6087E-4	5.2312E-5
0.7	4.0631E-3	1.0549E-3	3.4431E-4	1.5423E-4
0.9	2.8669E-3	1.7899E-3	1.7280E-3	7.2341E-4

Table 4: The absolute errors of the approximate solutions for Example 7.4.

8. Conclusion

In this paper, a new operational matrix of Itô stochastic integration for Wilson wavelets was derived and applied for solving a class of nonlinear stochastic Itô-Volterra integral equations. A new computational method based on these basis functions and their operational matrices of integration and Itô stochastic integration was proposed to solve the problem under study. In the proposed method, a new technique for computing nonlinear terms in such problems was presented. Also some useful theorems for Wilson wavelets were derived and used to solve problems under consideration. The convergence analysis of the Wilson expansion was proved. Applicability and accuracy of the proposed method was checked by some numerical examples. Moreover, the results of the proposed method were in a good agreement with the exact solutions.

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