Wavelets and Linear Algebra 4(1) (2017) 11 - 21

Vali-e-Asr University of Rafsanjan

# Wavelets and Linear Algebra

http://wala.vru.ac.ir

# Determination of Subrepresentations of the Standard Higher Dimensional Shearlet Group

**M. Zare[a], R. A. Kamyabi-Gol[a,*], Z. Amiri[a]**

[a] *Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Islamic Republic of Iran.*

## Article Info

*Article history:*
Received 20 July 2016
Accepted 17 December 2016
Available online 01 July 2017
Communicated by Abdolaziz Abdollahi

*Keywords:*
Semi direct product, Orbit, Shearlet transform.

*2000 MSC:*
22D10, 42C15.

## Abstract

This paper is devoted to definition standard higher dimension shearlet group $\mathbb{S} = \mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ and determination of square-integrable subrepresentations of this group. Also we give a characterisation of admissible vectors associated with the Hilbert spaces corresponding to each subrepresentation.



## 1. Introduction

Wavelet systems have been a popular method to analyze multidimensional data; however, these systems do not yield any information about directional components. To solve this problem, several approaches have been suggested in the context of directional signal analysis such as ridgelets

*Corresponding author

*Email addresses:* zare.masume@gmail.com (M. Zare), kamyabi@um.ac.ir (R. A. Kamyabi-Gol), za_am10@stu.um.ac.ir (Z. Amiri)

http://doi.org/10.22072/wala.2017.23366 

[6], curvelets [5], contourlets and surfacelet [9], shearlets [12, 16], and many others. Among all these approaches, the shearlet transform stands out because it is related to group theory, i.e., this transform can be derived from a square-integrable representation of the shearlet group (see [7]). Therefore, in the context of the shearlet transform, all the powerful tools of group representation theory can be exploited. In addition, shearlets have been shown to yield (almost) optimally sparse representations which are either built on band-limited or on compactly supported functions [11, 14] and have been implemented [15]. It is becoming more common for higher dimensional data sets to appear, which need to be analyzed. For analyzing data in $\mathbb{R}^n, n \geq 3$, the shearlet transform has to be generalized to higher dimensions. Finding optimal representations of signals in higher dimensions is currently the subject of the researches by Dahlke et all [8]. Although wavelets, provide to be a satisfactory tool in one dimension, do not provide any directional information. So there is an important motivation to obtain directional representations which capture directional features, like orientations of curves in images, while providing sparse decompositions. Hense Dahlke et all in [8] introduced higher dimensional shearlet transform. Irreducible unitary representations of a locally compact group $G$, are the basic building blocks of the harmonic analysis associated to $G$. In fact by Gelfand-Raikov Theorem a locally compact group always has enough irreducible representations to separate points [10]. It is known that the standard unitary representations of shearlet group (2-D) and standard higher dimensional shearlet group are not square-integrable or even irreducible. In this paper we determine all irreducible and square-integrable sub-representations of these groups, where recently in [3] the authors characterize irreducible as well as square-integrable subrepresentations of the standard shearlet group representation in 2-D.

More precisely this paper is structured as follow. In Section 2, the class of semi-direct product of locally compact groups and their representations are introduced. In section 3, we state some preliminaries about higher dimensional shearlet group, its unitary representation and the associated shearlet transform and introduce standard higher dimension shearlet group. Section 4 is devoted to determination of irreducible subrepresentations of standard higher dimensional shearlet group representation and characterizing admissibility condition for any vector associated to the Hilbert spaces corresponding to these subrepresentations.

## 2. Preliminaries and notation

We shall use the following conventions throughout the paper. For two locally compact groups $H$ and $K$, let $h \mapsto \tau_h$ be a homomorphism of $H$ into the group of automorphisms of $K$ denoted by $\mathrm{Aut}(K)$. Also assume that the mapping $(h, k) \mapsto \tau_h k$, from $H \times K$ (endowed with the product topology) onto $K$ is continuous. Then the set $H \times K$ with the operations:

$$(h, k)(h', k') := (hh', k\tau_h(k')),$$

and

$$(h, k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1}),$$

is a locally compact group. This group is denoted by $H \ltimes_\tau K$ and called the semi direct product of $H$ and $K$, respectively. The left Haar measure of $G = H \ltimes_\tau K$ is $d\mu_{G_\tau}(h, k) = \delta(h)d\mu_H(h)d\mu_K(k)$, where $d\mu_H$ and $d\mu_K$ are the left Haar measures on $H$ and $K$, respectively and $\delta$ is a positive continuous homomorphism on $H$ which is given by

$$d\mu_K(k) = \delta(h) d\mu_K(\tau_h(k)).$$

For more details on semi direct product groups see [1, 13]. The unitary representation of $G = H \ltimes_\tau K$ on the Hilbert space $L^2(K)$, denoted by $(U, L^2(K))$, is defined by

$$U(h,k)f(y) = \delta(h)^{\frac{1}{2}} f(\tau_{h^{-1}}(yk^{-1})),$$

for all $f \in L^2(K), (h,k) \in G$. This representation is called quasi-regular representation of $G$. It is worthwhile to note that this representation generally is not irreducible [1, 2]. Assume that $K$ is also Abelian and consider $\widehat{K}$ as the dual group of $K$. To build an irreducible subrepresentation of $U$, one can define an action of $H$ on $\widehat{K}$ by

$$H \times \widehat{K} \longrightarrow \widehat{K}; \qquad (h, w) = w \circ \tau_{h-1}.$$

For a fixed $\omega \in \widehat{K}$, the orbit $O_w$ and the stabilizer $H^w$ of $w$ are defined by

$$O_w = \left\{ w \circ \tau_{h^{-1}}; h \in H \right\}, \qquad H^w := \left\{ h \in H; wo\tau_{h^{-1}} = w \right\}.$$

One can easily see that $H^w$ is a closed subgroup of $H$ and $O_w$ is an $H$-invariant subset in $\widehat{K}$, that is $H^w h \subseteq H^w$, for all $h \in H$.

## 3. Standard higher dimensional sherlet group

For analysing data in $\mathbb{R}^n, n \geq 3$, Dahlke et all [8] generalized two dimensional shearlet transform to higher dimensions, in the fallowing method.

Let $I_n$ denote the $n \times n$ identity matrix, also $0_n$, the vector with $n$ entries. For $a \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $s \in \mathbb{R}^{n-1}$

$$A_a = \begin{pmatrix} a & 0_{n-1} \\ 0_{n-1}^T & sgn(a)|a|^{\frac{1}{n}} I_{n-1} \end{pmatrix} \qquad and \qquad S_s = \begin{pmatrix} 1 & s \\ 0_{n-1}^T & I_{n-1} \end{pmatrix}.$$

The choice of $S_s$ lead shearlet transform to be a square integrable group representation. In order to have directional selectivity, the dilation factors at the diagonal of $A_a$ is chosen in an anisotropic way, i.e., if the first diagonal entry is $a$, the other ones should increase less than linearly in $a$ as $a \to \infty$. The set $\mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ endowed with the operation

$$(a, s, t) \circ (a', s', t') = (aa', s + |a|^{1-\frac{1}{n}} s', t + S_s A_a t'),$$

is a locally compact group $\mathbb{S}$, which is called full higher dimensional shearlet group. The left and right Haar measures on $\mathbb{S}$ are given by

$$\mathrm{d}\mu_l(a, s, t) = \frac{1}{|a|^{n+1}} \mathrm{d}a \mathrm{d}s \mathrm{d}t \qquad and \qquad \mathrm{d}\mu_r(a, s, t) = \frac{1}{|a|} \mathrm{d}a \mathrm{d}s \mathrm{d}t.$$

For $f \in L^2(\mathbb{R}^n)$ the map $\pi : \mathbb{S} \to \mathcal{U}(L^2(\mathbb{R}^n))$, defined by

$$\pi(a, s, t) f(x) = f_{a,s,t}(x) := |a|^{\frac{1}{2n} - 1} f\left(A_a^{-1} S_s^{-1} (x - t)\right),$$

is a unitary representation of locally compact group $\mathbb{S}$ on the Hilbert space $L^2(\mathbb{R}^n)$, with respect to the Haar measure $d\mu_l$. It is worthwhile to know that this representation is irreducible square integrable representation. For more details on full higher dimensional shearlet group see [8]. Recall that a non trivial function $\psi \in L^2(\mathbb{R}^n)$ is called admissible with respect to $\pi$ if

$$\int_{\mathbb{S}} | \langle \psi, \pi(a,s,t)\psi \rangle |^2 \, d\mu_l(a,s,t) < \infty. \tag{3.1}$$

If there exists at least one admissible vector $\psi \in L^2(\mathbb{R}^n)$ with respect to $\pi$ , then $\pi$ is called square-integrable [1]. In the sequel, by a square-integrable representation, we mean irreducible square-integrable representation.

Although the representation of full higher dimensional shearlrt group $\mathbb{R}^* \times \mathbb{R}^{n-1} \times \mathbb{R}^n$, is square integrable, but the representation of standard higher dimensional shearlet group $\mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^n$, which we will define in the sequel with similar method, is not square integrable in general.

Now we are ready to define standard higher dimensional sherlet group. Consider $n \times n$ dilation matrices, depend on a parameter $a \in \mathbb{R}^+$, defined by

$$\mathbf{A}_a = \begin{pmatrix} a & 0_{n-1} \\ 0_{n-1}^T & a^{\frac{1}{n}} I_{n-1} \end{pmatrix},$$

in which $I_n$ is $n \times n$ identity matrix. The dilation factor at the diagonal of $A_a$ is chosen in an anisotropic way. This choice of $A_a$ enables us to detect special directional information. The $n \times n$ shear matrices is defined by

$$\mathbf{S}_s = \begin{pmatrix} 1 & s \\ 0_{n-1}^T & I_{n-1} \end{pmatrix},$$

where $s \in \mathbb{R}^{n-1}$. The set of shear matrices form a subgroup of $GL_n(\mathbb{R})$, all real $n \times n$ non-singular matrices . The similar calculation like full higher dimensional shearlet group show that, the set $\mathbb{R}^+ \times \mathbb{R}^{n-1} \times \mathbb{R}^n$ equipped with the group operations

$$(a,s,t)o(a',s',t') = (aa', s + a^{1-\frac{1}{n}} s', t + S_s A_a t'),$$

and

$$(a,s,t)^{-1} = (a^{-1}, -a^{\frac{1}{n}-1} s, -A_a^{-1} S_s^{-1} t),$$

is a locally compact group. Its left and right Haar measures are given by

$$d\mu_l(a,s,t) = \frac{1}{a^{n+1}} \, da \, ds \, dt,$$

and

$$d\mu_r(a,s,t) = \tfrac{1}{a} \, da \, ds \, dt,$$

respectively. In the sequel, by higher dimensional shearlet group $\mathbb{S}$, we mean standard higher dimensional shearlet group. We regard our higher dimensional shearlet group, the semi direct product group $\mathbb{S} = H \ltimes_\tau \mathbb{R}^n$, in the following form

$$\mathbb{S} = \Big\{(M, t); M \in H, t \in \mathbb{R}^n\Big\},$$

where $H$ is the group of the matrices

$$H = \Big\{S_s A_a; s \in \mathbb{R}^{n-1}, a \in \mathbb{R}^+\Big\} \subseteq GL(n, \mathbb{R}), \tag{3.2}$$

and the homomorphism $\tau : H \to Aut(\mathbb{R}^n)$ is defined by

$$\tau_M(t) = Mt = S_s A_a t. \tag{3.3}$$

So for any $\psi \in L^2(\mathbb{R}^n)$, the quasi-regular representation of $\mathbb{S} = H \ltimes_\tau \mathbb{R}^n$ is defined by

$$\pi(a, s, t)f(x) = f_{a,s,t}(x) := (detA_a)^{\frac{-1}{2}} f\Big(A_a^{-1} S_s^{-1}(x - t)\Big) \tag{3.4}$$

$$= a^{\frac{1}{2n}-1} f\Big(A_a^{-1} S_s^{-1}(x - t)\Big), \tag{3.5}$$

which is a unitary representation of $\mathbb{S}$ on $U(L^2(\mathbb{R}^n))$.

A function $\psi \in L^2(\mathbb{R}^n)$ that fulfils the admissibility condition (3.1), is called a continuous shearlet and the transform

$$\mathcal{SH}_\psi : L^2(\mathbb{R}^n) \longrightarrow L_2(\mathbb{S}),$$

defined by

$$\mathcal{SH}_\psi f(a, s, t) := \langle f, \psi_{a,s,t}\rangle = f * \psi^*_{a,s,0}(t),$$

is called a continuous shearlet transform.

The admissibility condition usually yields to a resolution of the identity that leads to reconstruction of a signal $f \in L^2(\mathbb{R}^n)$ from the representation coefficients $(\langle\psi, \pi(a, s, t)\psi\rangle)_{(a,s,t)\in\mathbb{S}}$.

## 4. Main results

Let $GL(n, \mathbb{R})$ denote the group of invertible $n \times n$ real matrices with the usual topology. It is well known that any subgroup of the group $GL(n, \mathbb{R})$ has a natural action on $\mathbb{R}^n$ via $(x, M) \longmapsto Mx$. The action of $GL(n, \mathbb{R})$ on the dual group $\widehat{\mathbb{R}^n}$ is $(k, M) \mapsto M^T k$, where $M^T$ denotes the transpose of matrix $M$. Let $H$ be an n-dimensional closed subgroup of $GL(n, \mathbb{R})$. Note that for any $\gamma_0 \in \widehat{\mathbb{R}^n}$, its orbit under $H$ is $O_{\gamma_0} = \{M^{-T}\gamma_0; M \in H\}$. We say that $H$ has an open free $H$-orbit in $\widehat{\mathbb{R}^n}$, if there exists a $\gamma_0 \in \widehat{\mathbb{R}^n}$ such that $O_{\gamma_0}$ is an open set in $\widehat{\mathbb{R}^n}$ and furthermore, $M^{-T}k = k$ for any $k \in O_{\gamma_0}$ implies that $M = I_n$. In other words, the stabilizer of any $k \in O_{\gamma_0}$ is trivial. Note that for any $H \ltimes_\tau \mathbb{R}^n$ where $H$ is an n-dimensional subgroup of $GL(n, \mathbb{R})$, it is not always the case that open free orbit of $H$ exists in $\widehat{\mathbb{R}^n}$. However, if there does exist one such orbit, then $\widehat{\mathbb{R}^n}$ is disjoint union of such orbits [1].

Let $U$ be an open subset of $\widehat{\mathbb{R}^n}$ and $L^2(U)$ denote the closed subspace of $L^2(\widehat{\mathbb{R}^n})$ consisting of elements supported on $U$ and consider $\mathcal{H}_U^2 = F^{-1}(L^2(U))$, where $F^{-1}(L^2(U))$ is the inverse Fourier transform of $L^2(U)$. Then $\mathcal{H}_U^2$ can be thought of as a generalized Hardy space [4]. It is worthwhile to note that if $H$ is closed subgroup of $GL(n,\mathbb{R})$, then $U$ is free $H$-orbit in $\widehat{\mathbb{R}^n}$ and if $\rho$ is a unitary representation of $H \ltimes_\tau \mathbb{R}^n$ on $L^2(\mathbb{R}^n)$, then $\rho|_{\mathcal{H}_U^2}$ is a square integrable representation of $H \ltimes_\tau \mathbb{R}^n$ on $\mathcal{H}_U^2$ [4, Theorem 1]. Also an element $\psi \in \mathcal{H}_U^2$, is admissible if and only if $\widehat{\psi}\Psi_\gamma^{\frac{1}{2}} \in L^2(U)$, where $\Psi_\gamma = \Psi_H \circ \alpha_\gamma^{-1}$, in which $\Psi_H(h) = \frac{\Delta_H(h)}{\delta(h)}$ and $\alpha_\gamma : H \to U$ is a homeomorphism defined by $\alpha_\gamma(h) = \gamma h$ [4, Corollary 1]. The main goal of this section is to determine all square-integrable subrepresentations of higher dimensional shearlet group. To do this aim, first in the next theorem we shall show that the action of $H$ on $\mathbb{R}^n$ is free.

**Theorem 4.1.** *Let $H$ be an $n$-dimensional closed subgroup of $GL(n,\mathbb{R})$ defined in (3.2). Then the action of $H$ on $\widehat{\mathbb{R}^n}$ given by $M.\gamma := \gamma o \tau_{M^{-1}}$ is free, that is, for any $\gamma \in \widehat{\mathbb{R}^n}$, $H^\gamma = \{I_n\}$.*

*Proof.* Using equations (3.3) we have

$$\begin{aligned} \gamma o \tau_{M^{-1}}(t) &= \gamma(M^{-1}t) = \gamma(A_a^{-1}S_s^{-1}t) \\ &= \gamma(S_{\frac{-s}{\sqrt[n]{a^{n-1}}}} A_{\frac{1}{a}} t) = e^{2\pi i \gamma . S_{\frac{-s}{\sqrt[n]{a^{n-1}}}} A_{\frac{1}{a}} t} \\ &= e^{2\pi i A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt[n]{a^{n-1}}}} \gamma . t} = A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt[n]{a^{n-1}}}} \gamma(t), \end{aligned}$$

where $a \in \mathbb{R}^+$ and $s = (s_1, \cdots, s_{n-1}) \in \mathbb{R}^{n-1}$. Therefore we have

$$\begin{aligned} M.\gamma &= A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt[n]{a^{n-1}}}} \gamma \\ &= \begin{pmatrix} \frac{1}{a} & 0_{n-1} \\ 0^T_{n-1} & a^{\frac{-1}{n}} I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0_{n-1} \\ \frac{-s^T}{\sqrt[n]{a^{n-1}}} & I_{n-1} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{a} & 0_{n-1} \\ \frac{-s^T}{a} & a^{\frac{-1}{n}} I_{n-1} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}, \end{aligned}$$

i.e.,

$$\begin{pmatrix} \frac{1}{a} & 0 & 0 & \cdots & 0 \\ \frac{-s_1}{a} & \frac{1}{\sqrt[n]{a}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-s_{n-2}}{a} & 0 & \cdots & \frac{1}{\sqrt[n]{a}} & 0 \\ \frac{-s_{n-1}}{a} & 0 & \cdots & 0 & \frac{1}{\sqrt[n]{a}} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1}{a} \\ \frac{-s_1\gamma_1}{a} + \frac{\gamma_2}{\sqrt[n]{a}} \\ \vdots \\ \frac{-s_{n-1}\gamma_1}{a} + \frac{\gamma_n}{\sqrt[n]{a}} \end{pmatrix}.$$

So $M.\gamma = \gamma$ yields $a = 1$, also $s = 0$ as a vector in $\mathbb{R}^{n-1}$, i.e., M is the identity matrix for any $\gamma \in \widehat{\mathbb{R}^n}$. Thus $H^\gamma = \{I_n\}$ and so the action of $H$ on $\widehat{\mathbb{R}^n}$ is free. □

Now by the procedure in the proof of Theorem 4.1, we can find all orbits of the action of $H$ on $\widehat{\mathbb{R}^n}$. Note that the structure of the orbits depend on the choice of $\gamma_i,\ i = 1, ..., n$, to be positive, negative or zero. The non-zero orbits are classified as two non-zero invariant subsets $A_+ := O_{(1,0,..,0)}$ and $A_- := O_{(-1,0,...,0)}$, in which

$$O_{(1,0,...,0)} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_1 > 0, x_i \in \mathbb{R} \quad for \quad i = 2, ..., n \right\},$$

and

$$O_{(-1,0,...,0)} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, x_1 < 0, x_i \in \mathbb{R} \quad for \quad i = 2, ..., n \right\}.$$

Note that $A_+$ and $A_-$ are subsets of positive measure in $\widehat{\mathbb{R}^n}$. Now we are ready to determine all square integrable subrepresentations of unitary representation of higher dimensional shearlet group. Let $\pi$ be the unitary representation of $\mathbb{S}$ defined in (3.4), we identify all square integrable subrepresentations of $\pi$.

**Theorem 4.2.** *Square-integrable subrepresentations of higher dimensional shearlet group are precisely the following two subrepresentations*

$$\pi_+ : \mathbb{S} \longrightarrow U(\mathcal{H}^2_{A_+}), \qquad \pi_+(a, s, t)\psi = \psi_{(a,s,t)},$$

*and*

$$\pi_- : \mathbb{S} \longrightarrow U(\mathcal{H}^2_{A_-}), \qquad \pi_-(a, s, t)\phi = \phi_{(a,s,t)}.$$

*Proof.* As we proved in Theorem 4.1, the action defined on $\widehat{\mathbb{R}^n}$ is free for any $\gamma \in \widehat{\mathbb{R}^n}$. Therefore $A_+$ and $A_-$ are free H-orbits also with positive measure in $\widehat{\mathbb{R}^n}$. Hence using [4, Theorem 1], $\pi_+$ and $\pi_-$ are square integrable representations of $\mathbb{S}$ on $\mathcal{H}^2_{A_+}$ and $\mathcal{H}^2_{A_-}$ respectively. On the other hand since $A_+$ and $A_-$ are only invariant subsets of $\widehat{\mathbb{R}^n}$ with positive measure, so $\pi_+ : \mathbb{S} \longrightarrow U(\mathcal{H}^2_{A_+})$ and $\pi_- : \mathbb{S} \longrightarrow U(\mathcal{H}^2_{A_-})$ are the only square integrable subrepresentations of $\pi$ defined by $\pi_+(a, s, t)\psi = \psi_{a,s,t}$ and $\pi_-(a, s, t)\phi = \phi_{a,s,t}$, where

$$\psi_{a,s,t}(x) = a^{\frac{1}{2n}-1} f\Big(A_a^{-1} S_s^{-1}(x - t)\Big),$$

and

$$\phi_{a,s,t}(x) = a^{\frac{1}{2n}-1} f\Big(A_a^{-1} S_s^{-1}(x - t)\Big).$$

□

In the next theorem all admissible vectors in $\mathcal{H}^2_{A_+}$ and $\mathcal{H}^2_{A_-}$ are characterized, by using orbits, in the other word we shall state admissibility condition for any vector in $\mathcal{H}^2_{A_+}$ and $\mathcal{H}^2_{A_-}$.

**Theorem 4.3.** *An element* $\psi \in \mathcal{H}^2_{A_+}$ *(or* $\in \mathcal{H}^2_{A_-}$*) is admissible if and only if*

$$\int_{\widehat{\mathbb{R}^n}} \frac{|\widehat{\psi}(w)|^2}{w_1^n}\, dw < \infty,$$

*where* $w = (w_1, \cdots, w_n)^T \in \widehat{\mathbb{R}^n}$.

*Proof.* Recall that the higher dimensional shearlet group $\mathbb{S}$, is a type of semi direct product group denoted by $H \ltimes_\tau \mathbb{R}^n$, where $H$ is locally compact group with group action $(S_s A_a)o(S_{s'}A_{a'}) = S_s A_a S_{s'} A_{a'}$ such that

$$A_a = \begin{pmatrix} a & 0_{n-1} \\ 0^T_{n-1} & a^{1/n} I_{n-1} \end{pmatrix}, \qquad a \in \mathbb{R}^+,$$

$$S_s = \begin{pmatrix} 1 & s \\ 0^T_{n-1} & I_{n-1} \end{pmatrix}, \qquad s \in \mathbb{R}^{n-1}.$$

Therefore it is easily to compute that the modular function on $H$, is

$$\Delta_H : H \longrightarrow (0, \infty); \qquad \Delta_H(S_s A_a) = \frac{(\sqrt[n]{a})^{n-1}}{a^{n-1}}.$$

Since $\psi_H(h) = \dfrac{\Delta_H(h)}{det(h)} = \frac{1}{a^n}$, where $h = S_s A_a \in H$, so $\psi_{(1,0,\ldots,0)}(\omega_1, ..., \omega_n) = \psi_H o \alpha^{-1}_{(1,0,\ldots,0)}(w_1, ..., w_n) = \frac{1}{w_1^n}$. Hence [4, corollary 1] complete the proof i.e; $\psi$ is admissible if and only if $\int_{\widehat{\mathbb{R}^n}} \frac{|\widehat{\psi}(w)|^2}{w_1^n}\, dw < \infty$. □

Now we give an example and present these results. It can be useful to state that in [3], the square-integrable sub-representations of the standard shearlet group representation in 2-D is investigated.

**Example 4.4.** In this example we determine all irreducible and square-integrable subrepresentations of standard 3-D dimensional shearlet group.

Let $\mathbb{S} = (\mathbb{R}^+ \times \mathbb{R}^2) \ltimes_\lambda \mathbb{R}^3$. The action of $(\mathbb{R}^+ \times \mathbb{R}^2)$ on $\widehat{\mathbb{R}^3}$ given by $(a, s).\gamma := \gamma \circ \lambda_{(a,s)^{-1}}$ is free for any $a \in \mathbb{R}^+$ and $s = (s_1, s_2) \in \mathbb{R}^2$, indeed for any $\gamma \in \widehat{\mathbb{R}^3}$ we have $H^\gamma = (1, 0, 0)$. Since

$$\begin{aligned} \gamma \circ \lambda_{(a,s)^{-1}}(t) &= \gamma(S_{\frac{-s}{\sqrt[3]{a^2}}} A_{\frac{1}{a}} t) \\ &= e^{2\pi i \gamma . S_{\frac{-s}{\sqrt[3]{a^2}}} A_{\frac{1}{a}} t} \\ &= e^{2\pi i \gamma . A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt[3]{a^2}}} t} \\ &= A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt[3]{a^2}}} \gamma(t), \end{aligned}$$

we have

$$\begin{aligned}(a,s).\gamma := \gamma\circ\lambda_{(a,s)^{-1}} &= A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt[3]{a^2}}}\gamma \\ &= \begin{pmatrix}\frac{1}{a} & 0 & 0\\ 0 & \frac{1}{\sqrt[3]{a}} & 0\\ 0 & 0 & \frac{1}{\sqrt[3]{a}}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ \frac{-s_1}{\sqrt[3]{a^2}} & 1 & 0\\ \frac{-s_2}{\sqrt[3]{a^2}} & 0 & 1\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{a} & 0 & 0\\ \frac{-s_1}{a} & \frac{1}{\sqrt[3]{a}} & 0\\ \frac{-s_2}{a} & 0 & \frac{1}{\sqrt[3]{a}}\end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{a}\gamma_1\\ \frac{-s_1}{a}\gamma_1+\frac{1}{\sqrt[3]{a}}\gamma_2\\ \frac{-s_2}{a}\gamma_1+\frac{1}{\sqrt[3]{a}}\gamma_3\end{pmatrix}.\end{aligned}$$

Therefore $\begin{pmatrix}\frac{1}{a}\gamma_1\\ \frac{-s_1}{a}\gamma_1+\frac{1}{\sqrt[3]{a}}\gamma_2\\ \frac{-s_2}{a}\gamma_1+\frac{1}{\sqrt[3]{a}}\gamma_3\end{pmatrix} = \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}$, yields $(a,(s_1,s_2)) = (1,(0,0))$.

The above mentioned action has 11 orbits as follows:

$$\begin{aligned}
O_{(0,0.0)} &= \{(0,0,0)\},\\
O_{(\gamma_1,\gamma_2,\gamma_3)} &= \{(x,y,z)\in\mathbb{R}^3 : x>0\}, \gamma_1>0,\\
O_{(\gamma_1,\gamma_2,\gamma_3)} &= \{(x,y,z)\in\mathbb{R}^3 : x<0\}, \gamma_1<0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : y>0\}, \gamma_2>0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : y<0\}, \gamma_2<0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : z>0\}, \gamma_3>0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : z<0\}, \gamma_3<0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : y>0, z>0\}, \gamma_2>0, \gamma_3>0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : y>0, z<0\}, \gamma_2>0, \gamma_3<0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : y<0, z>0\}, \gamma_2<0, \gamma_3>0,\\
O_{(0,\gamma_2,\gamma_3)} &= \{(0,y,z)\in\mathbb{R}^3 : y<0, z<0\}, \gamma_2<0, \gamma_3<0.
\end{aligned}$$

All the non-zero measure orbits are classified as two orbits:

$$O_{(1,0,0)} = \Big\{(x,y,z)\in\mathbb{R}^3 : x>0\Big\},$$

and

$$O_{(-1,0,0)} = \Big\{(x,y,z)\in\mathbb{R}^3 : x<0\Big\}.$$

Define $A_+ := O_{(1,0,0)}$ and $A_- := O_{(-1,0,0)}$. Then $\sigma_+ : \mathbb{S} \to \mathcal{U}(\mathcal{H}^2_{A_+})$ and $\sigma_- : \mathbb{S} \to \mathcal{U}(\mathcal{H}^2_{A_-})$ defined by

$$\begin{aligned}\sigma_+(a,s,t)\psi(x) &= a^{\frac{1}{2n}-1}\psi\Big(A_a^{-1}S_s^{-1}(x-t)\Big)\\ &= a^{\frac{-5}{6}}\psi\Big(\begin{pmatrix}\frac{1}{a} & \frac{-s}{a}\\ 0_2^T & \frac{1}{\sqrt[3]{a}}I_2\end{pmatrix}(x-t)\Big),\end{aligned}$$

and

$$\begin{aligned}\sigma_-(a,s,t)\phi(x) &= a^{\frac{1}{2n}-1}\phi(A_a^{-1}S_s^{-1}(x-t))\\ &= a^{\frac{-5}{6}}\phi\Big(\begin{pmatrix}\frac{1}{a} & \frac{-s}{a}\\ 0_2^T & \frac{1}{\sqrt[3]{a}}I_2\end{pmatrix}(x-t)\Big),\end{aligned}$$

are precisely the square-integrable subrepresentations of 3-D standard shearletb group, for $\psi \in \mathcal{H}^2_{A_+}$ and $\varphi \in \mathcal{H}^2_{A_-}$. Therefore the standard representation of this group is direct sum of two irreducible representations, in fact, $\sigma = \sigma_+ \oplus \sigma_-$. Also $\psi \in \mathcal{H}^2_{A_+}$ (or $\in \mathcal{H}^2_{A_-}$) is admissible if and only if $\int_{\mathbb{R}^3} \frac{|\widehat{\psi}(\xi_1,\xi_2,\xi_3)|^2}{\xi_1^2} d\xi_1 d\xi_2 d\xi_3 < \infty$.

## References


[1] S.T. Ali, J.P. Antoine and J.P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, New York: Springer-Verlag, 2000.

[2] A.A. Arefijamaal and R.A. Kamyabi-Gol, A characterization of square-integrable representations associated with CWT, *J. Sci., Islam. Repub. Iran,* **18**(2) (2007), 159-166.

[3] V. Atayi and R.A. Kamyabi-Gol, On the characterization of subrepresentations of shearlet group, *Wavelets and Linear Algebra,* **2**(1) (2015), 1-9.

[4] D. Bernier and K.F. Taylor, Wavelet from square-integrable representation, *SIAM J. Math. Anal.,* **27**(2) (1996), 594-608.

[5] E.J. Candes and D.L. Donoho, Curvelets: A surprisingly effective nonadaptive representation for objects with edges, *Saint-Malo Proceedings*, 2000.

[6] E.J. Candes and D.L. Donoho, Ridgelets: a key to higher-dimensional intermittency, *Philos. Trans. R. Soc. Lond., A, Math. Phys. Eng. Sci.,* **357**(1760) (1999), 2495-2509.

[7] S. Dahlke, G. Kutyniok, C. Sagiv, H.G. Stark and G. Teschke, The uncertainty principle associated with the continuous shearlet transform, *Int. J. Wavelets Multiresolut. Inf. Process.,* **6**(2) (2008), 157-181.

[8] S. Dahlke and G. Teschke, The continuous shearlet transform in arbitrary space dimensions, *J. Fourier Anal. Appl.,* **16** (2010), 340-364.

[9] M.N. Do and M. Vetterli, The contourlet transform: an efficient directional multiresolution image representation, *IEEE Trans. Image Process.,* **14**(12) (2005), 2091-2106.

[10] G.B. Folland, *A Course in Abstract Harmonic Analysis*, Boca Katon: CRC Press, 1995.

[11] K. Guo and D. Labate, Optimally sparse multidimensional representation using shearlets, *SIAM J. Math. Anal.,* **39**(1) (2007), 298-318.

[12] K. Guo, G. Kutyniok and D. Labate, Sparse multidimensional representations using anisotropic dilation and shear operators, *International Conference on the Interaction between Wavelets and Splines,* Athens, 2005.

[13] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis,* vol. 1, Springer-Verlag, Berlin, 1963.

[14] G. Kutyniok and W. Lim, *Compactly Supported Shearlets,* vol. 13, Approximation Theory XIII: San Antonio 2010. Springer Proceedings in Mathematics, Springer, New York, 2011.

[15] G. Kutyniok, W. Lim and X. Zhuang, Digital shearlet transforms, *Appl. Comput. Harmon. Anal.*, (2012), 239–282.

[16] D. Labate, W. Lim, G. Kutyniok and G. Weiss, Sparse multidimensional representation using shearlets, *SPIE Proc. 5914, SPIE, Bellingham*, (2005), 254-262.