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A Necessary Condition for a Shearlet System to be a Frame via Admissibility

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Abstract

Necessary conditions for shearlet and cone-adapted shearlet systems to be frames are presented with respect to the admissibility condition of generators.

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1. Introduction and Preliminaries

Shearlets were introduced by Guo, Kutyniok, Labate, Lim and Weiss in [8, 14] and developed by some others in e.g. [10, 13] as the first directional representation system which allows a unified treatment of the continuum and digital world similar to wavelets. Shealets were derived within a larger class of affine-like systems, composite wavelets, using shearing to control directional selectivity. In contrast to other x-lets which mostly utilize the geometry of the data, shearlet systems form an *affine system*, generated by dilations and translations of a generator, where the dilation matrix is the product of a parabolic scaling matrix and a shear matrix. This makes the shearlet approach more remunerative for obtaining the anisotropic and directional features of multidimensional data [13]. This property provides additional simplicity of construction and a connection with the theory of square integrable group representations of the affine group [1, 2, 4, 5, 12]. Of particular importance for the shearlet transform is the situations under which any vector in $L^2(\mathbb{R}^2)$ can be reconstructed from shearlet atoms. Admissibility condition is a sufficient condition for this facility.

Discrete and cone-adapted discrete shearlet systems are studied by Kutyniok and Labate in [11, 13]. They have derived sufficient conditions in [11] for a discrete shearlet system to form a frame for $L^2(\mathbb{R}^2)$, whereas in this paper, we establish a necessary condition for both discrete and cone-adapted discrete shearlet systems to be frames via admissibility. In fact, we provide a relation between shearlet frames and admissibility condition of the generators.

We propose here some preliminaries and notation about shearlets. We define the shearlet group $\mathbb{S}$, as the semi-direct product

$$(\mathbb{R}^+ \times \mathbb{R}) \times \mathbb{R}^2$$

equipped with group multiplication given by

$$(a, s, t).(a', s', t') = (aa', s + s'\sqrt{a}, t + S_s A_a t'),$$

where the *parabolic scaling matrices* A_a and the *shearing matrix* S_s are given by

$$A_a = \begin{bmatrix} a & 0 \\ 0 & a^{\frac{1}{2}} \end{bmatrix}, \qquad S_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}.$$

The left-invariant Haar measure of this group is $\frac{da}{a^3} ds dt$. Let $\psi \in L^2(\mathbb{R}^2)$. The *continuous shearlet system* associated with ψ is defined by

$$\left\{\psi_{a,s,t} = T_t D_{A_a} D_{S_s} \psi : \ a > 0, \ s \in \mathbb{R}, \ t \in \mathbb{R}^2\right\}, \tag{1.1}$$

where T and D are translation and dilation operators, respectively defined as $T_t f(x) = f(x-t)$, $D_B f(x) = |detB|^{-\frac{1}{2}} f(B^{-1}x)$, where $t \in \mathbb{R}$ and B is an invertible 2×2 matrix. The *continuous shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is the mapping

$$f \mapsto \mathcal{SH}_\psi \qquad f(a, s, t) = \langle f, \psi_{a,s,t} \rangle, \qquad (a, s, t) \in \mathbb{S}.$$

One of our concerns in shearlet theory is the reconstruction formula which is associated with the admissibility condition on ψ.

A *discrete shearlet system* associated with ψ is defined by

$$\Big\{\psi_{j,k,m} = a_0^{-\frac{3}{4}j}\psi(S_k A_{a_0^{-j}} \cdot - m) : \ j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\Big\}, \ \ a_0 > 0. \tag{1.2}$$

The *discrete shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is the mapping defined by

$$f \mapsto \mathcal{SH}_\psi f(j,k,m) = \langle f, \psi_{j,k,m}\rangle, \qquad (j,k,m) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^2.$$

Definition 1.1. If $\psi \in L^2(\mathbb{R}^2)$ satisfies

$$c_\psi := \int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{\xi_1^2} d\xi_1 d\xi_2 < \infty, \tag{1.3}$$

it is called an admissible shearlet. We denote by c_ψ^+, c_ψ^- the following formulas

$$c_\psi^+ = \int_0^\infty \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{\xi_1^2} d\xi_2 d\xi_1, \qquad c_\psi^- = \int_{-\infty}^0 \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{\xi_1^2} d\xi_2 d\xi_1. \tag{1.4}$$

Here, we recall the definitions of a cone-adapted discrete shearlet system and transform from [13]. For $\phi,\ \psi,\ \tilde{\psi} \in L^2(\mathbb{R}^2)$ and $c = (c_1, c_2) \in (\mathbb{R}^+)^2$, the *cone-adapted discrete shearlet system* is defined by

$$\Phi(\phi; c_1) \cup \Psi(\psi; c) \cup \tilde{\Psi}(\tilde{\psi};\ c), \tag{1.5}$$

where

$$\Phi(\phi; c_1) = \Big\{\phi_m = \phi(\cdot - c_1 m) : m \in \mathbb{Z}^2\Big\},$$

$$\Psi(\psi; c) = \Big\{\psi_{j,k,m} = a_0^{\frac{3}{4}j}\psi(S_k A_{a_0^j} \cdot - M_c m) : j \geq 0, |k| \leq [a_0^{\frac{j}{2}}], m \in \mathbb{Z}^2\Big\},$$

$$\tilde{\Psi}(\tilde{\psi}; c) = \Big\{\tilde{\psi}_{j,k,m} = a_0^{\frac{3}{4}j}\tilde{\psi}(S_k^T \tilde{A}_{a_0^j} \cdot - \tilde{M}_c m) : j \geq 0, |k| \leq [a_0^{\frac{j}{2}}], m \in \mathbb{Z}^2\Big\},$$

with

$$M_c = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \quad \tilde{M}_c = \begin{bmatrix} c_2 & 0 \\ 0 & c_1 \end{bmatrix}.$$

The system $\Phi(\phi; c_1)$ is associated with $\mathcal{R}$ and the systems $\Psi(\psi; c)$ and $\tilde{\Psi}(\tilde{\psi}; c)$ are associated with $\mathcal{E}_1 \cup \mathcal{E}_3$ and $\mathcal{E}_2 \cup \mathcal{E}_4$, respectively, where

$$\mathcal{R} = \Big\{(\xi_1,\xi_2) : |\xi_1|, |\xi_2| \leq 1\Big\},$$

$$\mathcal{E}_1 \cup \mathcal{E}_3 = \Big\{(\xi_1,\xi_2) : |\frac{\xi_2}{\xi_1}| \leq 1, |\xi_1| > 1\Big\}, \qquad \mathcal{E}_2 \cup \mathcal{E}_4 = \Big\{(\xi_1,\xi_2) : |\frac{\xi_2}{\xi_1}| > 1, |\xi_2| > 1\Big\},$$

$$\tilde{\psi}(\xi_1,\xi_2) = \psi(\xi_2,\xi_1).$$

The *cone-adapted discrete shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is the mapping defined by

$$f \mapsto \mathcal{SH}_{\phi,\psi,\tilde{\psi}}\ f\Big(m'', (j,k,m), (j',k',m')\Big) = \Big(\langle f, \phi_{m''}\rangle, \langle f, \psi_{j,k,m}\rangle, \langle f, \tilde{\psi}_{j',k',m'}\rangle\Big),$$

with
$$\left(m'', (j,k,m), (j',k',m')\right) \in \mathbb{Z}^2 \times \Lambda \times \Lambda,$$
where
$$\Lambda = \mathbb{N}_0 \times \{-[a_0^{\frac{j}{2}}], \dots, [a_0^{\frac{j}{2}}]\} \times \mathbb{Z}^2.$$
We define for $C \subseteq \mathbb{R}^2$, $L^2(C)^\vee = \{f : f \in L^2(\mathbb{R}^2) : \mathrm{supp}\widehat{\mathrm{f}} \subseteq \mathrm{C}\}$.

In a discrete shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$, in order to have a numerically stable reconstruction algorithm for f from the coefficients $\langle f, \psi_{j,k,m}\rangle$, we require that $\{\psi_{j,k,m}\}_{j,k,m}$ constitutes a frame. In this paper, using several ideas in [7] we establish a relation between shearlet frames and admissibility condition. The manuscript is organized as follows. In Section 2, we give a necessary condition via admissibility, for a discrete shearlet system to be a frame. In fact, we show that if a discrete shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ is a frame, then ψ is admissible. In Section 3, we establish such a condition for cone-adapted discrete shearlet systems. Finally, we give a similar result for higher dimensions.

2. The necessary condition for discrete shearlet systems

In this section, we will consider a discrete shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ as defined in (1.2) and we establish a necessary condition for this system to be a frame. The system $\{\psi_{j,k,m}\}_{j,k,m}$ is called a shearlet frame for $L^2(\mathbb{R}^2)$, if there exist constants $0 < A \leq B < \infty$ such that for all $f \in L^2(\mathbb{R}^2)$,
$$A\|f\|^2 \leq \sum_{j,k,m} |\langle f, \psi_{j,k,m}\rangle|^2 \leq B\|f\|^2. \tag{2.1}$$
Recall that an operator E is called of trace-class if $\sum_n |\langle Ee_n, e_n\rangle|$ is finite for all orthonormal bases $\{e_n\}$. The trace of E is defined to be
$$TrE = \sum_n \langle Ee_n, e_n\rangle.$$

Theorem 2.1. *If the discrete shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ constitutes a frame for $L^2(\mathbb{R}^2)$ with frame bounds A, B, then*
$$\alpha A \leq \int_0^\infty \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi_1,\xi_2)|^2}{\xi_1^2} d\xi_2 d\xi_1 \leq \alpha B, \tag{2.2}$$
for some constant $\alpha > 0$, i.e. ψ is an admissible shearlet.

Proof. Let $\{\psi_{j,k,m}\}_{j,k,m}$ constitute a frame with bounds A, B and $\{e_l\}_l$ be an orthonormal basis for $L^2(\mathbb{R}^2)$. Put $f = e_l$ in (2.1). Then for coefficients $c_l \geq 0$ with $\sum_l c_l\|e_l\|^2 < \infty$, we obtain
$$A\sum_l c_l\|e_l\|^2 \leq \sum_l c_l \sum_{j,k,m} |\langle e_l, \psi_{j,k,m}\rangle|^2 \leq B\sum_l c_l\|e_l\|^2. \tag{2.3}$$
If C is any positive trace-class operator, then $C = \sum_l c_l\langle ., e_l\rangle e_l$ and $\sum_l c_l = TrC > 0$. We have therefore, by (2.3)
$$A\, TrC \leq \sum_{j,k,m} \langle C\psi_{j,k,m}, \psi_{j,k,m}\rangle \leq B\, TrC. \tag{2.4}$$

Suppose $\operatorname{supp}(\widehat{h}) \subseteq [0,\infty) \times \mathbb{R}$ and $\int_0^\infty \int_{\mathbb{R}} \frac{|\widehat{h}(\xi)|^2}{\xi_1^2} d\xi_2 d\xi_1 < \infty$ (e.g. h may be a classical shearlet, see [13]). We consider

$$C = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle ., h_{a,s,t} \rangle h_{a,s,t} c(a,s,t) \frac{dtdsda}{a^3}, \tag{2.5}$$

where $h_{a,s,t}$ is defined as in (1.1) and

$$c(a,s,t) = \begin{cases} w(\frac{|s|}{a}, \frac{|t|}{a}), & 1 \leq a \leq a_0 \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

with $t = (t_1, t_2) \in \mathbb{R}^2$ and w positive and integrable i.e. $\int_{\mathbb{R}} \int_{\mathbb{R}^2} w(|s|,|t|) dtds < \infty$. We then have

$$C = \int_1^{a_0} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle ., h_{a,s,t} \rangle h_{a,s,t} w(\frac{|s|}{a}, \frac{|t|}{a}) dtds \frac{da}{a^3}.$$

So

$$\sum_{j,k,m} \langle C\psi_{j,k,m}, \psi_{j,k,m} \rangle = \sum_{j,k,m} \int_1^{a_0} \int_{\mathbb{R}} \int_{\mathbb{R}^2} w(\frac{|s|}{a}, \frac{|t|}{a}) |\langle \psi_{j,k,m}, h_{a,s,t} \rangle|^2 dtds \frac{da}{a^3}. \tag{2.7}$$

We calculate

$$\begin{aligned} \langle \psi_{j,k,m}, h_{a,s,t} \rangle &= a_0^{-\frac{3}{4}j} . a^{-\frac{3}{4}} \int \psi\Big(S_k A_{a_0^{-j}}(x-m)\Big) . \overline{h\Big(A_a^{-1} S_s^{-1}(x-t)\Big)} dx \\ &= a_0^{\frac{3}{4}j} . a^{-\frac{3}{4}} \int \psi(y) . \overline{h}\Big(A^{-1}_{aa_0^{-j}} S^{-1}_{s\sqrt{a_0^{-j}}+k}(y - S_k A_{a_0^{-j}}(t-m))\Big) dy \\ &= \Big\langle \psi, h_{aa_0^{-j}, s\sqrt{a_0^{-j}}+k, S_k A_{a_0^{-j}}(t-m)} \Big\rangle, \end{aligned} \tag{2.8}$$

where in the second equality above, we have chosen the change of variable $y = S_k A_{a_0^{-j}}(x-m)$. After the change of variables,

$$a' = aa_0^{-j}, \quad s' = s\sqrt{a_0^{-j}} + k, \quad t' = S_k A_{a_0^{-j}}(t-m),$$

the sum in (2.7) becomes

$$\begin{aligned} &\sum_{j,k,m} \int_{a_0^{-j}}^{a_0^{-j+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} w\Big(\frac{|a_0^{\frac{j}{2}} s' - a_0^{\frac{j}{2}} k|}{a_0^j a'}, \frac{|A^{-1}_{a_0^{-j}} S_k^{-1} t + m|}{a_0^j a'}\Big) \Big|\langle \psi, h_{a',s',t'} \rangle\Big|^2 a_0^{\frac{3}{2}j} dt' a_0^{\frac{j}{2}} ds' \frac{a_0^j da'}{a'^3 a_0^{3j}} \\ &= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} \Big|\langle \psi, h_{a,s,t} \rangle\Big|^2 \sum_{k,m} w\Big(\frac{|a_0^{-\frac{j}{2}}(s-k)|}{a}, \frac{|A^{-1}_{a_0^{-j}} S_k^{-1} t + m|}{a_0^j a}\Big) dtds \frac{da}{a^3}. \end{aligned} \tag{2.9}$$

Now consider w as

$$w(s,t) = \lambda^3 e^{-\lambda^2 \pi s^2} e^{-\lambda^2 \pi t_1^2} e^{-\lambda^2 \pi t_2^2}, \quad s \in \mathbb{R}, t = (t_1, t_2) \in \mathbb{R}^2.$$

By a similar argument as in the proof of [6, Lemma 2.2], we get

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^2} w(\alpha s+\beta,\gamma t+\eta)dtds - w_{max} &\\ &\leq \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^2} w(\alpha m+\beta,\gamma n+\eta)\\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} w(\alpha s+\beta,\gamma t+\eta)dtds + w_{max}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^2} w(s,t)dtds - (\alpha|det\gamma|)w_{max} &\\ &\leq (\alpha|det\gamma|)\sum_{m}\sum_{n} w(\alpha m+\beta,\gamma n+\eta)\\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}^2} w(s,t)dtds + (\alpha|det\gamma|)w_{max},\end{aligned}$$

where $\alpha = \frac{a_0^{-\frac{j}{2}}}{a}$, $|det\gamma| = \frac{1}{a_0^j a}|detA^{-1}_{a_0^j}S^{-1}_k| = \frac{a_0^{\frac{j}{2}}}{a}$.

Then, we have

$$\sum_{m}\sum_{n} w(\alpha m+\beta,\gamma n+\eta) = a^2+\rho(a,s,t),$$

such that $|\rho(a,s,t)| \leq w(0,0) = \lambda^3$. Therefore continuing from (2.9), (2.7) will be

$$\begin{aligned}\sum_{j,k,m}\langle C\psi_{j,k,m},\psi_{j,k,m}\rangle &= \int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}\left|\langle\psi,h_{a,s,t}\rangle\right|^2\left(a^2+\rho(a,s,t)\right)dtds\frac{da}{a^3}\\ &= \int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}\left|\langle\psi,h_{a,s,t}\rangle\right|^2 dtds\frac{da}{a}+R,\end{aligned}\tag{2.10}$$

where $R = \int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}|\langle\psi,h_{a,s,t}\rangle|^2\rho(a,s,t)dtds\frac{da}{a^3}$. Note that R is bounded. Indeed,

$$\begin{aligned}|R| &\leq \int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}\left|\langle\psi,h_{a,s,t}\rangle\right|^2\left|\rho(a,s,t)\right|dtds\frac{da}{a^3}\\ &\leq \lambda^3\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}\left|\langle\psi,h_{a,s,t}\rangle\right|^2 dtds\frac{da}{a^3}\\ &= \lambda^3\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}\left|\psi*h^*_{a,s,0}(t)\right|^2 dtds\frac{da}{a^3}\\ &= \lambda^3\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}\left|\widehat{\psi}(\xi)\right|^2.\left|\widehat{h^*}_{a,s,0}(\xi)\right|^2 d\xi ds\frac{da}{a^3}\\ &= \lambda^3\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}^2}\left|\widehat{\psi}(\xi)\right|^2.a^{-\frac{3}{2}}.\left|\widehat{h}\left(a\xi_1,\sqrt{a}(\xi_2+s\xi_1)\right)\right|^2 d\xi dsda,\end{aligned}\tag{2.11}$$

in which $h^*(x) = \overline{h(-x)}$. Moreover, the first term in (2.10), using the Plancherel theorem,is computed as follows

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left|\langle \psi, h_{a,s,t}\rangle\right|^2 dtds\frac{da}{a} &= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left|\widehat{\psi}(\xi)\right|^2 .a^{\frac{1}{2}}.\left|\widehat{h}\Big(a\xi_1,\ \sqrt{a}(\xi_2 + s\xi_1)\Big)\right|^2 d\xi dsda \\ &= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} \left|\widehat{\psi}(\xi)\right|^2 .\frac{1}{{\xi_1}^2}\left|\widehat{h}(w_1, w_2)\right|^2 d\xi dw_2 dw_1 \\ &= \int_{\mathbb{R}} \int_0^\infty \frac{|\widehat{\psi}(\xi_1, \xi_2)|^2}{\xi_1^2} d\xi_1 d\xi_2 \int_0^\infty \int_{\mathbb{R}} \left|\widehat{h}(w_1, w_2)\right|^2 dw_2 dw_1 \\ &+ \int_{\mathbb{R}} \int_{-\infty}^0 \frac{|\widehat{\psi}(\xi_1, \xi_2)|^2}{\xi_1^2} d\xi_1 d\xi_2 \int_{-\infty}^0 \int_{\mathbb{R}} \left|\widehat{h}(w_1, w_2)\right|^2 dw_2 dw_1 \\ &= c_\psi^+ \left\|\widehat{h}\right\|^2,\end{aligned}$$

where $\xi = (\xi_1, \xi_2)$, $w_1 = a\xi_1$, $w_2 = \sqrt{a}(\xi_2 + s\xi_1)$. Furthermore,

$$\begin{aligned} TrC &= \int_1^{a_0} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \|h\|^2 w\Big(\frac{|s|}{a}, \frac{|t|}{a}\Big) dtds\frac{da}{a^3} \\ &= \|h\|^2 \ln a_0 \int_{\mathbb{R}} \int_{\mathbb{R}^2} w(|s|, |t|) dtds. \end{aligned} \tag{2.12}$$

Since $\int_{\mathbb{R}} \int_{\mathbb{R}^2} w(|s|, |t|)dsdt = 1$, then by (2.12), $TrC = \|h\|^2 \ln a_0$. Hence by (2.4)

$$A\Big(\|h\|^2 \ln a_0\Big) \le c_\psi^+ \|h\|^2 + R \le B\Big(\|h\|^2 \ln a_0\Big), \tag{2.13}$$

where $|R| \le \lambda^3 \|\widehat{\psi}\|^2 (c_h^+ + c_h^-)$. If we divide (2.13) by $\|h\|^2$ and let λ tend to zero, then the result follows by considering $\alpha := \ln a_0$. □

In the following example, we give a Parseval shearlet frame which is admissible by Theorem 2.1.

Example 2.2. Let $\psi_1 \in L^2(\mathbb{R})$ be a Lemarie'-Meyer wavelet that satisfies the discrete Caldero'n condition

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}_1(2^{-j}w)|^2 = 1,$$

with $\widehat{\psi}_1 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_1 \subseteq [-\frac{1}{2}, -\frac{1}{16}] \cup [\frac{1}{16}, \frac{1}{2}]$, Consider $\psi_2 \in L^2(\mathbb{R})$ is a bump function such that $\|\widehat{\psi}_2\|_2 = 1$ and for all $w \in [-1, 1]$,

$$\sum_{k=-1}^{1} |\widehat{\psi}_2(w + k)|^2 = 1,$$

where $\widehat{\psi}_2 \in C^\infty(\mathbb{R})$ and supp $\widehat{\psi}_2 \subseteq [-1, 1]$. Suppose $\psi \in L^2(\mathbb{R}^2)$ is given by

$$\widehat{\psi}(\xi_1, \xi_2) = \widehat{\psi}_1(\xi_1)\widehat{\psi}_2(\frac{\xi_2}{\xi_1}).$$

By [13, Proposition 2], the shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ as defined in (1.2) with $a_0 = 2$ is a Parseval frame for $L^2(\mathbb{R}^2)$. So by Theorem 2.1, we have $C^+_\psi = C^-_\psi = \ln 2$.

In [3] the continuous shearlet transform is generalized to higher dimensions. Here we give the discrete version and state our main result in this setting. In fact, for $\psi \in L^2(\mathbb{R}^d)$, we define the discrete shearlet system as

$$\Big\{\psi_{j,k,m} = (a_0^{-\frac{j}{2}})^{2-\frac{1}{d}}\psi\Big(S_k A_{a_0^{-j}}(x - m)\Big) : j \in \mathbb{Z}, k \in \mathbb{Z}^{d-1}, m \in \mathbb{Z}^d\Big\},\ a_0 > 0,$$

where

$$A_{a_0^{-j}} = \begin{bmatrix} a_0^{-j} & 0^T_{d-1} \\ 0_{d-1} & \operatorname{sgn}(a_0^{-j})|a_0^{-j}|^{\frac{1}{d}}.I_{d-1} \end{bmatrix}, \qquad S_k = \begin{bmatrix} 1 & k^T \\ 0_{d-1} & I_{d-1} \end{bmatrix}.$$

Proposition 2.3. *If the system $\{\psi_{j,k,m}\}_{j,k,m}$ constitutes a frame for $L^2(\mathbb{R}^d)$ with frame bounds A, B, then ψ is admissible, in the sense that*

$$\alpha A \le \int_0^\infty \int_{\mathbb{R}^{d-1}} \frac{|\widehat{\psi}(\xi_1, \xi_2)|^2}{\xi_1^d} d\xi_2 d\xi_1 \le \alpha B, \tag{2.14}$$

for some constant $\alpha > 0$, ((2.14) is the admissibility condition appeared in [3, Theorem 2.4]).

The proof of Proposition 2.3 is straightforward and therefore is omitted.

The sufficient condition for the shearlet system $\{\psi_{j,k,m}\}_{j,k,m}$ to be a frame for $L^2(\mathbb{R}^2)$ is proposed in [11, Theorem 3.1].

3. The necessary condition for cone-adapted discrete shearlet systems

Similar to Theorem 2.1 a necessary condition can be given for a cone-adapted discrete shearlet system to be a frame. For convenience we denote the cone-adapted discrete shearlet system (1.5) by $\{g_\alpha\}_\alpha$. We define a cone-adapted discrete shearlet system $\{g_\alpha\}_\alpha$ to be a frame for $L^2(\mathbb{R}^2)$ if there exists $0 < A, B < \infty$ such that

$$A\|f\|^2 \le \sum_{m''} |\langle f, \phi_{m''}\rangle|^2 + \sum_{j,k,m} |\langle f, \psi_{j,k,m}\rangle|^2 + \sum_{j',k',m'} |\langle f, \psi_{j',k',m'}\rangle|^2 \le B\|f\|^2, \tag{3.1}$$

for all $f \in L^2(\mathbb{R}^2)$.

The following theorem is our main result of this section which is a necessary condition via admissibility for a cone-adapted discrete shearlet system to be a frame.

Theorem 3.1. *If the cone-adapted discrete shearlet system $\{g_\alpha\}_\alpha$ is a frame for $L^2(\mathbb{R}^2)$, then there exists $\Delta \subseteq \mathbb{R}^2$ such that the following admissibility condition holds*

$$A\zeta \le |\widehat{\phi}(\xi)|^2 + c^+_\psi + c^+_{\tilde{\psi}} \le B\zeta, \qquad \xi \in \Delta. \tag{3.2}$$

Proof. Let the system $\{g_\alpha\}_\alpha$ constitute a frame with bounds A, B. Consider $\{e_l\}_l$ an orthonormal basis for $L^2(\mathbb{R}^2)$. Put $f = e_l$ in (3.1). Then for coefficients $c_l \geq 0$ with $\sum_l c_l \|e_l\|^2 < \infty$, we obtain

$$A \sum_l c_l \|e_l\|^2 \leq \sum_l c_l \sum_\alpha |\langle e_l, g_\alpha\rangle|^2 \leq B \sum_l c_l \|e_l\|^2. \tag{3.3}$$

If C is any positive trace-class operator, then as in the proof of Theorem 2.1

$$A\, TrC \leq \sum_\alpha \langle C g_\alpha, g_\alpha\rangle \leq B\, TrC. \tag{3.4}$$

Suppose that $h \in L^2(\mathbb{R}^2)$, with $\mathrm{supp}(\hat{\mathrm{h}}) \subseteq [0,\infty) \times \mathbb{R}$ and $\int_0^\infty \int_{\mathbb{R}} \frac{|\widehat{h}(\xi)|^2}{\xi_1^2} d\xi_2 d\xi_1 < \infty$. Also assume that for $a \in \mathbb{R}^+ - \{1\}$, $s \in \mathbb{R} - \{0\}$ and $t \in \mathbb{R}^2$, we have $h_{a,s,t} \in L^2((\mathcal{E}_1 \cup \mathcal{E}_3) \cup (\mathcal{E}_2 \cup \mathcal{E}_4))^\vee$, and for $a = 1, s = 0, t \in \mathbb{R}$, we have $h_{a,s,t} \in L^2(\mathcal{R})^\vee$.

Consider

$$C = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle ., h_{a,s,t}\rangle h_{a,s,t} c(a,s,t) dt ds \frac{da}{a^3} + \int_{\mathbb{R}^2} \langle ., h_{1,0,t}\rangle h_{1,0,t} c(1,0,t) dt, \tag{3.5}$$

in which $c(a,s,t)$ is defined as (2.6) for $1 < a \leq a_0$, $s \in \mathbb{R} - \{0\}$, $t = (t_1, t_2) \in \mathbb{R}^2$ and $c(1,0,t) = \lambda^2 e^{-\pi\lambda^2 |t|^2}$. Then we have

$$\sum_\alpha \langle C g_\alpha, g_\alpha\rangle = \sum_{m''} \langle C\phi_{m''}, \phi_{m''}\rangle + \sum_{j,k,m} \langle C\psi_{j,k,m}, \psi_{j,k,m}\rangle + \sum_{j',k',m'} \langle C\tilde{\psi}_{j',k',m'}, \tilde{\psi}_{j',k',m'}\rangle. \tag{3.6}$$

By definition of C as in (3.5), we obtain

$$\begin{aligned} \sum_{m''} \langle C\phi_{m''}, \phi_{m''}\rangle &= \sum_{m''} \Big\langle \int_{\mathbb{R}^2} \langle \phi_{m''}, h_{1,0,t}\rangle h_{1,0,t} c(1,0,t) dt, \phi_{m''} \Big\rangle \\ &= \sum_{m''} \int_{\mathbb{R}^2} \Big| \langle \phi_{m''}, h_{1,0,t}\rangle \Big|^2 c(1,0,t) dt \\ &= \sum_{m''} \int_{\mathbb{R}^2} \Big| \langle \phi, h_{1,0,t}\rangle \Big|^2 \lambda^2 e^{-\pi\lambda^2 |t+m''|^2} dt, \end{aligned}$$

where $\sum_{m''} \lambda^2 e^{-\pi\lambda^2|t+m''|^2} = 1 + \rho(t)$, such that $|\rho(t)| \leq \lambda^2$. Hence we have

$$\begin{aligned} \sum_{m''} \langle C\phi_{m''}, \phi_{m''}\rangle &= \int_{\mathbb{R}^2} |\langle \phi, h_{1,0,t}\rangle|^2 dt + \int_{\mathbb{R}^2} |\langle \phi, h_{1,0,t}\rangle|^2 \rho(t) dt \\ &= \int_{\mathbb{R}^2} |\langle \phi, h_{1,0,t}\rangle|^2 dt + R_1, \end{aligned}$$

where $R_1 = \int_{\mathbb{R}^2} |\langle \phi, h_{1,0,t}\rangle|^2 \rho(t)dt$. Also R_1 is bounded, since

$$\begin{aligned} |R_1| &\leq \lambda^2 \int_{\mathbb{R}^2} |\langle \phi, h_{1,0,t}\rangle|^2 dt \\ &= \lambda^2 \int_{\mathbb{R}^2} |(\phi * h^*)(t)|^2 dt \\ &\leq \lambda^2 \|\widehat{\phi}\|^2 \|\widehat{h}\|^2 < \infty. \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^2} |\langle \phi, h_{1,0,t}\rangle|^2 dt = \int_{\mathbb{R}^2} |\widehat{\phi}(\xi)|^2 |\widehat{h}(\xi)|^2 d\xi.$$

So

$$\sum_{m''} \langle C\phi_{m''}, \phi_{m''}\rangle = \int_{\mathbb{R}^2} |\widehat{\phi}(\xi)|^2 |\widehat{h}(\xi)|^2 d\xi + R_1.$$

Also, similar to the proof of Theorem 2.1 for $\psi_{j,k,m}$ and $\tilde{\psi}_{j',k',m'}$, we have

$$\sum_{j,k,m} \langle C\psi_{j,k,m}, \psi_{j,k,m}\rangle = c_{\psi}^{+} \int_{\mathbb{R}^2} |\widehat{h}(\xi)|^2 d\xi + R_2,$$

where $R_2 = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\langle \psi, h_{a,s,t}\rangle|^2 \rho(a,s,t) dt ds \frac{da}{a^3}$ and

$$\sum_{j',k',m'} \langle C\tilde{\psi}_{j',k',m'}, \tilde{\psi}_{j',k',m'}\rangle = c_{\tilde{\psi}}^{+} \int_{\mathbb{R}^2} |\widehat{h}(\xi)|^2 d\xi + R_3,$$

where $R_3 = \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\langle \tilde{\psi}, h_{a,s,t}\rangle|^2 \rho(a,s,t) dt ds \frac{da}{a^3}$. Then

$$\sum_{\alpha} \langle C g_\alpha, g_\alpha\rangle = \int_{\mathbb{R}^2} \left(|\widehat{\phi}(\xi)|^2 + c_{\psi}^{+} + c_{\tilde{\psi}}^{+} \right) |\widehat{h}(\xi)|^2 d\xi + R_1 + R_2 + R_3.$$

Furthermore,

$$\begin{aligned} TrC &= \sum_n \langle C e_n, e_n\rangle \\ &= \sum_n \langle \int_1^{a_0} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle e_n, h_{a,s,t}\rangle h_{a,s,t} c(a,s,t) dt ds \frac{da}{a^3}, e_n\rangle \\ &+ \sum_n \langle \int_{\mathbb{R}^2} \langle e_n, h_{1,0,t}\rangle h_{1,0,t} c(1,0,t) dt, e_n\rangle \\ &= \int_1^{a_0} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \sum_n |\langle e_n, h_{a,s,t}\rangle|^2 c(a,s,t) dt ds \frac{da}{a^3} \\ &+ \int_{\mathbb{R}^2} \sum_n |\langle e_n, h_{1,0,t}\rangle|^2 c(1,0,t) dt \\ &= \|h_{a,s,t}\|^2 \int_1^{a_0} \int_{\mathbb{R}} \int_{\mathbb{R}^2} c(a,s,t) dt ds \frac{da}{a^3} + \|h_{1,0,t}\|^2 \int_{\mathbb{R}^2} c(1,0,t) dt \\ &= \|h\|^2 . \zeta, \end{aligned}$$

where $\zeta = \int_1^{a_0}\int_{\mathbb{R}}\int_{\mathbb{R}^2} c(a,s,t)dtds\frac{da}{a^3} + \int_{\mathbb{R}^2} c(1,0,t)dt = \ln a_0 + 1$. Hence by (3.4)

$$A(\|h\|^2.\zeta) \leq \int_{\mathbb{R}^2}(|\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+)|\widehat{h}(\xi)|^2 d\xi + R_1 + R_2 + R_3 \leq B(\|h\|^2.\zeta).$$

Since $\|h\| = \|\widehat{h}\|$, so

$$\int_{\mathbb{R}^2} A\zeta|\widehat{h}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^2}(|\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+)|\widehat{h}(\xi)|^2 d\xi + R_1 + R_2 + R_3 \leq \int_{\mathbb{R}^2} B\zeta|\widehat{h}(\xi)|^2 d\xi.$$

Let λ tend to zero, then

$$\int_{\mathbb{R}^2} A\zeta|\widehat{h}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^2}(|\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+)|\widehat{h}(\xi)|^2 d\xi \leq \int_{\mathbb{R}^2} B\zeta|\widehat{h}(\xi)|^2 d\xi. \tag{3.7}$$

By (3.7), we have

$$\int_{\mathbb{R}^2}(|\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+ - B\zeta)|\widehat{h}(\xi)|^2 d\xi \leq 0, \tag{3.8}$$

and

$$\int_{\mathbb{R}^2}(|\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+ - A\zeta)|\widehat{h}(\xi)|^2 d\xi \geq 0. \tag{3.9}$$

Now since $|\widehat{h}(\xi)|^2 > 0$, then by (3.8) there exists $\Delta_1 \subseteq \mathbb{R}^2$ such that for all $\xi \in \Delta_1$, we have

$$|\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+ - B\zeta \leq 0,$$

and by (3.9) there exists $\Delta_2 \subseteq \mathbb{R}^2$ such that for all $\xi \in \Delta_2$, we have

$$|\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+ - A\zeta \geq 0.$$

Consider $\Delta := \Delta_1 \cap \Delta_2$, then for all $\xi \in \Delta$ we have

$$A\zeta \leq |\widehat{\phi}(\xi)|^2 + c_\psi^+ + c_{\tilde{\psi}}^+ \leq B\zeta.$$

□

Example 3.2. Consider

$$C_1 = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : |\frac{\xi_2}{\xi_1}| \leq 1\}\ , \quad C_2 = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : |\frac{\xi_2}{\xi_1}| > 1\}.$$

Define

$$f(x) = \begin{cases} 0 & , \quad x < 0 \\ 35x^4 - 84x^5 + 70x^6 - 20x^7 & , \quad 0 \leq x < 1 \\ 1 & , \quad x \geq 1 \end{cases},$$

$$v(u) = \begin{cases} \sqrt{f(1+u)} & , \quad u \leq 0 \\ \sqrt{f(1-u)} & , \quad u > 0 \end{cases}.$$

It is obvious that $f, v \in C(\mathbb{R})$, also supp $v \subset [-1, 1]$ and

$$|v(u-1)|^2 + |v(u)|^2 + |v(u+1)|^2 = 1, \qquad \text{for } |u| \le 1. \tag{3.10}$$

In addition, we have $v(0) = 1$ and by (3.10),

$$\sum_{m=-2^j}^{2^j} |v(2^j u - m)|^2 = 1, \qquad \text{for } |u| \le 1. \tag{3.11}$$

Let ϕ be given by

$$\hat{\phi}(\xi) = c\, \xi^2 e^{-\frac{1}{2}(5\xi)^2} \ , \quad \xi \in \mathbb{R},$$

in which we have chosen c so that $0 \le \hat{\phi} \le 1$. ($e.g.$ $c = 33.9264$) and supp $\hat{\phi} \subseteq [-1, 1]$, ($\hat{\phi}(1) = 1.2647 \times 10^{-4}$).

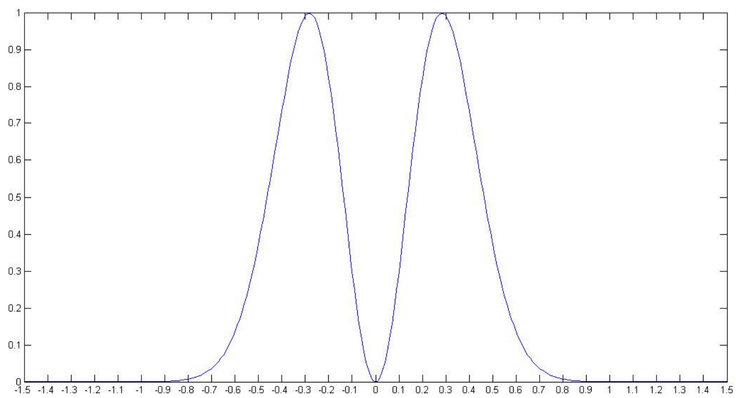

Figure 1: The graph of $\hat{\phi}$ (c=33.9)

We consider

$$\hat{\Phi}(\xi_1, \xi_2) = \hat{\phi}(\xi_1)\hat{\phi}(\xi_2), \quad (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{3.12}$$

Then

$$0 \le \hat{\Phi}(\xi_1, \xi_2) \le 1. \tag{3.13}$$

Now, define

$$\hat{\Psi}(\xi_1, \xi_2) = \hat{\psi}(\xi_1)\hat{\psi}(\xi_2),$$

where

$$\hat{\psi}(\xi) = \begin{cases} (\xi - 1)^2 e^{-(\xi-1)^2}, & \xi > 1 \\ (\xi + 1)^2 e^{-(\xi+1)^2}, & \xi < -1 \\ 0 \qquad , & -1 \le \xi \le 1 \end{cases} ,$$

$(\xi_1, \xi_2) \in \mathbb{R}^2$.

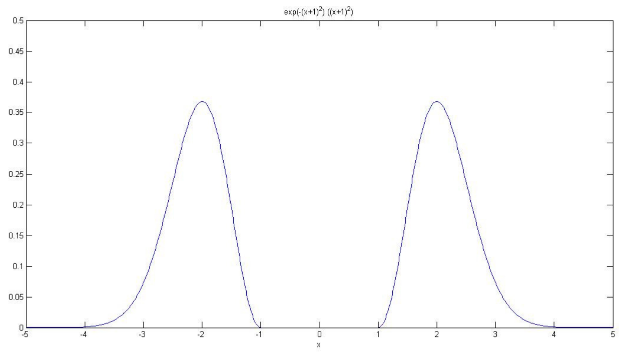

Figure 2: The graph of $\hat{\psi}$

By definition of $\hat{\Psi}$, it is clear that for $\xi = (\xi_1, \xi_2)$

$$0 \leq \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 < \infty, \qquad \text{for } \xi_{\rm i} \in \text{supp } \hat{\psi}\ , \ \ \text{i} = 1, 2.$$

Infact, there exists a positive constant $b \in \mathbb{R}$ such that

$$0 \leq \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 \leq b. \tag{3.14}$$

Now, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ consider the following cone-adapted shearlet system for $L^2(\mathbb{R}^2)$:

$$\begin{aligned} \{\Phi(\cdot - k) : k \in \mathbb{Z}^2\} \cup \{\Psi^{(d)}_{j,k,m} : j \geq 0, |k| < 2^j, m \in \mathbb{Z}^2, d = 1, 2\} \\ \cup \{\tilde{\Psi}_{j,k,m} : j \geq 0, k = \pm 2^j, m \in \mathbb{Z}^2\}, \end{aligned} \tag{3.15}$$

where

$$\hat{\Psi}^{(1)}_{j,k,m}(\xi) = 2^{-\frac{3}{2}j}\hat{\Psi}(2^{-2j}\xi)V_1(\xi A_1^{-j} S_1^{-k})e^{2\pi i \xi A_1^{-j} S_1^{-k} m},$$

and

$$A_1 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V_1(\xi_1, \xi_2) = v(\frac{\xi_2}{\xi_1}), \quad \xi \in \mathbb{R}^2,$$

$$\hat{\Psi}^{(2)}_{j,k,m}(\xi) = 2^{-\frac{3}{2}j}\hat{\Psi}(2^{-2j}\xi)V_2(\xi A_2^{-j} S_2^{-k})e^{2\pi i \xi A_2^{-j} S_2^{-k} m},$$

and

$$A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad V_2(\xi_1, \xi_2) = v(\frac{\xi_1}{\xi_2}), \quad \xi \in \mathbb{R}^2.$$

Also, for $j > 0$, define

$$(\tilde{\Psi}_{j,k,m})^\wedge(\xi) = \begin{cases} 2^{-\frac{3}{2}j - \frac{1}{2}}\hat{\Psi}(2^{-2j}\xi_1, 2^{-2j}\xi_2)v(2^j\frac{\xi_2}{\xi_1} - k)e^{2\pi i \xi 2^{-1} A_1^{-j} S_1^{-k} m}, & \text{if } \xi \in \mathrm{C}_1 \\ 2^{-\frac{3}{2}j - \frac{1}{2}}\hat{\Psi}(2^{-2j}\xi_1, 2^{-2j}\xi_2)v(2^j\frac{\xi_1}{\xi_2} - k)e^{2\pi i \xi 2^{-1} A_1^{-j} S_1^{-k} m}, & \text{if } \xi \in \mathrm{C}_2, \end{cases},$$

and for $j = 0,\ m \in \mathbb{Z}^2,\ k = \pm 1$,

$$(\tilde{\Psi}_{0,k,m})^{\wedge}(\xi) = \begin{cases} \hat{\Psi}(\xi_1, \xi_2) v(\frac{\xi_2}{\xi_1} - k) e^{2\pi i \xi m}, & \text{if } \xi \in \mathrm{C}_1 \\ \hat{\Psi}(\xi_1, \xi_2) v(\frac{\xi_1}{\xi_2} - k) e^{2\pi i \xi m}, & \text{if } \xi \in \mathrm{C}_2 \end{cases}.$$

The following calculations show that the shearlet system (3.15) is a frame for $L^2(\mathbb{R}^2)$.

For $f \in L^2(\mathbb{R}^2)$, we observe that

$$\sum_{d=1}^{2} \sum_{j \geq 0} \sum_{|k|<2^j} \sum_{m \in \mathbb{Z}^2} |\langle f, \Psi^{(d)}_{j,k,m} \rangle|^2 + \sum_{j \geq 0} \sum_{k = \pm 2^j} \sum_{m \in \mathbb{Z}^2} |\langle f, \tilde{\Psi}_{j,k,m} \rangle|^2$$

$$= \sum_{j \geq 0} \sum_{|k|<2^j} \sum_{m \in \mathbb{Z}^2} \left(|\langle \hat{f}, \hat{\Psi}^{(1)}_{j,k,m} \rangle|^2 + |\langle \hat{f}, \hat{\Psi}^{(2)}_{j,k,m} \rangle|^2 \right) + \sum_{j \geq 0} \sum_{k = \pm 2^j} \sum_{m \in \mathbb{Z}^2} |\langle \hat{f}, (\tilde{\Psi}_{j,k,m})^{\wedge} \rangle|^2$$

$$= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 \left(\sum_{|k|<2^j} |v(2^j \frac{\xi_2}{\xi_1} - k)|^2 + \sum_{|k|<2^j} |v(2^j \frac{\xi_1}{\xi_2} - k)|^2 \right) d\xi$$

$$+ \int_{C_1} |\hat{f}(\xi)|^2 \textstyle\sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 |v(2^j(\frac{\xi_2}{\xi_1} - 1))|^2 d\xi$$

$$+ \int_{C_1} |\hat{f}(\xi)|^2 \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 |v(2^j(\frac{\xi_2}{\xi_1} + 1))|^2 d\xi$$

$$+ \int_{C_2} |\hat{f}(\xi)|^2 \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 |v(2^j(\frac{\xi_1}{\xi_2} - 1))|^2 d\xi$$

$$+ \int_{C_2} |\hat{f}(\xi)|^2 \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 |v(2^j(\frac{\xi_1}{\xi_2} + 1))|^2 d\xi$$

$$= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 \sum_{|k| \leq 2^j} (|v(2^j \frac{\xi_2}{\xi_1} - k)|^2 \chi_{C_1}(\xi) + |v(2^j \frac{\xi_1}{\xi_2} - k)|^2 \chi_{C_2}(\xi)) d\xi$$

$$= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j \geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 d\xi,$$

the last equality results from $C_1 \cap C_2 = \emptyset$ and (3.11).

Finally, using (3.12), for any $f \in L^2(\mathbb{R}^2)$ we have

$$\sum_{n\in\mathbb{Z}^2} |\langle f, \Phi(\cdot - n)\rangle|^2 + \sum_{d=1}^{2}\sum_{j\geq 0}\sum_{|k|<2^j}\sum_{m\in\mathbb{Z}^2} |\langle f, \Psi^{(d)}_{j,k,m}\rangle|^2 + \sum_{j\geq 0}\sum_{k=\pm 2^j}\sum_{m\in\mathbb{Z}^2} |\langle f, \tilde{\Psi}_{j,k,m}\rangle|^2$$

$$= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 |\hat{\Phi}(\xi)|^2 d\xi + \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \sum_{j\geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 d\xi$$

$$= \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left(|\hat{\Phi}(\xi)|^2 + \sum_{j\geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 \right) d\xi.$$

It follows from (3.14) there exists a positive constant $a \in \mathbb{R}$, so that

$$a \leq |\hat{\Phi}(\xi)|^2 + \sum_{j\geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 \leq (b+1).$$

(Note that $a \neq 0$. Indeed, for $\xi_1 = \xi_2 = 1$, $\sum_{j\geq 0} |\hat{\Psi}(2^{-2j}\xi)|^2 = 0$ but $|\hat{\phi}(1)| = 1.2647 \times 10^{-4} \neq 0$). Hence

$$a\|f\|^2 \leq \sum_{n\in\mathbb{Z}^2} |\langle f, \Phi(\cdot - n)\rangle|^2 + \sum_{d=1}^{2}\sum_{j\geq 0}\sum_{|k|<2^j}\sum_{m\in\mathbb{Z}^2} |\langle f, \Psi^{(d)}_{j,k,m}\rangle|^2$$

$$+ \sum_{j\geq 0}\sum_{k=\pm 2^j}\sum_{m\in\mathbb{Z}^2} |\langle f, \tilde{\Psi}_{j,k,m}\rangle|^2 \leq (b+1)\|f\|^2.$$

So by Theorem 3.1 the admissibility condition (3.2) holds for the shearlet frame (3.15), i.e. there exist $\zeta \in \mathbb{R}^+$, so that for $\xi \in \operatorname{supp} \hat{\phi}$, we have

$$a'\zeta \leq |\widehat{\phi}(\xi)|^2 + c^+_{\psi^{(1)}} + c^+_{\psi^{(2)}} + c^+_{\tilde{\psi}} \leq (b+1)\zeta.$$

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